Wavelets and Adaptive Grids for PDE

Sônia Maria Gomes

soniag@ime.unicamp.br

Cancun, April 28th, 2003

Motivation

⇒ Solutions to many interesting flow problems may exhibit localized singular features:

- sharp transition layers
- propagating steep fronts
- pronounced spikes

⇒ Approximations of these problems present a challenging computational task

⇒ Uniform grid is not a practical option: high resolution is only needed where irregularities occur.

⇒ Improvements in accuracy and computational efficiency may be obtained by economically adapting the grid points to the numerical solution.

Some Basic Principles in Wavelet Analysis

- The information is organized in different scale levels
- The information at a certain level is obtained from the information at the previous coarser level and the addition of details (wavelet coefficients).
- Wavelet coefficientes may be interpreted as local approximation errors: they are significant in the presence of strong gradiends
- Wavelet coefficientes can be used as local regularity indicators.

Main Objectives: Application to Evolution Problems

$$\frac{\partial \mathcal{U}}{\partial t} = L\mathcal{U}$$

$L(\mathcal{U}) \to$ differential operator acting on spatial variables.

Suppose that at time $t_n = n\Delta t$ an sparse representation $(\mathcal{M}^n, \mathcal{U}^n)$ for the approximate solution is given, in which $\mathcal{U}^n$ is formed by the numerical solution discrete values associated to an adaptive mesh $\mathcal{M}^n$.

Next time step: $(\mathcal{M}^{n+1}, \mathcal{U}^{n+1})$

$\Rightarrow$ Extension (refinement): $(\mathcal{M}^n, \mathcal{U}^n) \xrightarrow{\mathcal{E}} (\mathcal{M}^{n+}, \mathcal{U}^{n+})$

$\Rightarrow$ Time evolution: $(\mathcal{M}^{n+}, \mathcal{U}^{n+}) \to (\mathcal{M}^{n+}, \breve{\mathcal{U}}^{n+1})$.

$\Rightarrow$ Truncation (coarsening): $(\mathcal{M}^{n+}, \breve{\mathcal{U}}^{n+1}) \xrightarrow{\mathcal{T}_\epsilon} (\mathcal{M}^{n+1}, \mathcal{U}^{n+1})$.

Outlines

- Part One: Wavelets and Finite Differences
  - ⇒ Adaptive Construction of Block–Structured Grids
  - ⇒ Operations on Block-Structured Grids
  - ⇒ Application to Evolution Problems
- Part Two: Wavelets and Discontinuous Galerkin Method
  - ⇒ Discontinuous Glerkin method with implicit diffusivity
  - ⇒ h-adaptivity using wavelets
  - ⇒ Some applications in CFD problems
- Conclusions

Part One: Wavelets and Finite Differences

In colaboration with

Margarete O. Domingues and Lilliam A. Díaz

Sparse point representation – SPR (Holmstrom - 1997)

$\Rightarrow$ Motivation: Create an adaptive finite difference strategy that combines:

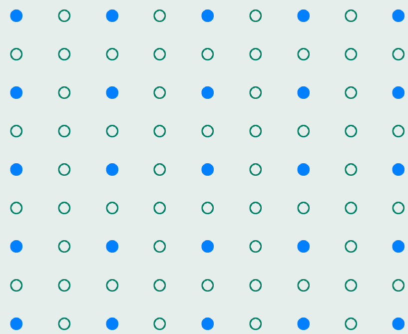
- simplicity, stability and accuracy of FD methods
- ability of wavelet coefficients in the characterization of local regularity of functions

$\Rightarrow$ Ideas: Represent functions by the point values associated to their significant wavelet coefficients

- coarse grid in smooth regions,
  fine grid close to irregularities.
- at each point, spatial derivatives are discretized by uniform FD $\rightarrow$ step size proportional to the point local scale
- if a stencil is not present in the grid, it is approximated from coarser scales by an interpolating subdivision scheme.

Hierarchical Uniform Grids

- The grids: $\mathcal{X}^\ell = \{(x_k^\ell, y_m^\ell) = (kh_x^\ell, mh_y^\ell) \in I \times I\}$
- $\mathcal{X}^\ell \subset \mathcal{X}^{\ell+1}$: $(x_k^\ell, y_m^\ell) \in \mathcal{X}^\ell \Rightarrow (x_{2k}^{\ell+1}, y_{2m}^{\ell+1}) \in \mathcal{X}^{\ell+1}$
- Points in $\mathcal{X}^{\ell+1}$



$\bullet$ $\longrightarrow$points of $\mathcal{X}^\ell$

$\circ$ $\longrightarrow$points of $\mathcal{X}^{\ell+1} \setminus \mathcal{X}^\ell$

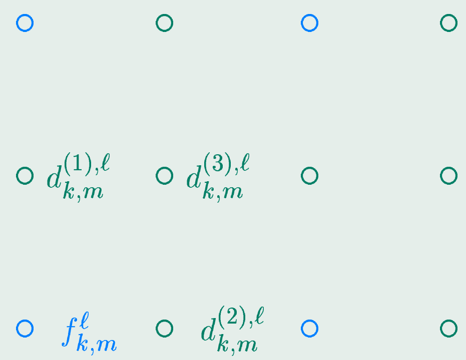
Interpolatory Wavelet Coefficients

$f^{\ell}_{k,m} = f(x^{\ell}_{k}, y^{\ell}_{m}) = f^{\ell+1}_{2k,2m}$

wavelet coefficients: Interpolation errors

$$
\begin{aligned}
d^{(1)\,\ell}_{k,m} &= f^{\ell+1}_{2k,2m+1} - \tilde{f}^{\ell+1}_{2k,2m+1} \\
d^{(2)\,\ell}_{k,m} &= f^{\ell+1}_{2k+1,2m} - \tilde{f}^{\ell+1}_{2k+1,2m} \\
d^{(3)\,\ell}_{k,m} &= f^{\ell+1}_{2k+1,2m+1} - \tilde{f}^{\ell+1}_{2k+1,2m+1}
\end{aligned}
$$

differences between the values of f in $\mathcal{X}^{\ell+1} \setminus \mathcal{X}^{\ell}$ and the ones obtained by interpolation, using the values of f in $\mathcal{X}^{\ell}$



$\circ \rightarrow$ position of wavelets coefficients

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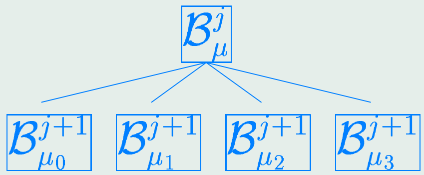
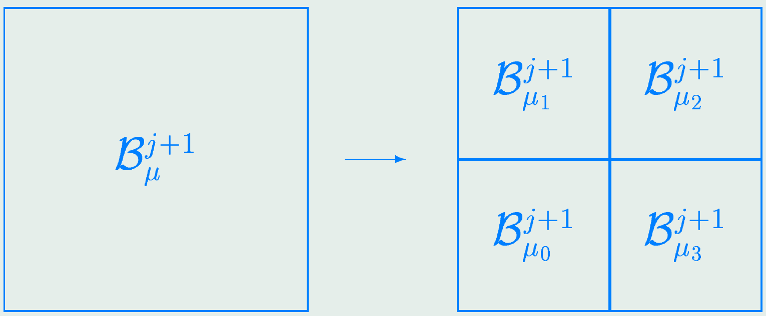
Quad-tree

Tree structure



Grid structure



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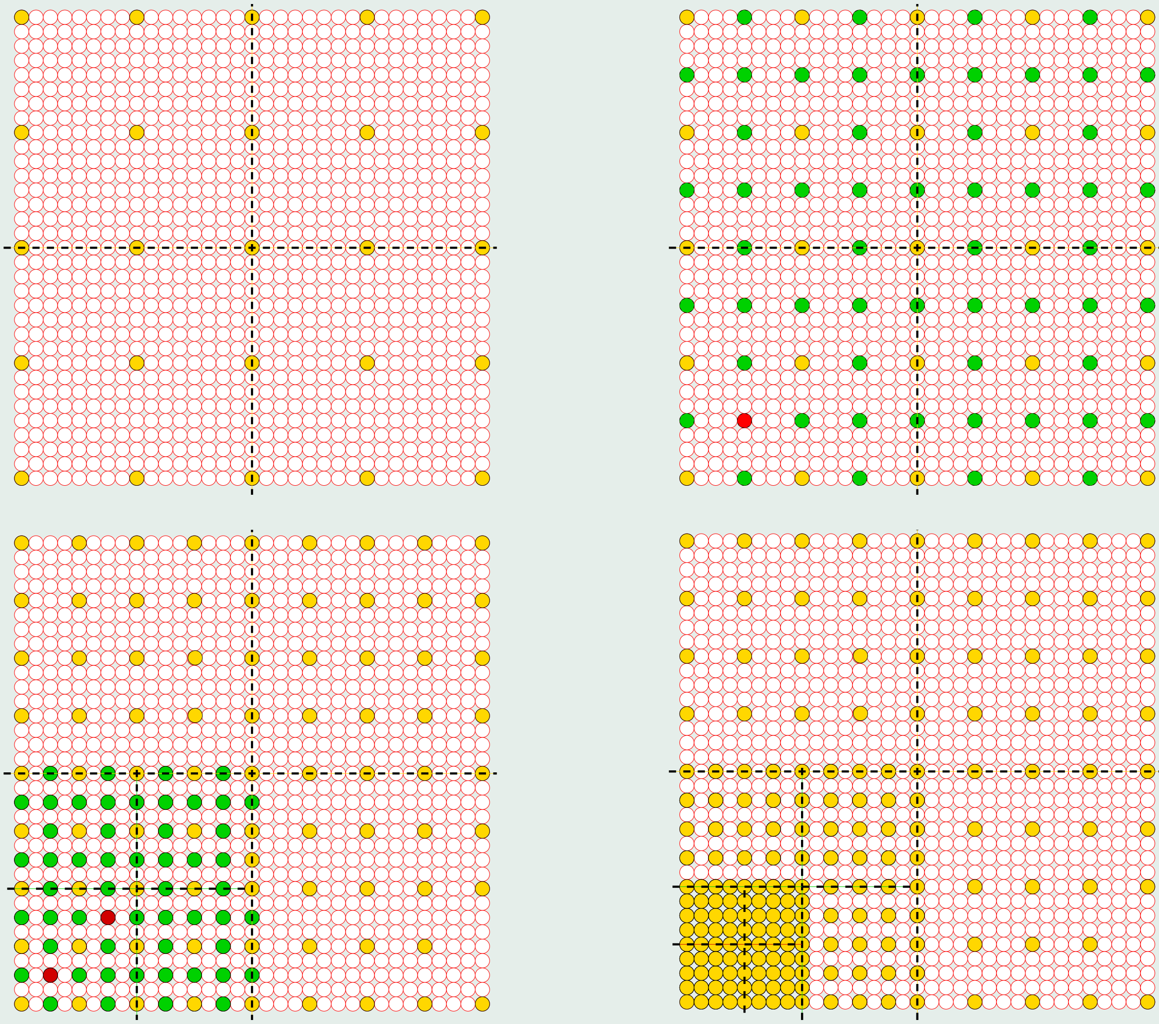
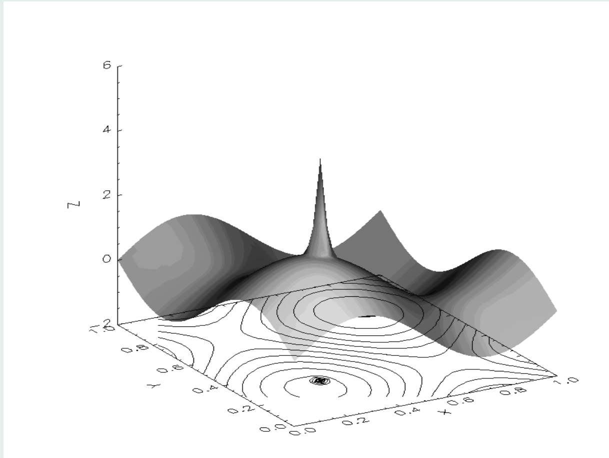
Construction Ideas of $\mathcal{M}_\epsilon \subset \mathcal{X}^5$

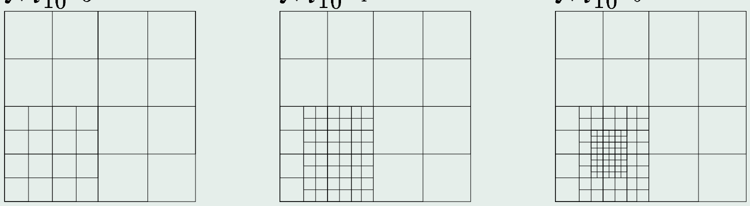


Example: ABR for the spike function



$$f(x,y) = 3\ \exp^{-2500.0\,(\,(x-0.3)^2+(y-0.3)^2)} + sen(2\pi x) + sen(2\pi y)$$

$\mathcal{M}_{10^{-3}}$ $\mathcal{M}_{10^{-4}}$ $\mathcal{M}_{10^{-5}}$

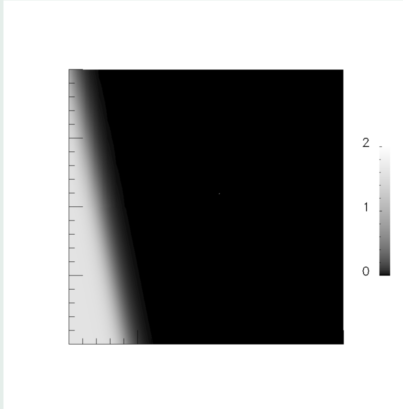


ϵ	$\sharp blocks$	$\sharp(\mathcal{M}_\epsilon)$	$\sharp(\mathcal{X}^\ell)$	$\sharp(\mathcal{M}_\epsilon)/\sharp(\mathcal{X}^\ell)$
10^{-5}	100	102400	2097152	0.05
10^{-4}	64	65536	262144	0.25
10^{-3}	28	28672	65536	0.43

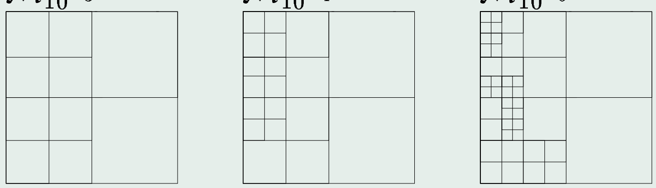
ABR for oblique-front function

$$f(x,y) = 1 - tanh(25x + 5(y-1)),$$

with abrupt changes close to the line $25x + 5(y-1) = 0$



$\mathcal{M}_{10^{-3}}$ $\mathcal{M}_{10^{-4}}$ $\mathcal{M}_{10^{-5}}$



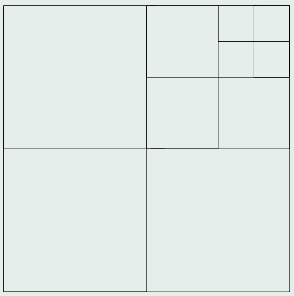
ϵ	$\sharp$ blocks	$\sharp(\mathcal{M}_\epsilon)$	$\sharp(\mathcal{X}^j)$	$\sharp(\mathcal{M}_\epsilon)/\sharp(\mathcal{X}^\ell)$
10^{-5}	44	45056	262144	0.17
10^{-4}	19	19456	65536	0.30
10^{-3}	10	10240	16384	0.60
10^{-2}	4	4096	4096	1.00

Operations on Block-Structured Grids

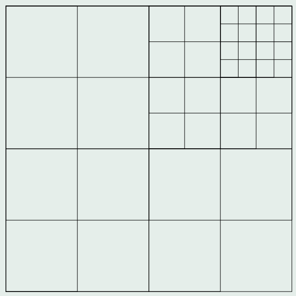
- Tree Extension (Grid Refinement) :

$$\mathcal{A} \xrightarrow{\mathcal{E}} \tilde{\mathcal{A}}$$

Grid $\mathcal{M}$



Grid $\tilde{\mathcal{M}}$



- Tree Reduction (Grid Coarsening):

$$\tilde{\mathcal{A}} \xrightarrow{\mathcal{T}_\epsilon} \mathcal{A}$$

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- Functional Operations

  Operations between two functions represented in block–structured grids are straightforward point-wise evaluations if their grids coincide.

  Otherwise, extend both grids in order to get representations in a common grid.

- Differentiation

  The idea is to use finite difference operators with uniform spacing in each block.

  To avoid demanding search procedures, the process of block construction should consider the addition of needed extra rows and columns around the block boundaries (ghost points).

Examples: Advection Equation

$$\frac{\partial \mathcal{U}}{\partial t} + \left(\frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{U}}{\partial y} \right) = 0,$$

for $t \geq 0$, $(x, y) \in (0,1)\times(0,1)$, with periodic boundary condition and initial condition:

$$\mathcal{U}(x, y, 0) = exp^{-300((x-0.5)^2+(y-0.5)^2)} + 0.2 \sin(2\pi x) + \sin(2\pi y)$$

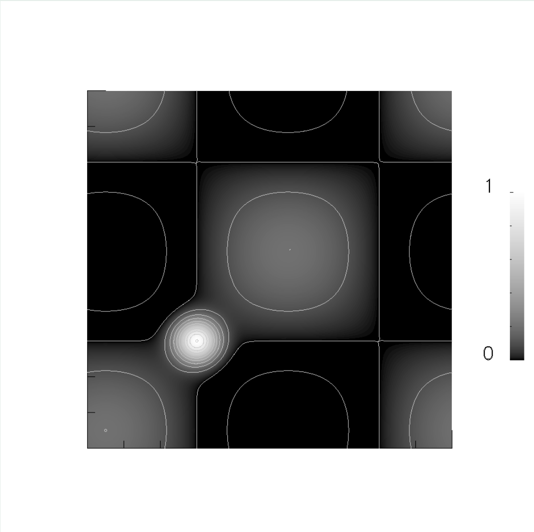
Exact solution: spike moving along the diagonal $x = y$, without changing its format.

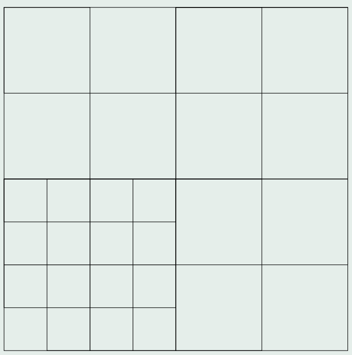
Parameters

- $\epsilon = 10^{-3}$, $N_x = N_y = 32$ and $\lambda = h^n_{\min}/\Delta t^n = 5 \times 10^{-2}$
- interpolation, finite differences and Runge-Kutta $\rightarrow$ 4-th order

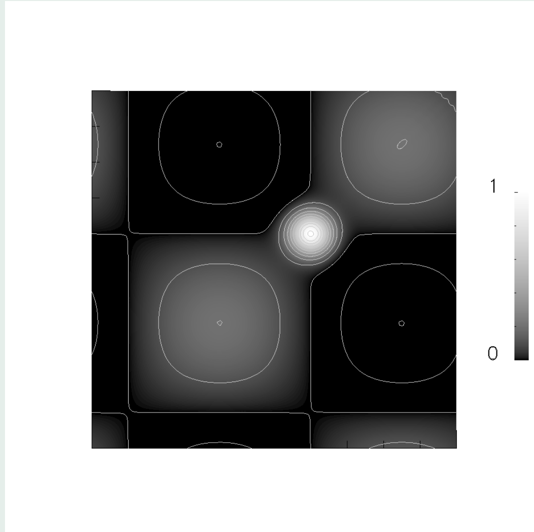
Advecting spike

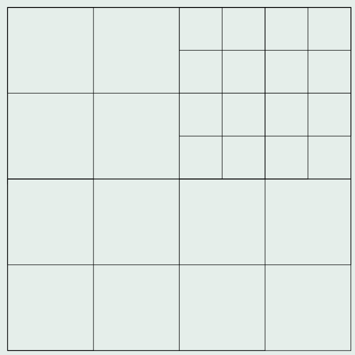
$t = 0.2$

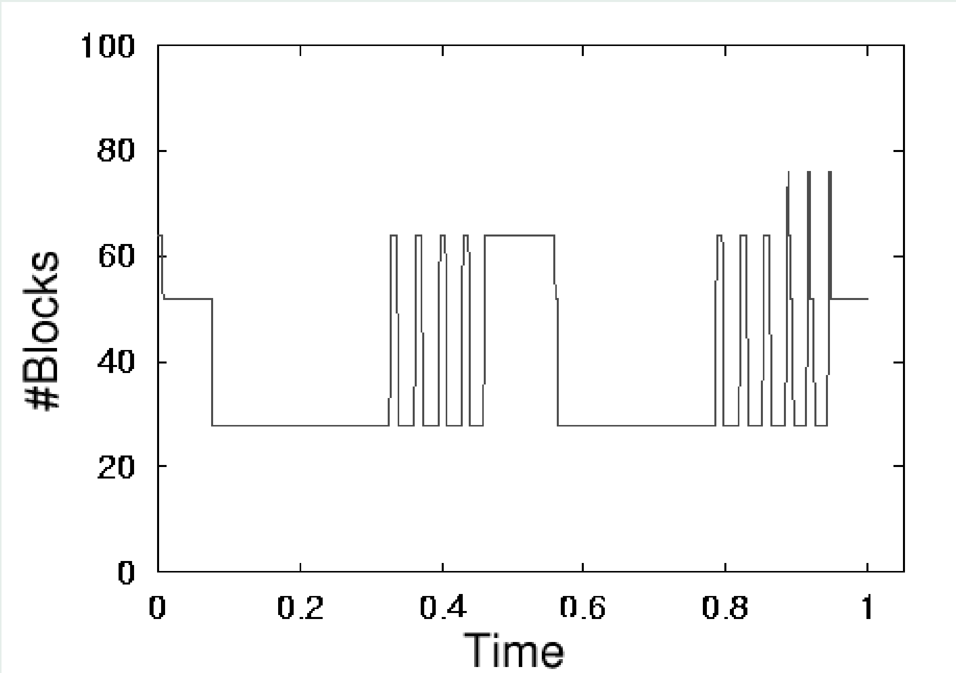




$t = 0.9$







$\sharp$ blocks in the adaptive meshes

Oblique Front: Advection-diffusion equation

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{U}}{\partial y} - \frac{\partial^2 \mathcal{U}}{\partial x^2} - \frac{\partial^2 \mathcal{U}}{\partial y^2} = \mathbf{F}$$

$t \geq 0$, $(x, y) \in (0, 1) \times (0, 1)$ with appropriate boundary conditions and the forcing term.

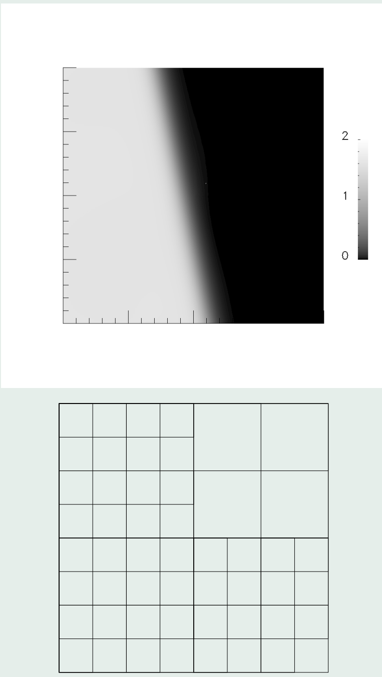
Exact solution $\mathcal{U} = 1 - tanh\Big(25(x-t) + 5(y-1)\Big)$, describing a propagating steep front moving to the right.

Parameters

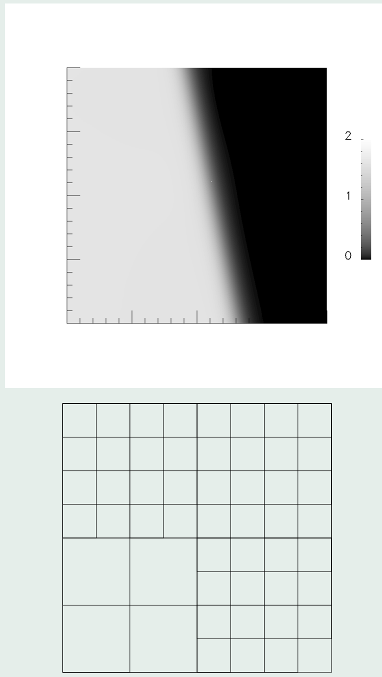
$\boxed{\epsilon = 10^{-2},\ N_x = N_y = 16 \text{ and } \lambda = 10^{-3}}$

Propagation of Oblique front

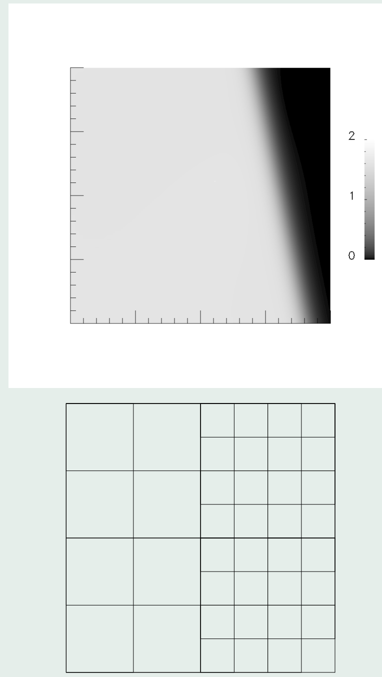
$t = 0.35$



$t = 0.45$



$t = 0.70$



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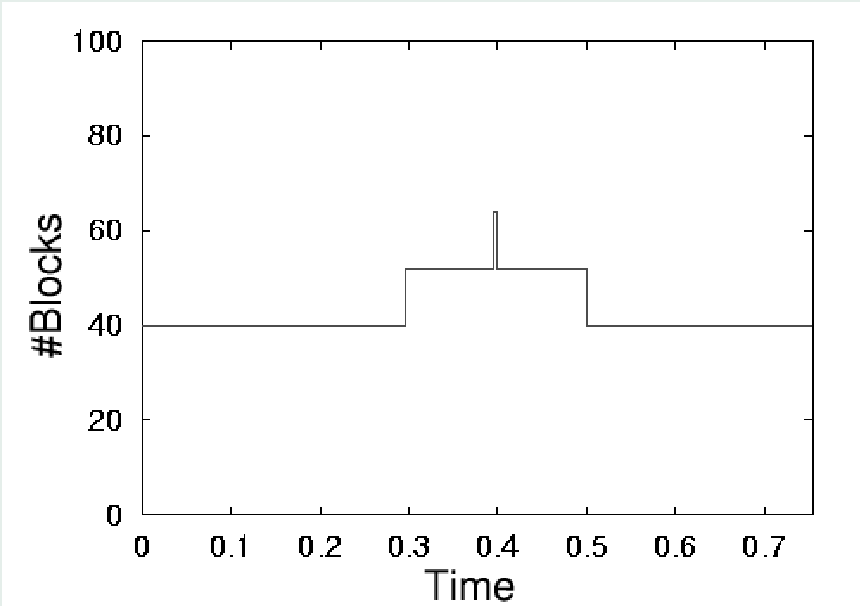


$\sharp$ blocks in the adaptive meshes

Sharp Transition Layers: Burgers' equation:

$$\frac{\partial \mathcal{U}}{\partial t} + \mathcal{U}\,\frac{\partial \mathcal{U}}{\partial x} + \mathcal{U}\,\frac{\partial \mathcal{U}}{\partial y} - \mu\,\frac{\partial^2 \mathcal{U}}{\partial x^2} - \mu\,\frac{\partial^2 \mathcal{U}}{\partial y^2} = 0,$$

$t \geq 0$, $(x, y) \in [0, 1] \times [0, 1]$, with periodic boundary conditions and initial data

$$\mathcal{U}(x, y, 0) = sen(2\pi x)\ sen(2\pi y).$$

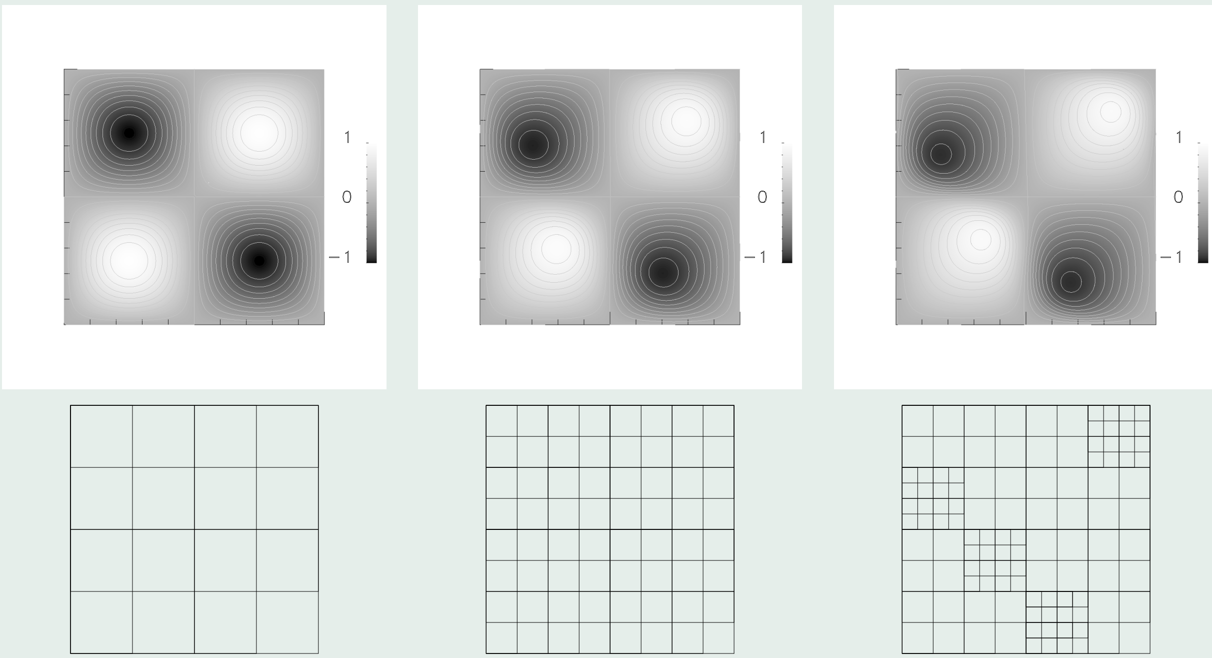
negative and positive features move on opposite directions, producing sharp transition layers, as time evolves.

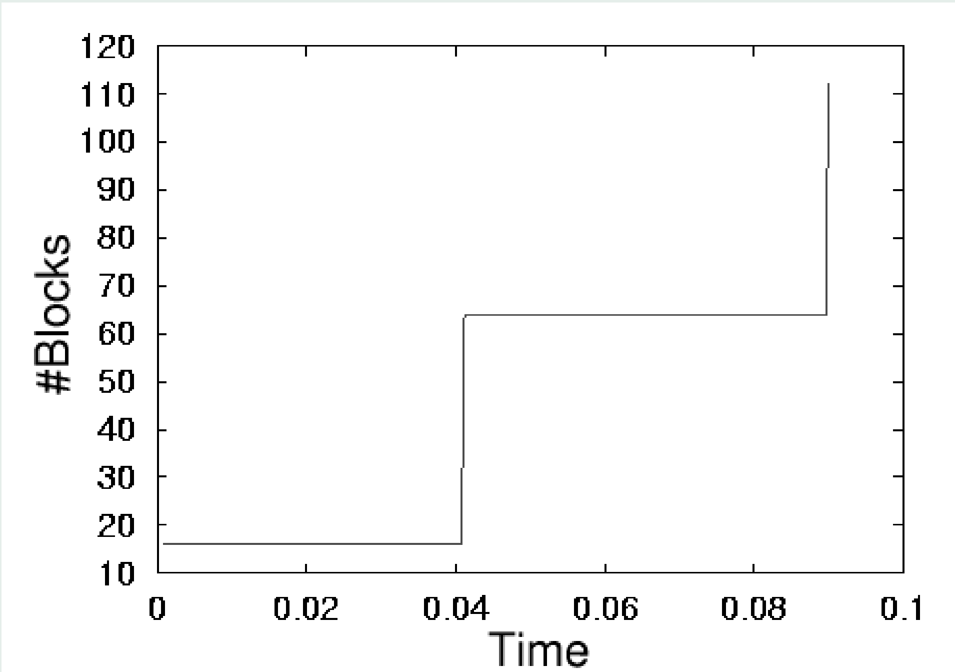
Parameters

$\boxed{\mu = 10^{-2},\ \epsilon = 10^{-5},\ N_x = N_y = 32 \text{ and } \lambda = 5 \times 10^{-2}}$

Formation of Sharp Transition Layers

t=0 t=0.05 t=0.09





$\sharp$ blocks in the adaptive meshes

Conclusion

⇒ An adaptive finite–difference scheme for PDEs based on block–structured grids, which are dinamically generated by wavelet representation techniques.

⇒ Algorithms and data structure are formulated by using abstract concepts borrowed from quad trees.

⇒ Given any desired accuracy, the method is intended to produce simulations with automatic grid refinement, as required by the numerical solution, without penalizating the computational complexity.

⇒ The method, as stated, faces the typical dilema of adaptive solvers with explicit time discretization. Since, for stability, Δt is adjusted to the current finest scale level, the effect of any grid refinement is the increment of the total number of time steps.

Part Two
Discontinuous Galerkin Method with Implicit Diffusivity and h-Adaptivity Based on Wavelets

In colaboration with

Jorge Díaz Calle and Philippe R. B. Devloo

Conservation Laws

$$\frac{\partial \mathbf{u}}{\partial t}(t,x) + \nabla . f(\mathbf{u}(t,x)) = 0 \qquad \mathbf{x} \in \Omega \subset R^d, \quad t > 0$$

$$\mathbf{u}(t,\mathbf{x}) = (u_1(t,\mathbf{x}), \cdots, u_m(t,\mathbf{x})), \text{conserved quantities}$$

$$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \cdots, f_d(\mathbf{u})), \text{flux function}$$

Initial condition: $\mathbf{u}(0,x) = \mathbf{u}_0(x), x \in \Omega$ + apropriate boundary conditions

Discontinuous Galerkin Method

- Approximating space

  $M_h \to$ partion of Ω; $C \in M_h \to$ computational cells

  $V_h \to$ space of piecewise polynomials of fixed degree $p \geq 0$

  $$w \in V_h \Rightarrow w\,|_C \in \Pi_p$$

  $B_{V_h} = \{\varphi_i(\mathbf{x}), \quad i = 1, 2, \ldots, N\}$ basis spanning V_h

- Approximate solution

$$\mathbf{u}(t, \mathbf{x}) = \sum_{i=1}^{N} \mathbf{u}_i(t) \varphi_i(\mathbf{x})$$

- Galerkin Formulation: Find $\mathbf{u} \in V_h$ such that $\forall \varphi_j$ and C

$$\frac{\partial}{\partial t}\int_C \varphi_j \mathbf{u} - \int_C \nabla\varphi_j.\mathbf{f}(\mathbf{u}) + \int_{\partial C} \varphi_j \hat{\mathbf{f}}(\mathbf{u}).\tilde{\eta} = 0$$

$\hat{\mathbf{f}}(u) \rightarrow$ numerical flux

- Time discretization + stabilization

  Runge-Kutta + slope limiters (Cockburn and Shu)

  very small time-steps $(CFL < 1/(p+1))$

  slope limiters depend on the geometry and p

  Runge-Kutta + SUPG diffusive term

  does not work

  Implicit Euler scheme + SUPG difusive term

  satability does not depend on p

- Implicit time discretization + SUPG diffusive term

$$\sum_{i=1}^{N} \mathbf{u}_i^{n+1} \int_C \left[\varphi_i \varphi_j - \delta \Delta t_n \left(\nabla \varphi_j . \beta \right) \sum_{s=1}^{d} f_s^{'}(\mathbf{u^{n+1}}) \frac{\partial}{\partial \mathbf{x_s}} \varphi_{\mathbf{i}} \right] =$$

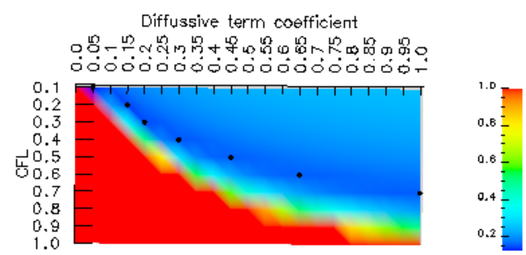
$$\int_C \mathbf{u}^n \varphi_j + \Delta t_n \left\{ \int_C \nabla \varphi_j . \mathbf{f}(\mathbf{u}^n) - \int_{\partial C} \varphi_j \hat{\mathbf{f}}(\mathbf{u}^n) . \eta_C \right\} .$$

Diffusive coefficient: δ; $\beta = (\beta_1, \beta_2)$ (Bonhaus)

Jacobian matrices are estimated from the values of the solution in the previous time step. They have block-diagonal structure

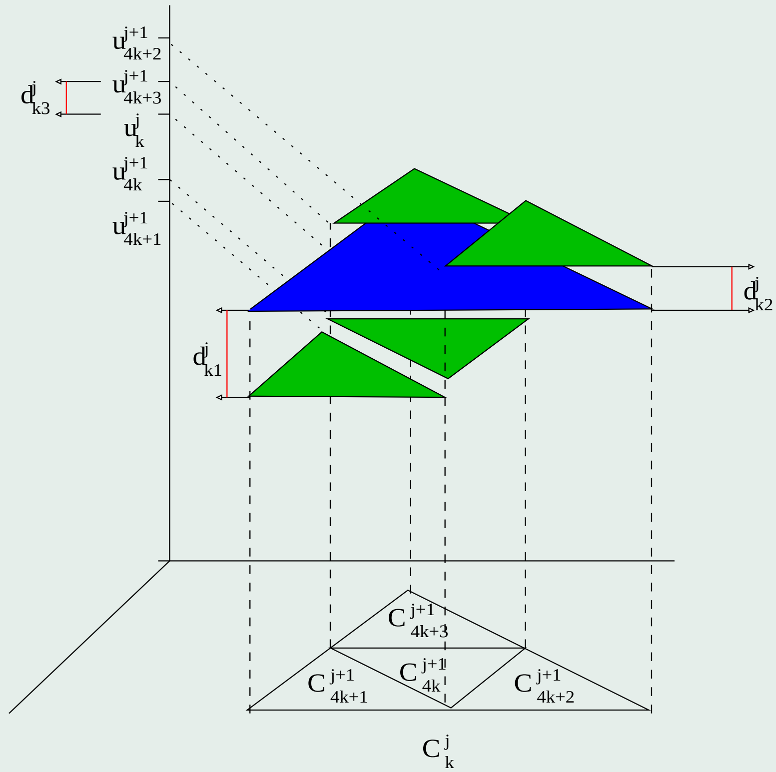
Stability analysis: Does not depend on $p = 1, 2, 3$

$$\delta \geq \frac{10}{3} CFL^2 - \frac{2}{3} CFL + \frac{1}{10} \quad \text{linear advection eq.}$$



p=3 : Errors to discontinuous and implicit scheme

Haar Wavelets: Local Measure of Gradient Variation



$$
\begin{aligned}
|\ d_\mu\ | \ &\leq\ \inf_{q\in\Pi_0} \|\mathbf{u}-q\|_{L^\infty(\overline{C})} \\
&\leq\ C\,2^{-j}\,\|\mathbf{u}\|_{C^1(\overline{C})}
\end{aligned}
$$

Adaptive Strategy

- Transient problem

  Given the numerical solution $\mathbf{u}(t^n, x)$ based on the adapted grid M^n

  1. Compute one level of wavelet coefficients d_λ
  2. If $| d_\lambda | > \epsilon$, the corresponding element and neighbors are refined
  3. If $| d_\lambda | < \theta$ within an element, then it is coarsened.
  4. The coarsest grid and highest level of refinement are stablished a priori

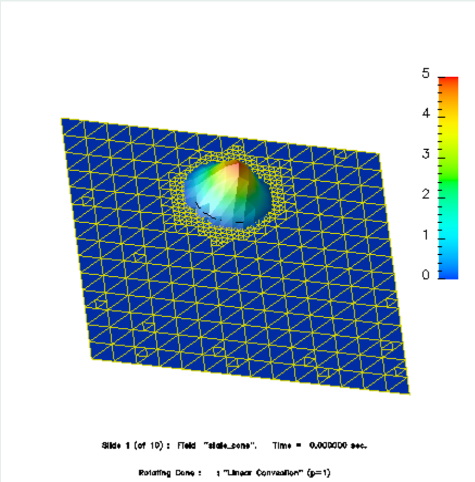
- Stationary problem

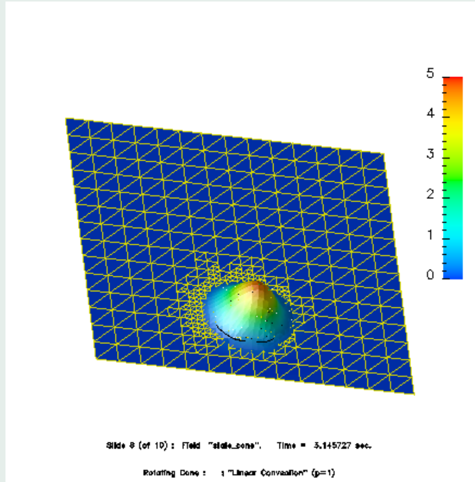
  1. Consider the corresponding transient problem
  2. Iterate on a given gid until iteration error is less than a prescribed parameter
  3. Apply the wavelet-based refinement strategy to update the grid and repeat the process

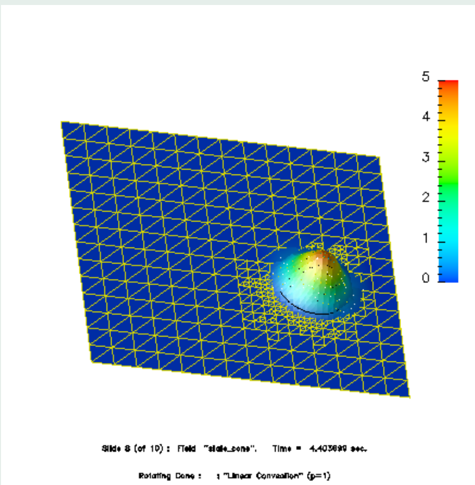
Rotating Cone

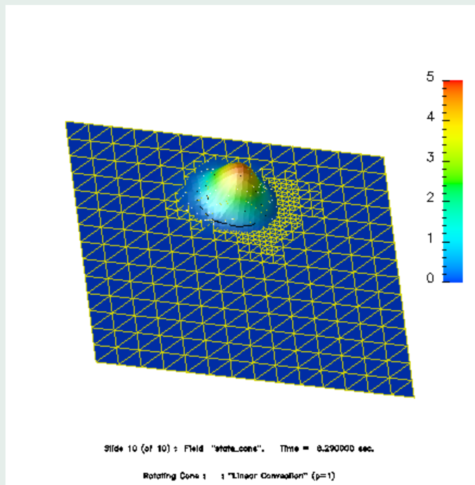
$$f_1(\mathbf{u}) = -y\mathbf{u}, \quad f_2(\mathbf{u}) = x\mathbf{u}, \quad \Omega = [-5,5]\times[-5,5], \quad 0 \le t \le 2\pi$$

Parameters : $p = 1$, $CFL = 0.6$ and $\delta = 0.1$









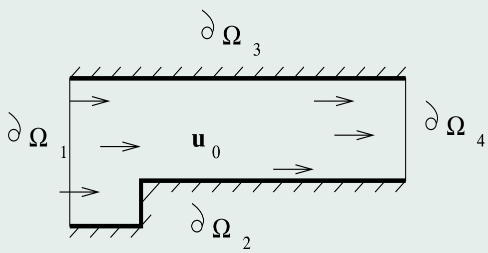
cone	uniform mesh	adaptive mesh
CPU time	7.5 units	2 units
$\sharp$ elements	5000	872
$(\Delta x)_{min}$	0.2	0.179
$\|\text{error}\|_{L_1}$	0.128	0.0656

Euler Equations of Gas Dynamics

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} \quad f_1(\mathbf{u}) = \begin{pmatrix} \rho u \\ \rho v^2 + p \\ \rho v \\ u(E+p) \end{pmatrix} \quad f_2(\mathbf{u}) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}$$

$$E = \frac{p}{\gamma-1} + \frac{1}{2}(u^2+v^2)$$

Backward Facing Step



Estado inicial do fluido

- Initial condition $\mathbf{u}_0$: $\rho_0 = 1.4, u_0 = 3, v_0 = 0, p_0 = 1$
- Boundary conditions:

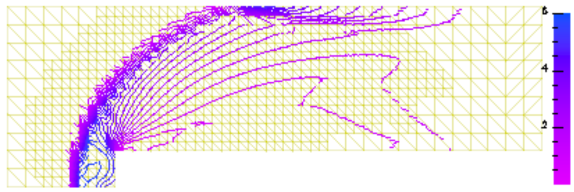
  At $\partial\Omega_2$ and $\partial\Omega_3$: wall

  At $\partial\Omega_1$: $\mathbf{u} = \mathbf{u}_0$

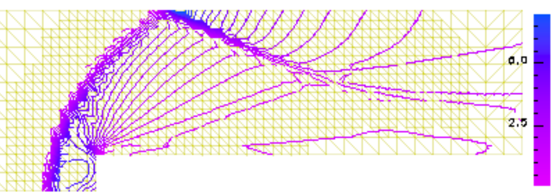
  At $\partial\Omega_4$: free flow

- parameters :$p = 1$, $CFL = 0.4$ and $\delta = 0.75$

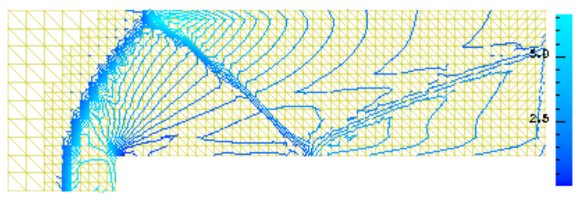
Slide 10 (of 10) : "density". Time = 0.100 sec. p=1



Slide 5 (of 10) : "density". Time = 0.600 sec. p=1



Slide 1 (of 10) : "density". Time = 0.900 sec. p=1



Slide 9 (of 10) : "density". Time = 1.960 sec. p=1

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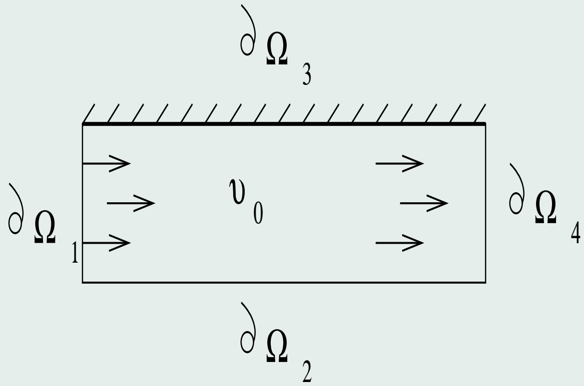
Reflecting Shock



- Initial condition $\mathbf{u}_0$: $\rho_0 = 1.0, u_0 = 2.9, v_0 = 0, p_0 = 0.71$
- Boundary conditions:

  At $\partial\Omega_2$: $\rho_1 = 1.7, u_1 = 2.61, v_1 = 0.50, p_1 = 1.52$

  At $\partial\Omega_1$: $\mathbf{u} = \mathbf{u}_0$

  At $\partial\Omega_3$: wall

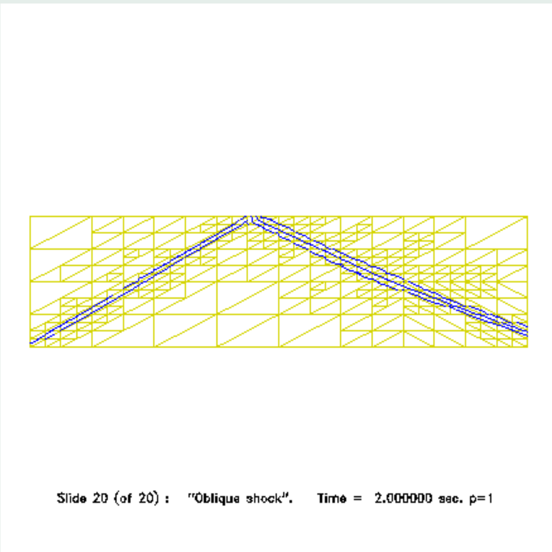
  At $\partial\Omega_4$: free flow

- parameters :$p = 1$, $CFL = 0.5$ and $\delta = 0.5$

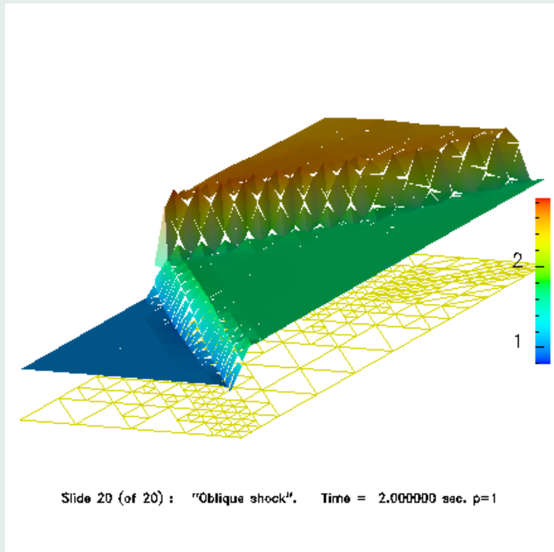
Reflecting Shock: Steady State

Reflecting shock	uniform	adaptive (NE)	adaptive (E)
CPU time	14 units	8 units	5 units
$\sharp$ elements	2080	493	591
$(\Delta x)_{min}$	0.0625	0.0625	0.0625

$p = 1$



$p = 2$



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Conclusions

- Stabilization of a scheme that uses Euler in time and discontinuous Galerkin in space is obtained by the introduction of an implicit diffusive term
- Definition of an h-adaptive strategy that uses wavelet coefficients as regularity indicators
- Savings in memory and CPU time are ilustated for typical CFD test problems
- Implementation in a computational framework based on object oriented philosophy