Characterization and Continuity of Fuzzy Morphological Associative Memories on Complete Lattice-Ordered Double Monoids

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Abstract

Fuzzy associative memories (FAMs) are between-cube fuzzy system. They are often defined as artificial neural networks whose inputs, outputs, and connection weights are fuzzy valued. Recently, Valle and Sussner observed that many FAM models are equipped with neurons that perform elementary operations of mathematical morphology such as dilation or erosion. Thus, they can be classified as fuzzy morphological associative memories (FMAMs). Although complete lattices provide a general framework for FMAMs, in this paper we note that these models can be completely characterized in a mathematical structure called clodum or complete lattice-ordered double monoid. Precisely, in a clodum, a between-cube fuzzy system - and consequently an FMAM model - can be viewed a fuzzy logic neural network if, and only if, it yields a mapping that performs either a dilation or an erosion that is invariant under a certain type of membership regraduations. Furthermore, we show that fuzzy learning by adjunction yields a continuous FMAM if all the association pairs are memorized correctly.

Keywords: Fuzzy associative memories, fuzzy learning by adjunction, artificial neural networks, mathematical morphology, complete lattices.

1 Introduction

Associative memories (AMs) are models inspired in the human brain ability to store and recall information [8, 10]. In other words, AM are input-output systems able to store a set $\{(\mathbf{x}^1, \mathbf{y}^1), \ldots, (\mathbf{x}^k, \mathbf{y}^k)\}$ of association pairs. In mathematical terms, an AM corresponds to a mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ denote the sets of input and output patterns, respectively. In principle, the mapping $\mathcal{A}$ is required to satisfy $\mathcal{A}(\mathbf{x}^\xi) = \mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$, which means that the item $\mathbf{y}^\xi$ can be retrieved upon presentation of $\mathbf{x}^\xi$. In addition, such as the brain, AMs should be able to retrieve a memorized information from a possible incomplete or corrupted item. Therefore, the mapping $\mathcal{A}$ should be continuous in the sense that $\mathcal{A}(\tilde{\mathbf{x}}^\xi)$ equals $\mathbf{y}^\xi$ even for noisy or incomplete versions $\tilde{\mathbf{x}}^\xi$ of $\mathbf{x}^\xi$. This feature makes AMs suitable for a wide variety of applications such as classification [21, 34], biometric technologies [32, 33], image processing [7, 25, 26], and prediction [23, 20]. Due to the biological motivation, most AM models are given by artificial neural networks [8, 10].

We speak of a *fuzzy associative memory* (FAM) if the mapping $\mathcal{A}$ correspond to a fuzzy system and the patterns $\mathbf{x}^\xi \in [0,1]^n$ and $\mathbf{y}^\xi \in [0,1]^m$ represent fuzzy sets for every $\xi = 1, \ldots, k$ [12, 16]. Thus, in theory, a FAM model is a between-cube continuous fuzzy system. In general, FAMs are given by *fuzzy logic neural networks* (FLNN) [16]. For example, the famous max-min and max-product FAMs of Kosko are equipped with fuzzy logical neurons that compute the maximum of minimums and the maximum of products, respectively [12]. Similarly, the *generalized FAMs* (GFAMs) of Chung and Lee and the *implicative fuzzy associative memories* (IFAMs) have neurons that compute the maximum of triangular norms [2, 22]. Despite successful applications of FLNN-FAMs to problems such as backing up a truck and trailer [6, 11, 12] and forecasting the average monthly streamflow of a large hydroelectric plant [20, 24, 23], we are faced with a number of mathematical questions. For example, which properties an FLNN-FAM exhibit? Are FLNN-FAMs indeed continuous mappings?

Recently, Valle and Sussner observed that many FLNN-FAMs are equipped with fuzzy neurons that perform an elementary operation of mathematical morphology such as dilation or erosion [24, 29]. Thus, they can be classified as belonging to the broad class of *fuzzy morphological associative memories* (FMAMs). In this paper, we point out that the most important FMAM models in the literature, including GFAMs and IFAMs, can be well defined in a mathematical structure called *clodum* or *complete lattice-ordered double monoid* [13]. Also, we note that they yield a mapping that is invariant under a certain type of membership regraduations. Conversely, we have that if an FMAM $\mathcal{A} : [0,1]^n \rightarrow [0,1]^m$ performs either a dilation or an erosion invariant under membership regraduations, then it is given by a FLNN. Finally, we address the continuity of FMAMs on a clodum. Precisely, we show that an FMAM model trained by means of the *fuzzy learning by adjunction* (FLA) is continuous if and only if $\mathcal{A}(\mathbf{x}^\xi) = \mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$. In other words, an FMAM with FLA is continuous if and only if it is able to memorize all the association pairs $(\mathbf{x}^\xi, \mathbf{y}^\xi)$ correctly.

The paper is organized as follows. Section 2 briefly reviews the clodum framework. Section 3 introduces the classes of max-$\star$ and min-$\star$ FMAMs. This section also provides a proposition which characterizes these FMAM models. Section 4 briefly recalls a recording recipe which can be used for the storage of patterns in max-$\star$ and min-$\star$ FMAMs. Section 5 establishes a relationship between the continuity of max-$\star$ FMAMs and the fact that the model is able to store correctly the association pairs. The paper finishes with the concluding remarks in Section 6.

2 A Brief Review on Mathematical Morphology and Clodums

Mathematical morphology (MM) is a theory that is concerned with the processing and analysis of objects using operators and functions based on topological and geometrical concepts [9, 18, 19]. This theory can be very well conducted in complete lattices [9, 19].

A *complete lattice* $\mathbb{L}$ constitutes a partially ordered set in which every subset has an infimum and a supremum in $\mathbb{L}$ [1, 9]. The infimum and the supremum of $X \subseteq \mathbb{L}$ are denoted by $\bigwedge X$ and $\bigvee X$, respectively. We speak of an *infinitely distributive complete lattice* if the following equations hold true for every $a \in \mathbb{L}$ and $X \subseteq \mathbb{L}$:

$$a \wedge \left(\bigvee X\right) = \bigvee_{x \in X} (a \wedge x) \quad \text{and} \quad a \vee \left(\bigwedge X\right) = \bigwedge_{x \in X} (a \vee x)\,. \tag{1}$$

The unit interval $[0,1]$ and the class of all fuzzy sets defined on a finite universe of discourse, i.e., the hypercube $[0,1]^n$, are examples of infinitely distributive complete lattices [14].

Two important operators of MM are erosion and dilation [9, 19]. Given complete lattices $\mathbb{L}$ and $\mathbb{M}$, an *erosion* is a mapping $\varepsilon : \mathbb{L} \rightarrow \mathbb{M}$ that commutes with the infimum operation. Similarly, an operator $\delta : \mathbb{L} \rightarrow \mathbb{M}$ that commutes with the supremum is called a *dilation*. Mathematically, an operator ε represents an erosion and an operator δ represents a dilation if, and only if, the following equalities hold for every subset $X \subseteq \mathbb{L}$:

$$\varepsilon\left(\bigwedge X\right) = \bigwedge_{x \in X} \varepsilon(x) \quad \text{and} \quad \delta\left(\bigvee X\right) = \bigvee_{x \in X} \delta(x)\,. \tag{2}$$

In the context of fuzzy logic and fuzzy sets, we usually consider the infinitely distributive complete lattice $[0,1]$ equipped with triangular norms and co-norms, which are used to define the intersection and the union of fuzzy sets [14, 16]. Thus, let us consider an infinitely distributive complete lattice $\mathbb{L}$ with two binary operations $\star$ and $\star'$ such that both $(\mathbb{L}, \star)$ and $(\mathbb{L}, \star')$ are monoids [1]. In other words, suppose that the operations $\star$ and $\star'$ are associative and both have an identity element. For simplicity, let us also assume that these two operations are commutative. For example, we may consider $\mathbb{L} = [0,1]$ and identify the operations $\star$ and $\star'$ with a triangular norm and a triangular co-norm, respectively. Another example is given by the unit interval $[0,1]$ equipped with two kinds of uninorm operators [16, 31]. If, in addition, the operation $\star$ performs a dilation and the operation $\star'$ yields an erosion, then the mathematical structure $(\mathbb{L}, \vee, \wedge, \star, \star')$ is called *clodum* or *complete lattice-ordered double monoid*. Clodums have been introduced by Maragos in order to unify several approaches toward MM, including binary, fuzzy, and gray-scale MM [13]. This paper shows that clodums also provide an appropriate mathematical framework for FMAMs.

Finally, consider a clodum $([0,1], \vee, \wedge, \star, \star')$. Given $\alpha \in [0,1]$ and a fuzzy set $\mathbf{x} = [x_1, \ldots, x_n]^T \in [0,1]^n$, we define a *regraduation of* $\mathbf{x}$ *by* α as the fuzzy set $\mathbf{y} = [y_1, \ldots, y_n]^T \in [0,1]^n$ given by $y_j = \alpha \star x_j$ for every $j = 1, \ldots, n$. Note that $\mathbf{y}$ is obtained by redefining the membership values of the fuzzy set $\mathbf{x}$. We say that a between-cube mapping $\varphi : [0,1]^n \to [0,1]^m$ is *invariant under regraduations* if $\varphi(\alpha \star \mathbf{x}) = \alpha \star \varphi(\mathbf{x})$ for every $\alpha \in [0,1]$. Dually, a *dual regraduation of* $\mathbf{x}$ *by* α is the fuzzy set $[\alpha \star' x_1, \ldots, \alpha \star' x_n] \in [0,1]^n$ and an operator φ is called *invariant under dual regraduations* if $\varphi(\alpha \star' \mathbf{x}) = \alpha \star' \varphi(\mathbf{x})$ for all $\alpha \in [0,1]$.

3 Fuzzy Morphological Associative Memories Invariant under Regraduations

Let us begin this section by introducing two classes of FLNN-FAMs, referred to as the *class of max-$\star$ FMAMs* and the *class of min-$\star'$ FMAMs*, which subsumes several FAM models from the literature, including the FAMs of Kosko, the IFAMs, several GFAMs, and the FMAMs based on uninorm introduced by Valle and Sussner [24, 27, 29, 28].

Let $([0,1], \vee, \wedge, \star, \star')$ be a clodum. A max-$\star$ FMAM is a single-layer FLNN equipped with neurons that compute the maximum of the operation $\star$. Dually, a min-$\star'$ FMAM corresponds to a single-layer FLNN with neurons that compute the minimum of $\star'$. In mathematical terms, consider synaptic weight matrices $W, M \in [0,1]^{m \times n}$. Given an input pattern $\mathbf{x} \in [0,1]^n$, the output $\mathbf{y} \in [0,1]^m$ of a max-$\star$ FMAM $\mathcal{W}$ and the output $\mathbf{z} \in [0,1]^m$ of a min-$\star'$ FMAM $\mathcal{M}$ are given by the following equations where the symbol "$\circ$" denote a max-$\star$ product and "$\bullet$" represents a min-$\star'$ product:

$$\mathbf{y} = \mathcal{W}(\mathbf{x}) = W \circ \mathbf{x} \quad \text{and} \quad \mathbf{z} = \mathcal{M}(\mathbf{x}) = M \bullet \mathbf{x}. \tag{3}$$

The max-$\star$ and min-$\star'$ products of two matrices $A \in [0,1]^{n \times k}$ and $B \in [0,1]^{k \times n}$, denoted by $C = A \circ B \in [0,1]^{n \times m}$ and $D = A \bullet B \in [0,1]^{n \times m}$, are defined as follows:

$$c_{ij} = \bigvee_{\xi=1}^{k} (a_{i\xi} \star b_{\xi j}) \quad \text{and} \quad d_{ij} = \bigwedge_{\xi=1}^{k} \left(a_{i\xi} \star' b_{\xi j}\right). \tag{4}$$

Since the operation $\star$ constitutes a dilation on $[0,1]$ and $\star'$ is an erosion in the same lattice, the mappings $\mathcal{W}$ and $\mathcal{M}$ given by (3) represent a dilation and an erosion, respectively [24, 29]. Thus, both max-$\star$ FMAMs and min-$\star'$ FMAMs indeed belong to the class of fuzzy morphological neural networks.

The following proposition, due to Maragos [13], reveals that the mapping $\mathcal{W}$ given by (3) is also invariant under regraduations. Similarly, $\mathcal{M}$ is invariant under dual regraduations. As a consequence, most FLNN-FAMs yield a between-cube mapping that is either a dilation invariant under regraduations or an erosion invariant under dual regraduations. Furthermore, Proposition 1 tell us that the converse also holds true.

Proposition 1 *Let $([0,1], \vee, \wedge, \star, \star')$ be a clodum. A mapping $\mathcal{W} : [0,1]^n \rightarrow [0,1]^m$ is a dilation invariant under regraduations if, and only if, it is given by (3) for some $W \in [0,1]^{m \times n}$. Dually, a mapping $\mathcal{M} : [0,1]^n \rightarrow [0,1]^m$ is an erosion invariant under dual regraduations if, and only if, it is given by (3) for some $M \in [0,1]^{m \times n}$.*

In the context of FAMs, we can drawn the following conclusion from Proposition 1. Suppose we intent to synthesize a FAM model that is able to store a set of associations $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\} \subseteq [0,1]^n \times [0,1]^m$. In other words, we would like to determine a mapping $\mathcal{A} : [0,1]^n \rightarrow [0,1]^m$ such that $\mathcal{A}(\mathbf{x}^\xi) = \mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$. This hard problem can be simplified by imposing that the mapping $\mathcal{A}$ is a dilation invariant under regraduations in a clodum $([0,1], \vee, \wedge, \star, \star')$. Hence, we only have to determine an appropriate synaptic weight matrix W. Alternatively, we could impose that the FAM model yields an erosion invariant under dual regraduations. In this case, the problem of determining a between-cube mapping $\mathcal{A}$ also reduces to the easier problem of computing a matrix $M \in [0,1]^{m \times n}$.

4 A Brief Review on Fuzzy Learning by Adjunction

The synaptic weight matrix of a max-$\star$ FMAM given by (3) can be determined by means of *fuzzy learning by adjunction* (FLA), also called *implicative fuzzy learning* [22, 29]. Given a set of associations $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\}$, where each $\mathbf{x}^\xi \in [0,1]^n$ and $\mathbf{y}^\xi \in [0,1]^m$, FLA defines the synaptic weight matrix $W \in [0,1]^{m \times n}$ of a max-$\star$ FMAM by means of the following equation:

$$W = \bigvee \left\{ A \in [0,1]^{m \times n} : A \circ \mathbf{x}^\xi \leq \mathbf{y}^\xi, \ \forall\, \xi = 1, \ldots, k \right\}. \tag{5}$$

Note that FLA makes optimal use of the synaptic weights of the max-$\star$ FMAM model given by (3). Indeed, if there exist $A \in [0,1]^{m \times n}$ such that $A \circ \mathbf{x}^\xi = \mathbf{y}^\xi$ for all ξ, then W given by FLA also satisfies $W \circ \mathbf{x}^\xi = \mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$ and the inequality $A \leq W$ holds true.

Besides the optimality that we have just mentioned, the synaptic weight matrix $W = (w_{ij}) \in [0,1]^{m \times n}$ given by FLA can be easily computed by means of the following equation:

$$w_{ij} = \bigwedge_{\xi=1}^{k} (x_j^\xi \Rightarrow y_i^\xi), \quad \text{for all } j = 1, \ldots, n \text{ and } i = 1, \ldots, m. \tag{6}$$

Here, the symbol "$\Rightarrow$" denotes the residual implication associated with the operation $\star$ of the clodum $([0,1], \vee, \wedge, \star, \star')$, which is given by the following equation:

$$(x \Rightarrow y) = \bigvee \{z \in [0,1] : x \star z \leq y\}, \quad \forall\, x, y \in [0,1]. \tag{7}$$

Similarly, given a set of associations $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\} \subseteq [0,1]^n \times [0,1]^m$, the synaptic weight matrix $M \in [0,1]^{m \times n}$ of a min-$\star'$ FMAM given by (3) is defined as

$$M = \bigwedge \left\{ A \in [0,1]^{m \times n} : A \bullet \mathbf{x}^\xi \geq \mathbf{y}^\xi, \ \forall\, \xi = 1, \ldots, k \right\}. \tag{8}$$

Again, FLA makes optimal use of the synaptic weights of a min-$\star'$ FMAM given by (3). Precisely, if there exist $A \in [0,1]^{m \times n}$ such that $A \bullet \mathbf{x}^\xi = \mathbf{y}^\xi$ for all ξ, then M given by (8) satisfy $M \leq A$ and $M \bullet \mathbf{x}^\xi = \mathbf{y}^\xi$ for every $\xi = 1, \ldots, k$. Furthermore, the synaptic weights m_{ij} can be easily determined by means of the following equations:

$$m_{ij} = \bigvee_{\xi=1}^{k} (x_j^\xi \not\Leftarrow y_i^\xi), \quad \text{where} \quad (x \not\Leftarrow y) = \bigwedge \{z \in [0,1] : x \star' z \geq y\}. \tag{9}$$

We would like to recall that the symbol "$\not\Leftarrow$" denotes the residual co-implication associated to the operation $\star'$ in the clodum $([0,1], \vee, \wedge, \star, \star')$ [3].

5 The Continuity of max-$\star$ FMAMs with FLA

Let us now focus on the continuity of a max-$\star$ FMAM when the synaptic weight matrix are given by FLA. We believe that a similar result can be derived for min-$\star$ FMAMs with FLA using the duality principle of complete lattices [1, 9].

Consider a clodum $([0,1], \vee, \wedge, \star, \star')$. The *residual bi-implication* associated with the operation $\star$ is given by the following equation where "$\Rightarrow$" denotes the operator given by (7):

$$(x \Leftrightarrow y) = (x \Rightarrow y) \wedge (y \Rightarrow y), \quad \text{for all } x, y \in [0,1]. \tag{10}$$

We would like to recall that a residual bi-implication is reflexive, symmetric, and transitive with respect to the operation $\star$, i.e., $(x \Leftrightarrow y) \star (y \Leftrightarrow z) \leq (x \Leftrightarrow z)$ for all $x, y, z \in [0,1]$. Furthermore, if e denotes the identity of $\star$, then $(x \Leftrightarrow y) \geq e$ if and only if $x = y$. In view of these remarks, the residual bi-implication given by (10) can be used to define the following similarity measure between fuzzy sets $\mathbf{x} = [x_1, \ldots, x_n]^T \in [0,1]^n$ and $\mathbf{y} = [y_1, \ldots, y_n]^T \in [0,1]^n$:

$$\mathcal{S}(\mathbf{x}, \mathbf{y}) = \bigwedge_{i=1}^{n} (x_i \Leftrightarrow y_i). \tag{11}$$

The similarity measure given by (11) can be used to establish the following notion of continuity: A mapping $\mathcal{A} : [0,1]^n \to [0,1]^m$ is said to be continuous at $\mathbf{x}_0 \in [0,1]^n$ if the following inequality holds true for every $\mathbf{x} \in [0,1]^n$:

$$\mathcal{S}\big(\mathcal{A}(\mathbf{x}_0), \mathcal{A}(\mathbf{x})\big) \geq \mathcal{S}\big(\mathbf{x}_0, \mathbf{x}\big). \tag{12}$$

In other words, $\mathcal{A}$ is continuous at $\mathbf{x}_0$ if any point $\mathbf{x}$ is mapped by $\mathcal{A}$ into a point $\mathcal{A}(\mathbf{x})$ whose degree of similarity with $\mathcal{A}(\mathbf{x}_0)$ is greather than or equal to the degree of similarity between $\mathbf{x}_0$ and $\mathbf{x}$. We would like to point out that this notion of continuity corresponds to a reformulation of the continuity introduced by Perfilieva and Lehmke for fuzzy systems of IF-THEN rules [17]. Thus, we will refer to it as the *continuity in the sense of Perfilieva-Lehmke.*

Let us now return to the associative memory problem. Given a set of associations $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\} \subseteq [0,1]^n \times [0,1]^m$, a FAM $\mathcal{W} : [0,1]^n \to [0,1]^m$ should, in principle, satisfy the equation $\mathcal{W}(\mathbf{x}^\xi) = \mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$. The following proposition shows that if a max-$\star$ FMAM with FLA satisfies $\mathcal{W}(\mathbf{x}^\xi) = \mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$, then $\mathcal{W}$ is also continuous at every fundamental memory $\mathbf{x}^\xi$. Furthermore, Proposition 2 shows that the converse also holds true: If $\mathcal{W}$ is continuous at every $\mathbf{x}^\xi$, then the association pairs $(\mathbf{x}^\xi, \mathbf{y}^\xi)$ have been memorized correctly.

Proposition 2 *Consider a clodum $([0,1], \vee, \wedge, \star, \star')$. Given a set of associations $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\} \subseteq [0,1]^n \times [0,1]^m$, let $\mathcal{W} : [0,1]^n \to [0,1]^m$ denote the max-$\star$ FMAM whose synaptic weight matrix is given by FLA. We have that $\mathcal{W}(\mathbf{x}^\xi) = \mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$ if, and only if, $\mathcal{W}$ is continuous at every $\mathbf{x}^\xi$ in the sense of Perfilieva-Lehmke.*

We would like to point out that the previous proposition is analogous to Theorem 2 in [17] for fuzzy systems of IF-THEN rules when $\star$ corresponds to a left-continuous t-norm. A proof of Proposition (2) in general clodums can be found in [15].

Example 1 *Consider the unit interval $[0,1]$ equipped with the conjunctive and disjunctive 3Π uninorm as operations $\star$ and $\star'$ [5, 30]. In this case, the mathematical structure $([0,1], \vee, \wedge, \star, \star')$ constitutes a clodum. Moreover, recall that the operation $\star$ satisfies*

$$x \star y = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 1, \\ 0 & \text{if } x = 1 \text{ and } y = 0, \\ \frac{xy}{xy + (1-x)(1-y)} & \text{elsewhere}, \end{cases} \tag{13}$$

and its residual implication is given by the following equation for every $x, y \in [0,1]$ [4]:

$$(x \Rightarrow y) = \begin{cases} 1 & \text{if } x = 0 \text{ and } y = 0\,, \\ 1 & \text{if } x = 1 \text{ and } y = 1\,, \\ \frac{(1-x)y}{y(1-x)+x(1-y)} & \text{elsewhere}\,. \end{cases} \tag{14}$$

Suppose that we intent to store the set of associations $\{(\mathbf{x}^1, \mathbf{y}^1), \ldots, (\mathbf{x}^3, \mathbf{y}^3)\} \subseteq [0,1]^5 \times [0,1]^4$ *in which*

$$\mathbf{x}^1 = \begin{bmatrix} 0.5 \\ 0.6 \\ 0.8 \\ 0.6 \\ 0.7 \end{bmatrix}, \ \mathbf{x}^2 = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.3 \\ 0.0 \\ 0.3 \end{bmatrix}, \ \mathbf{x}^3 = \begin{bmatrix} 0.1 \\ 0.6 \\ 0.2 \\ 0.8 \\ 0.2 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}^1 = \begin{bmatrix} 0.94 \\ 0.86 \\ 1.00 \\ 0.94 \end{bmatrix}, \ \mathbf{y}^2 = \begin{bmatrix} 0.63 \\ 0.39 \\ 1.00 \\ 0.63 \end{bmatrix}, \ \mathbf{y}^3 = \begin{bmatrix} 0.63 \\ 0.73 \\ 1.00 \\ 0.50 \end{bmatrix}. \tag{15}$$

In this case, FLA yields the following synaptic weight matrix:

$$W = \begin{bmatrix} 0.94 & 0.53 & 0.80 & 0.30 & 0.80 \\ 0.85 & 0.39 & 0.60 & 0.40 & 0.60 \\ 1.00 & 1.00 & 1.00 & 1.00 & 1.00 \\ 0.90 & 0.40 & 0.80 & 0.20 & 0.80 \end{bmatrix}. \tag{16}$$

We first confirmed that the association pairs $(\mathbf{x}^\xi, \mathbf{y}^\xi)$ *have been correctly stored in the memory, i.e.,* $\mathcal{W}(\mathbf{x}^\xi) = \mathbf{y}^\xi$*, for* $\xi = 1, \ldots, 3$*. Therefore, in view of Proposition 2, the max-$\star$ FMAM* $\mathcal{W}$ *is continuous at* $\mathbf{x}^1, \mathbf{x}^2$*, and* $\mathbf{x}^3$*. Indeed, if we introduce the pattern* $\mathbf{x} = [0.6, 0.5, 0.8, 0.7, 0.6]^T$ *as input, then the output of the max-$\star$ FMAM* $\mathcal{W}$ *is the fuzzy set* $\mathbf{y} = [0.96, 0.90, 1.00, 0.94]^T$*. Note that the fuzzy set* $\mathbf{x}$ *is obtained by interchanging the entries* x_1 *and* x_2 *as well as* x_4 *and* x_5 *of* $\mathbf{x}^1$*. The degree of similarity between* $\mathbf{x}^1$ *and* $\mathbf{x}$ *is* $\mathcal{S}(\mathbf{x}, \mathbf{x}^1) = 0.39$*. In contrast, the output* $\mathbf{y}$ *agrees with* $\mathbf{y}^1$ *in the entries* y_3 *and* y_4*. The degree of similarity of these two fuzzy sets is* $\mathcal{S}(\mathbf{y}, \mathbf{y}^1) = 0.41$*, which is greater than* $\mathcal{S}(\mathbf{x}, \mathbf{x}^1)$*.*

6 Concluding Remarks

In this paper, we first pointed out that *clodums*, or *complete lattice ordered double monoids*, constitute an appropriate framework for the characterization of *fuzzy morphological associative memories* (FMAMs) as *fuzzy logic neural networks* (FLNNs). Precisely, we noted that many FLNN-FAMs in the literature belong to either the class of max-$\star$ FMAMs or the class of min-$\star'$ FMAMs. Then, we observed that a certain FAM model $\mathcal{W}$ performs a dilation invariant under regraduations of fuzzy patterns if, and only if, it corresponds to a max-$\star$ FMAM. Dually, a FAM model $\mathcal{M}$ is an erosion invariant under dual regraduations if, and only if, it belong to the class of min-$\star'$ FMAMs. Therefore, in a clodum, the problem of finding an appropriate FAM for a given association task corresponds to the much more easy problem of determining an appropriate synaptic weight matrix. In view of this fact, we briefly revised *fuzzy learning by adjunction* (FLA), also called implicative fuzzy learning, which can be effectively applied for storage of a set of associations in max-$\star$ and min-$\star'$ FMAMs. Recall that this recording recipe yields, in some sense, an optimal synaptic weight matrix and, consequently, the best max-$\star$ or min-$\star'$ FMAM in a clodum. Finally, we translated the notions of continuity of systems of fuzzy IF-THEN rules introduced by Perfilieva and Lehmke to the context of FAMs. As a consequence, we have that a max-$\star$ FMAM with FLA is continuous - in the sense of Perfilieva-Lehmke - if, and only if, it is able to store the association pairs correctly.

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