Theoretical and Computational Aspects of Quaternionic Multivalued Hopfield Neural Networks

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Hopfield Neural Network (HNN)

- Hopfield, 1982.

Historical Survey

Hopfield Neural Network (HNN)

- Hopfield, 1982.

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Complex-Valued Multistate Hopfield Neural Network

- Jankowski, Lozowski, and Zurada, 1996.

Hopfield Neural Network (HNN)

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Complex-Valued Multistate Hopfield Neural Network

- Jankowski, Lozowski, and Zurada, 1996.

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Quaternionic Multistate Hopfield Neural Network
or Multivalued Quaternionic HNN (MV-QHNN)

- Isokawa, Nishimura, Saitoh, Kamiura, and Matsui, 2008.
- Minemoto, Isokawa, Nishimura, and Matsui, 2015.

Outline of the Talk

Quaternions

A quaternion is an hypercomplex number

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ are the hyper-imaginary units.

Phase-angle representation:

$$q = |q|e^{\mathbf{i}\phi}e^{\mathbf{k}\psi}e^{\mathbf{j}\theta},$$

where $\phi \in [-\pi, \pi)$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\psi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Unique phase-angle representation:

$$\mathcal{A} = \left\{ q = |q|e^{\mathbf{i}\phi}e^{\mathbf{k}\psi}e^{\mathbf{j}\theta} : q \neq 0 \text{ and } |\psi| \neq \frac{\pi}{4} \right\}.$$

MV-QHNN models

Like the complex-valued model of Jankowski et al., in a MV-QHNN the phase-angle intervals are divided in sectors.

Resolution factors and phase quanta

Given positive integers K_1, K_2, and K_3, called **resolution factors**, the **phase quanta** are defined by

$$\Delta\phi = \frac{2\pi}{K_1}, \quad \Delta\theta = \frac{\pi}{K_2}, \quad \text{and} \quad \Delta\psi = \frac{\pi}{2K_3}.$$

Network dynamic

Given a quaternionic synaptic weight matrix W, neurons are updated asynchronously if

$$v_i(t) = \sum_{j=1}^{n} w_{ij} x_j(t),$$

has an unique phase-angle representation, i.e., $v_i(t) \in \mathcal{A}$.

MV-QHNN of Isokawa et al.

Let the state and activation of the ith neuron at time t be

$$x_i(t) = e^{\mathbf{i}\phi_i(t)} e^{\mathbf{k}\psi_i(t)} e^{\mathbf{j}\theta_i(t)} \quad \text{and} \quad v_i(t) = |v_i(t)| e^{\mathbf{i}\phi} e^{\mathbf{k}\psi} e^{\mathbf{j}\theta}.$$

The neuron is updated asynchronously according to

$$x_i(t+1) = \begin{cases} e^{\phi_i(t)\mathbf{i}} e^{\psi_l \mathbf{k}} e^{\theta_l \mathbf{j}}, \\ \text{or} \\ e^{\phi_l \mathbf{i}} e^{\psi_l \mathbf{k}} e^{\theta_i(t)\mathbf{j}} \end{cases}$$

where

$$\phi_l = -\pi + \Delta\phi \left\lfloor \frac{\pi + \phi}{\Delta\phi} \right\rfloor, \psi_l = -\frac{\pi}{4} + \Delta\psi \left\lfloor \frac{\pi/4 + \psi}{\Delta\psi} \right\rfloor, \theta_l = -\frac{\pi}{2} + \Delta\theta \left\lfloor \frac{\pi/2 + \theta}{\Delta\theta} \right\rfloor.$$

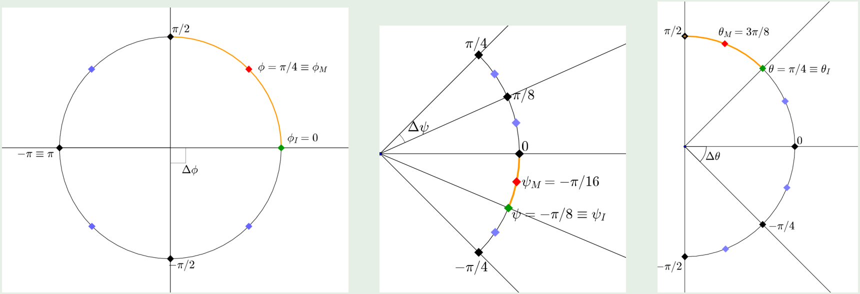
Note that the phase-angles of a quaternionic neuron are not updated simultaneously!

Numerical Instability

- The angles ϕ_I, ψ_I and θ_I are the clockwise edges of the arcs that contain ϕ, ψ and θ.
- A small perturbation can yield a significant change in ϕ_I, ψ_I or θ_I.

Example

The resolution factors $K_1 = K_2 = K_3 = 4$ yield the following circular sectors whose the boundaries are marked by black diamonds:



The conference paper contains examples in which:

- Rounding errors yield an absolute error far greater than the machine precision.
- The MV-QHNN converges to an unexpected stationary state.

Isokawa et al. claimed that the energy

$$E(\mathbf{x}) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{x}^H W \mathbf{x},$$

never increases if

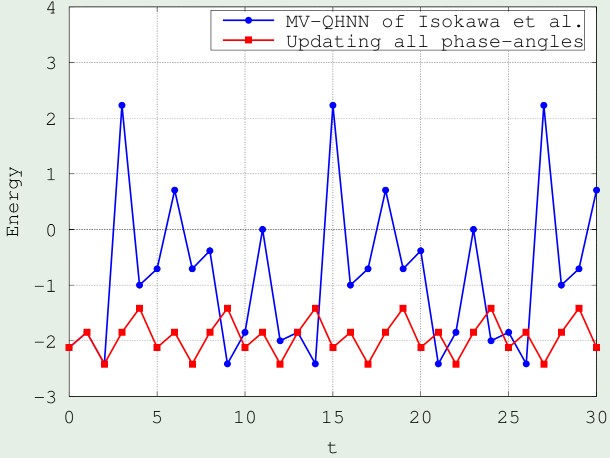
$$w_{ij} = \bar{w}_{ji} \quad \text{and} \quad w_{ii} \geq 0, \quad \forall i, j \in \{1, \ldots, n\}.$$

In the following example, the energy oscillates!

Example (Stability problem:)

Let $K_1 = K_2 = K3 = 4$,

$$W = \begin{bmatrix} 0 & 1 - 2\mathbf{j} - \mathbf{k} \\ 1 + 2\mathbf{j} + \mathbf{k} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}(0) = \begin{bmatrix} e^{-\frac{\pi}{4}\mathbf{j}} \\ 1 \end{bmatrix}.$$



Example (Stability problem:)

The energy at $\mathbf{x}(0)$ is

$$E(\mathbf{x}(0)) = -\frac{1}{2}\mathbf{x}^H(0)W\mathbf{x}(0) = -\frac{3\sqrt{2}}{2} = -2.1213.$$

The energy at $\mathbf{x}(1)$ is

$$E(\mathbf{x}(1)) = -\frac{1}{2}\mathbf{x}^H(1)W\mathbf{x}(1) = -\cos\left(\frac{\pi}{8}\right) = -0.92388.$$

The difference between the energies is

$$\Delta E = E(\mathbf{x}(1)) - E(\mathbf{x}(0)) = 1.1974.$$

Therefore, ΔE is positive even if we could work in exact arithmetic!

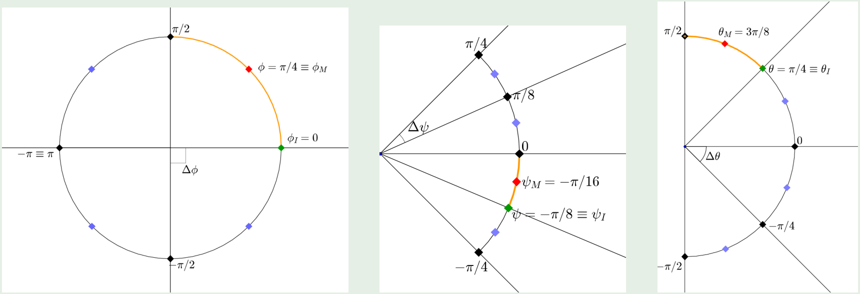
We address this issue in "*Comments on the Stability Analysis of Quaternionic Multistate Hopfield Neural Networks*".

MV-QHNN of Minemoto et al.

- Minemoto et al. presented an MV-QHNN which have been effectively used for the storage and recall of color images.
- The novel model is obtained by shifting the phase-angles ϕ_I, ψ_I, and θ_I by half of the phase quanta.

Example

The resolution factors $K_1 = K_2 = K_3 = 4$ yield the following circular sectors whose midpoints are marked by blue diamonds:



MV-QHNN of Minemoto et al.

Let the state and activation of the *i*th neuron at time *t* be

$$x_i(t) = e^{\mathbf{i}\phi_i(t)} e^{\mathbf{k}\psi_i(t)} e^{\mathbf{j}\theta_i(t)} \quad \text{and} \quad v_i(t) = |v_i(t)| e^{\mathbf{i}\phi} e^{\mathbf{k}\psi} e^{\mathbf{j}\theta}.$$

The neuron is updated asynchronously according to

$$x_i(t+1) = \begin{cases} e^{\phi_i(t)\mathbf{i}} e^{\psi_i(t)\mathbf{k}} e^{\theta_M \mathbf{j}}, \\ \text{or} \\ e^{\phi_i(t)\mathbf{i}} e^{\psi_M \mathbf{k}} e^{\theta_i(t)\mathbf{j}}, \\ \text{or} \\ e^{\phi_M \mathbf{i}} e^{\psi_i(t)\mathbf{k}} e^{\theta_i(t)\mathbf{j}} \end{cases}$$

where

$$\phi_M = \frac{1}{2}\left\{-2\pi + \Delta\phi\left(2\left\lfloor\frac{\pi+\phi}{\Delta\phi}\right\rfloor + 1\right)\right\}, \psi_M = \frac{1}{2}\left\{-\frac{\pi}{2} + \Delta\psi\left(2\left\lfloor\frac{\frac{\pi}{4}+\psi}{\Delta\psi}\right\rfloor + 1\right)\right.$$

and

$$\theta_M = \frac{1}{2}\left\{-\pi + \Delta\theta\left(2\left\lfloor\frac{\frac{\pi}{2}+\theta}{\Delta\theta}\right\rfloor + 1\right)\right\}.$$

MV-QHNN of Minemoto et al.

Numerical Stability:

- The phase-angles ϕ_M, ψ_M, and θ_M are insensitive to small perturbations in ϕ, ψ, and θ.
- Thus, the MV-QHNN of Minemoto et al. is numerically stable.

Stability Analysis:

Minemoto et al. claimed that the energy

$$E(\mathbf{x}) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{x}^H W\mathbf{x},$$

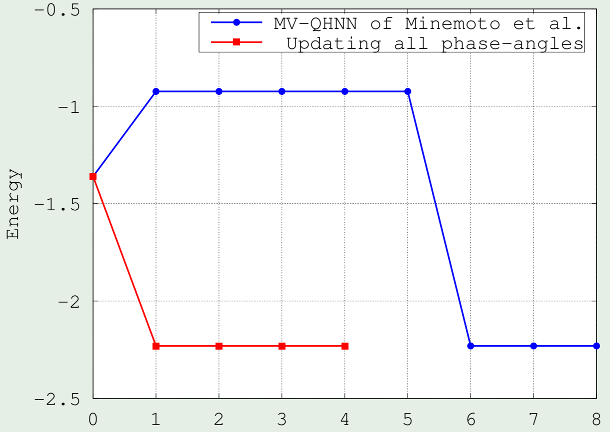
never increases if

$$w_{ij} = \bar{w}_{ji} \quad \text{and} \quad w_{ii} \geq 0, \quad \forall i,j \in \{1,\ldots,n\}.$$

Example (Stability problem:)

Let $K_1 = K_2 = K3 = 4$,

$$W = \begin{bmatrix} 0 & 1 - 2\mathbf{j} - \mathbf{k} \\ 1 + 2\mathbf{j} + \mathbf{k} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}(0) = \begin{bmatrix} e^{\frac{\pi}{4}\mathbf{i}} e^{\frac{\pi}{16}\mathbf{k}} e^{-\frac{\pi}{8}\mathbf{j}} \\ e^{\frac{\pi}{4}\mathbf{i}} e^{\frac{\pi}{16}\mathbf{k}} e^{\frac{\pi}{8}\mathbf{j}} \end{bmatrix}.$$



Example (Stability problem:)

The energy of the network at $\mathbf{x}(0)$ and $\mathbf{x}(1)$ are:

$$E(\mathbf{x}(0)) = -\frac{1}{2}\mathbf{x}^H(0)W\mathbf{x}(0) = -1.3604,$$

$$E(\mathbf{x}(1)) = -\frac{1}{2}\mathbf{x}^H(1)W\mathbf{x}(1) = -0.92388.$$

The difference between the energies at $t = 1$ and $t = 0$ is

$$\Delta E = E(\mathbf{x}(1)) - E(\mathbf{x}(0)) = 0.43651.$$

Despite ΔE positive, the MV-QHNN of Minemoto et al. reaches a stationary state after 6 steps.

In contrast, the energy of the Modified MV-QHNN does not increase!

Modified MV-QHNN of Minemoto et al.

Let the activation of the *i*th neuron at time *t* be

$$v_i(t) = |v_i(t)| e^{\mathbf{i}\phi} e^{\mathbf{k}\psi} e^{\mathbf{j}\theta}.$$

The neuron is updated asynchronously according to

$$x_i(t+1) = e^{\phi_M \mathbf{i}} e^{\psi_M \mathbf{k}} e^{\theta_M \mathbf{j}},$$

where

$$\phi_M = \frac{1}{2}\left\{-2\pi + \Delta\phi \left(2\left\lfloor \frac{\pi + \phi}{\Delta\phi} \right\rfloor + 1\right)\right\}, \psi_M = \frac{1}{2}\left\{-\frac{\pi}{2} + \Delta\psi \left(2\left\lfloor \frac{\frac{\pi}{4} + \psi}{\Delta\psi} \right\rfloor + 1\right)\right.$$

and

$$\theta_M = \frac{1}{2}\left\{-\pi + \Delta\theta \left(2\left\lfloor \frac{\frac{\pi}{2} + \theta}{\Delta\theta} \right\rfloor + 1\right)\right\}.$$

Note that all phase-angles are updated simultaneously!

Conjecture

We conjecture that, if

$$w_{ij} = \bar{w}_{ji} \quad \text{and} \quad w_{ii} \geq 0, \quad \forall i, j \in \{1, \ldots, n\},$$

then the energy

$$E(\mathbf{x}) = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{x}^H W \mathbf{x},$$

decreases if a neuron changes its state.

Thus, the modified MV-QHNN always reaches a stationary state.

Concluding Remarks

In this paper, we first revised the MV-QHNN of Isokawa et al.:

- It is numerically unstable.
- We cannot assure its convergence to a stationary state.

We also investigated the MV-QHNN of Minemoto et al.:

- It is numerically stable.
- Analogously, we cannot assure its convergence.

We proposed an MV-QHNN in which all phase-angles are updated.

- We conjecture the novel model always reaches a stationary state.

Keep in mind...

we should only consider the improved model from now on!

FAPESP CNPq

Thank you!