An Introduction to the Max-plus Projection Autoassociative Morphological Memory and Some of Its Variations

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Introduction

Autossociative Memories

- are systems designed for the storage and recall of $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subseteq \mathbb{R}^n$, called *fundamental memory set*.

Autassociative Morphological Memories:

- Use lattice-based operations from minimax algebra.
- Applications of AMMs include:
  - Restoration of corrupted images.
  - Vision-based self-localization in mobile robots.
  - Times-series prediction.

Organization of this Talk

Lattice-based Operations from Minimax Algebra

Definition (Max-Product and the min-product)

Let $A \in \mathbb{R}^{n \times k}$ and $B \in \mathbb{R}^{k \times n}$. The max-product of A by B is given by

$$C = A \boxdot\!\!\!\vee\, B \qquad \Longleftrightarrow \qquad c_{ij} = \bigvee_{\xi=1}^{k} \left(a_{i\xi} + b_{\xi j} \right).$$

The min-product of A by B is given by

$$C = A \boxdot\!\!\!\wedge\, B \qquad \Longleftrightarrow \qquad c_{ij} = \bigwedge_{\xi=1}^{k} \left(a_{i\xi} + b_{\xi j} \right).$$

Conjugation and Adjunction

Definition (Conjugate)

The conjugate of $A \in \mathbb{R}^{n \times k}$ is the matrix $A^* \in \mathbb{R}^{k \times n}$ given by

$$a^*_{ij} = -a_{ji}.$$

Proposition (Conjugation Relationship)

$$(A \boxed{\wedge} B)^* = B^* \boxed{\vee} A^* \quad \textit{and} \quad (A \boxed{\vee} B)^* = B^* \boxed{\wedge} A^*.$$

Proposition (Adjunction Relationship)

$$A \boxed{\vee} B \leq C \quad \Longleftrightarrow \quad B \leq A^* \boxed{\wedge} C \quad \Longleftrightarrow \quad A \leq C \boxed{\wedge} B^*.$$

Max-plus combination

Definition (Max-plus combination)

A vector

$$\mathbf{y} = \bigvee_{\xi=1}^{k} (\alpha_\xi + \mathbf{x}^\xi), \quad \alpha_\xi \in \mathbb{R},$$

is a max-plus combination of vectors from $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subseteq \mathbb{R}^n$. The set of all max-plus combinations from $\mathcal{X}$ is

$$\mathfrak{V}(\mathcal{X}) = \left\{ \mathbf{y} \in \mathbb{R}^n : \mathbf{y} = \bigvee_{\xi=1}^{k} (\alpha_\xi + \mathbf{x}^\xi), \alpha_j^\xi \in \mathbb{R} \right\}.$$

Maxinimax combination

Definition (Minimax combination)

A vector

$$\mathbf{z} = \bigwedge_{j=1}^{n} \bigvee_{\xi=1}^{k} (a_j^\xi + \mathbf{x}^\xi), \quad a_j^\xi \in \mathbb{R},$$

is a minimax combination of vectors from $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subseteq \mathbb{R}^n$. The set of all minimax combinations from $\mathcal{X}$ is

$$\mathfrak{S}(\mathcal{X}) = \left\{ \mathbf{z} \in \mathbb{R}^n : \mathbf{z} = \bigwedge_{j=1}^{n} \bigvee_{\xi=1}^{k} (a_j^\xi + \mathbf{x}^\xi), a_j^\xi \in \mathbb{R} \right\}.$$

Original AMM models

Definition (AMM $\mathcal{M}_{XX}$)

The AMM $\mathcal{M}_{XX} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\mathcal{M}_{XX}(\mathbf{x}) = M_{XX} \boxed{\wedge} \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

where $M_{XX} \in \mathbb{R}^{n \times n}$ is the *synaptic weight matrix*.

Definition (Recording Recipe)

Given a fundamental memory set $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$, M_{XX} is given by

$$M_{XX} = X \boxed{\vee} X^*,$$

where $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in \mathbb{R}^{n \times k}$.

From the conjugation relationship, we obtain the dual model:

Definition (AMM $\mathcal{W}_{XX}$)

The AMM $\mathcal{W}_{XX} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by

$$\mathcal{W}_{XX}(\mathbf{x}) = W_{XX} \boxdot' \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Given $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$, W_{XX} is determined by

$$W_{XX} = X \boxdot X^*,$$

where $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in \mathbb{R}^{n \times k}$.

However, we shall focus on the AMM $\mathcal{M}_{XX}$.

Proposition (Characterization of the AMM $\mathcal{M}_{XX}$)

The mapping $\mathcal{M}_{XX} : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\mathcal{M}_{XX}(\mathbf{x}) = \bigvee \{\mathbf{z} \in \mathfrak{S}(\mathcal{X}) : \mathbf{z} \leq \mathbf{x}\},$$

where $\mathfrak{S}(\mathcal{X})$ is the set of all minimax combinations of $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$.

Conclusion:

1. $\mathcal{M}_{XX}$ is idempotent.
2. Any minimax combination of $\mathbf{x}^1, \ldots, \mathbf{x}^k$ is a fixed point of $\mathcal{M}_{XX}$.
3. $\mathcal{M}_{XX}$ projects $\mathbf{x}$ downward into $\mathfrak{S}(\mathcal{X})$.
4. $\mathcal{M}_{XX}$ exhibits perfect recall of any undistorted vector $\mathbf{x}^\xi \in \mathcal{X}$.
5. $\mathcal{M}_{XX}$ has many spurious memories, i.e., any vector in $\mathfrak{S}(\mathcal{X}) \setminus \mathcal{X}$.
6. $\mathcal{M}_{XX}(\mathbf{x}) \leq \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.
   Thus, $\mathcal{M}_{XX}$ is suited for the reconstruction of patterns corrupted by dilative noise, i.e., $\mathbf{x} \geq \mathbf{x}^\xi$.

Generalized Kernel Method for AMMs

Idea:

- The AMM $\mathcal{M}_{XX}$ satisfies $\mathcal{M}_{XX}(\mathbf{x}) \leq \mathbf{x}$ (suited for dilative noise).
- Dually, $\mathcal{W}_{XX}(\mathbf{x}) \geq \mathbf{x}$ (suited for erosive noise).
- The idea is to combine the max-product and the min-product.
- Hopefully, we will be able to deal with dilative and erosive noise!

Definition (Generalized Kernel (Sussner, 2003))

A matrix $Z \in \mathbb{R}^{p \times k}$, $p \geq k$, is a *generalized kernel* for X if

$$W_{ZX} \boxdot_{\vee} (M_Z^X \boxdot_{\wedge} X) = X$$

where

$$W_{ZX} = X \boxdot_{\wedge} Z^* \quad \text{and} \quad M_Z^X = (Z \boxdot_{\vee} X^*) \boxdot_{\wedge} (X \boxdot_{\vee} X^*).$$

Definition (Generalized Kernel AMM (GK-AMM))

Given $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in \mathbb{R}^{n \times k}$ and a generalized kernel Z for X, the GK-AMM $\mathcal{Z} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\mathcal{Z}(\mathbf{x}) = W_{ZX} \boxdot (M_Z^X \boxdot' \mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

GK-AMM exhibited excellent noise tolerance for binary patterns!

Remark

The paper contains two other variations of the original AMM $\mathcal{M}_{XX}$. Namely,

1. The best-chebyshev approximation AMM (CBA-AMM).
2. The noise masking strategy.

Max-plus Projection AMM (max-plus PAMM)

Recall that:

Given $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$, the AMM $\mathcal{M}_{XX} : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\mathcal{M}_{XX}(\mathbf{x}) = \bigvee \{\mathbf{z} \in \mathfrak{S}(\mathcal{X}) : \mathbf{z} \leq \mathbf{x}\},$$

where $\mathfrak{S}(\mathcal{X})$ is the set of all minimax combinations of $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$.

Definition (Max-plus PAMM)

Given $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$, the max-plus PAMM $\mathcal{V}_{XX} : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\mathcal{V}_{XX}(\mathbf{x}) = \bigvee \{\mathbf{z} \in \mathfrak{V}(\mathcal{X}) : \mathbf{z} \leq \mathbf{x}\},$$

where $\mathfrak{V}(\mathcal{X})$ is the set of all max-plus combinations of $\mathcal{X}$.

Conclusion:

1. $\mathcal{V}_{XX}$ is idempotent.
2. Any max-plus combination of $\mathbf{x}^1, \ldots, \mathbf{x}^p$ is a fixed point of $\mathcal{V}_{XX}$.
3. $\mathcal{V}_{XX}$ projects $\mathbf{x}$ downward into $\mathfrak{V}(\mathcal{X})$.
4. $\mathcal{V}_{XX}$ exhibits perfect recall of any undistorted vector $\mathbf{x}^\xi \in \mathcal{X}$.
5. Since $\mathfrak{V}(\mathcal{X}) \subset \mathfrak{S}(\mathcal{X})$, $\mathcal{V}_{XX}$ has less spurious memories than $\mathcal{M}_{XX}$.
6. $\mathcal{V}_{XX}(\mathbf{x}) \leq \mathcal{M}_{XX}(\mathbf{x}) \leq \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$
   In words, $\mathcal{V}_{XX}$ has a better dilative noise tolerance than $\mathcal{M}_{XX}$.

Theorem (Formula to compute $\mathcal{V}_{XX}(\mathbf{x})$)

Let $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in \mathbb{R}^{n \times k}$*. For any input pattern* $\mathbf{x} \in \mathbb{R}^n$*, we have*

$$\mathcal{V}_{XX}(\mathbf{x}) = X \boxed{\vee} \boldsymbol{\alpha}, \quad \textit{where} \quad \boldsymbol{\alpha} = X^* \boxed{\wedge} \mathbf{x}.$$

Alternatively, the output of $\mathcal{V}_{XX}$ *can be expressed as*

$$\mathcal{V}_{XX}(\mathbf{x}) = \bigvee_{\xi=1}^{k} \bigwedge_{j=1}^{n} \left((x_j - x_j^\xi) + \mathbf{x}^\xi\right), \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Remark

The output of $\mathcal{M}_{XX}$ satisfies

$$\mathcal{M}_{XX}(\mathbf{x}) = \bigwedge_{j=1}^{n} \bigvee_{\xi=1}^{k} \left((x_j - x_j^\xi) + \mathbf{x}^\xi\right), \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

From

$$\mathcal{V}_{XX}(\mathbf{x}) = X \boxvee \boldsymbol{\alpha}, \quad \text{where} \quad \boldsymbol{\alpha} = X^* \boxwedge \mathbf{x},$$

we conclude that

$$\mathcal{V}_{XX}(\mathbf{x}) = X \boxvee \left(X^* \boxwedge \mathbf{x}\right).$$

In other words, we have

Theorem

The max-plus PAMM $\mathcal{V}_{XX}$ equals the GK-AMM $\mathcal{Z}$ with the generalized kernel $Z = X = [\mathbf{x}^1, \ldots, \mathbf{x}^k]$ in an hyperbox.

Proposition

The max-plus PAMM satisfies

$$\mathcal{V}_{XX}(\mathbf{x}) = \bigvee_{\xi=1}^{k} \big(\underbrace{\mathcal{A}(\mathbf{x}^{\xi}, \mathbf{x})}_{\alpha_\xi} + \mathbf{x}^{\xi} \big), \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is given by*

$$\mathcal{A}(\mathbf{y}, \mathbf{x}) = \bigwedge_{j=1}^{n} (x_j - y_j) = \mathbf{y}^* \boxtimes \mathbf{x}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Remark

We have $\mathcal{A}(\mathbf{y}, \mathbf{x}) \geq 0$ if and only if $\mathbf{x} \geq \mathbf{y}$.
In some sense, $\mathcal{A}(\mathbf{y}, \mathbf{x})$ measures the truth of the inequality $\mathbf{y} \leq \mathbf{x}$.

Theorem (Dual Representation of $\mathcal{V}_{XX}$)

Given $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$, the max-plus PAMM $\mathcal{V}_{XX}$ satisfies

$$\mathcal{V}_{XX}(\mathbf{x}) = \bigwedge \{\mathbf{y} \in \mathbb{R}^n : \mathcal{A}(\mathbf{x}^\xi, \mathbf{x}) \leq \mathcal{A}(\mathbf{x}^\xi, \mathbf{y}), \forall \xi \in 1, \ldots, k\},$$

for all input $\mathbf{x} \in \mathbb{R}^n$.

The dual representation of $\mathcal{V}_{XX}$ require further investigation!

Remark

The paper contains two other variations of the max-plus PAMM $\mathcal{V}_{XX}$:

1. The best-chebyshev approximation PAMM (CBA-PAMM).
2. The noise masking strategy.

Concluding Remarks

1. We briefly revised the original AMM models.
2. We also reviewed the generalized kernel AMMs (GK-AMMs).
3. We introduced the max-plus projection AMM $\mathcal{V}_{XX}$ by replacing the set of all minimax combination $\mathfrak{S}(\mathcal{X})$ by $\mathfrak{V}(\mathcal{X})$, the set of all max-plus combination of $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$.
4. Moreover, $\mathcal{V}_{XX}$ as less spurious memories than $\mathcal{M}_{XX}$.
5. In addition, $\mathcal{V}_{XX}$ corresponds to a certain GK-AMM.

Thank you!

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