RESOLUTION OF MAXWELL'S EQUATIONS IN A NON STAGGERED GRID MODEL

Pedro Pinho [(1)], Sônia Maria Gomes [(2)], Paulo Jorge Ferreira [(3)], José Rocha Pereira [(4)]
Margarete O. Domingues [(5)], Anamaria Gomide [(6)]

[(1)]ISEL, Rua Conselheiro Emídio Navarro, 1, 1950-062 Lisboa, Portugal, ppinho@deetc.isel.ipl.pt
[(2)]Unicamp, IMECC. Caixa Postal 6065. 13083-970 Campinas SP, Brasil, soniag@ime.unicamp.br
[(3)]Universidade de Aveiro, DETI/IEETA, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal, pjf@ieeta.pt
[(4)]Universidade de Aveiro, DETI/IT, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal, jrp@det.ua.pt
[(5)]INPE, LAC. Caixa Postal 515. 12245-970 São José dos Campos, SP, Brasil. margarete@lac.inpe.br
[(6)]Unicamp, IC. Caixa Postal 6176. 130874-971 Campinas, SP, Brasil. anamaria@ic.unicamp.br

Abstract— In this paper a scheme to obtain an adaptive method in space for the resolution of Maxwell's equations is presented. Using interpolating wavelets it is possible to obtain an adaptive grid allowing an economy of the computational resources. Using the non staggered grid model the stability factor is improved when compared with the classic FDTD and its value is greater that one. This factor is more limited with the increase of the interpolating polynomial. On the other hand the dispersion proprieties are more restricted, when compared with a staggered grid model.

Index Terms — Multiresolution, FDTD, interpolating wavelets, numerical dispersion, numerical stability.

1. INTRODUCTION

Currently, the numerical methods for the resolution of electromagnetic problems have a great acceptance and the obtained results are very good when compared with the measured results. There are a variety of methods, but the Yee´s FDTD (*Finite Difference Time Domain*) scheme is one of the most popular methods in computational electromagnetics [1]. The basis of the FDTD method is the two Maxwell curl equations in differential form in the time domain expressed by means of central finite differences. FDTD models a region of space by dividing it into cells. In each of these cells the value of the three components of the electric and magnetic fields are calculated in different points of the cell (staggered grid) and stored, for a given time. From these values, we can obtain a new set of values in a later instant by solving the equation in a recursive way, thus advancing in time until the steady state is reached.

Despite its simplicity and modeling versatility, the technique suffers however from serious limitations due to the use of uniformly dense grids leading to long computation times to simulate such grid. The vectors are dense and substantial computer resources are required to model electromagnetic problems with medium or large computational volumes. Finally the method is limited for the Courant stability factor.

To improve the performance of FDTD method, we can use the wavelet's theory for the resolution of Maxwell's equations. In this context the first method to appear was the MRTD (*Multiresolution Time Domain Technique*). This method use a Galerkin scheme, and there are two versions: S-MRTD and W-MRTD [2]. The first uses scaling functions only, the second uses scaling functions and wavelets. These methods tends to reduce the number of cells per wavelength allowing discretizations close the Nyquist limit, and tend to improve the numerical dispersion effect, when compared with the FDTD in a certain range of the CFL parameter [2]. With the W-MRTD it is possible to obtain adaptability by thresholding small wavelets contributions. However, for their multilevel wavelet version, the implementation complexity increases with the number of scale levels, seriously compromising the computational performance.

In this paper we present another type of adaptive strategy named SPR (Sparse Point Representation) that uses wavelet analysis using the second generation wavelets (biorthogonal interpolating wavelets) [3]. With this strategy is possible to obtain an adaptive mesh as a function of time that allows to an economy of resources and a relatively short or acceptable time of simulation. This grid is refined only in certain regions of the space, and less refined in other regions where the variation of the fields is smoother. The principle of the method is to represent the solution only through those points' values indicated by the significant wavelet coefficients, which are defined as interpolating errors. Using this method it is possible to use a number N_S of points, less than the N points of the original representation. One of the motivations for the use of wavelets for solving PDEs is its compression capability. For many types of signals, a small fraction of the coefficients is enough to achieve a good approximation to the function $f(x)$. It is acceptable to assume that this sparse representation could lead to efficient algorithms in terms of computer memory and the number of arithmetic operations.

When we apply this technique, there are two options, for the solution of Maxwell's equations: in the first each component of the field has one independent grid, as in the FDTD case (staggered grids). In the second the grid is common for both the electric and magnetic field. In [4] the author presents the first case. In the present paper we demonstrate some of the potential of interpolating wavelets for the resolution of Maxwell equations, in a non staggered grid model, in terms of the dispersion and stability proprieties, trough the results of a numerical simulation in one dimension.

2. NUMERICAL SCHEMES FOR MAXWELL EQUATIONS IN NON STAGGERED GRID.

Here we describe a 1-D example in order to demonstrate the method described above. Let us consider a TEM wave propagating along the x axis, the electric field is directed along the z axis and the magnetic field is directed along the y axis, being equal to zero at the initial instant. Maxwell's equations are given in Cartesian coordinates by the formulae:

$$\frac{\partial E_z}{\partial x} - \mu \frac{\partial H_y}{\partial t} = 0 \quad ; \quad \frac{\partial H_y}{\partial x} - \varepsilon \frac{\partial E_z}{\partial t} = 0 \tag{1}$$

where, ε is the electrical permittivity and μ the magnetic permeability. In a non staggered grid we use only a grid for the electric and magnetic field in the same point. This corresponds to take the following samples of the components of H and E:

$$\begin{aligned} E_k^n &\approx E(k\Delta x, n\Delta t) \\ H_k^n &\approx H(k\Delta x, n\Delta t) \end{aligned} \tag{2}$$

Based on these discrete values, approximations for the components of E and H at any point are defined by the following interpolation operators,

$$\begin{aligned} E(x, n\Delta t) &\approx \sum_k E_k^n \phi\left(\Delta x^{-1} x - k\right) \\ H(x, n\Delta t) &\approx \sum_k H_k^n \phi\left(\Delta x^{-1} x - k\right) \end{aligned} \tag{3}$$

The basic function $\phi(x)$ is chosen in the family of Dubuc-Delauriers scaling functions [5], which are the basis for the construction of interpolating multiresolution analysis. These scaling functions are identified by a parameter p associated to the interpolating subdivision schemes, based on central Lagrange polynomial interpolation of degree 2*p*-1, $p \geq 1$. They are symmetric, $\phi(-x) = \phi(x)$, and satisfy the interpolation property $\phi(k) = \delta_{0k}$.

Based on the interpolation of E and H, we consider the following approximations:

$$\begin{aligned} \frac{\partial E}{\partial x}(j\Delta x, n\Delta t) &\approx \frac{1}{\Delta x} \sum_k E_k^n \phi'(j-k) \\ &= \frac{1}{\Delta x} \sum_k E_k^n \beta(j-k) \\ \frac{\partial H}{\partial x}(j\Delta x, n\Delta t) &\approx \frac{1}{\Delta x} \sum_k H_k^n \phi'(j-k) \\ &= \frac{1}{\Delta x} \sum_k H_k^n \beta(j-k) \end{aligned} \tag{4}$$

where $\beta(u) = \phi'(u)$

Combining the last expressions and, considering the backward difference formula for the E field component and the forward difference formula for the H field component, to perform the temporal derivative, we obtain the following discrete equation system:

$$\begin{aligned} E_j^{n+1} &= E_j^n + \frac{\Delta t}{\varepsilon \Delta x} \sum_k H_k^n \beta(j-k) \\ H_j^{n+1} &= H_j^n + \frac{\Delta t}{\mu \Delta x} \sum_k E_k^{n+1} \beta(j-k) \end{aligned} \tag{5}$$

The Tab. 1 shows the coefficients $\beta(u)$ for $p = 1$ to $p = 4$.

Table 1 – Nonzero coefficients $\beta(u),\ u > 0$

u	p=1	p=2	p=3	p=4
0	0	0	0	0
1	1/2	2/3	272/365	1747/2203
2		-1/12	-53/365	-1483/7724
3			16/1095	399/11882
4			1/2920	-73/32823
5				128/743295
6				1/1189272

2.1 Dispersion and Stability

To avoid dispersion, the phase constant must be a linear function of the angular frequency ω. However, the numerical methods lead to dispersion (non-linear dependency in ω) the level of which depends on the adopted model. Let us consider plane wave solutions in the form $E(x,t) = Ee^{i(kx-wt)}$ and $H(x,t) = He^{i(kx-wt)}$, where ω is the frequency and k is the wave number in space. Replacing these expressions in the Maxwell's equations the conclusion is that plane waves solutions are admitted if the analytical relations holds, $\overline{v} = v_p / c = 1$, where $v_p = w/k$ and $c = 1/\sqrt{\mu\varepsilon}$. If we define the grid density $\delta = \Delta x \omega / c2\pi$, the wavelength $\lambda = cT$ where $T = 2\pi/\omega$ is the period, then for $\delta = 1/N$, $N = \lambda/\Delta x$ is the number of cells per wavelength. For the numerical scheme, plane waves solutions exist provided the numerical dispersion relation $\sin(s\pi\delta) = s\widetilde{\beta}(2\pi\delta/\overline{v})$ holds, where $s = c\Delta t/\Delta x$ is the CFL parameter and $\widetilde{\beta}(\eta) = \sum_{k \geq 0} \beta(k) \sin(k\eta)$. Consequently the stability condition reads $s \leq 1/k_\beta$, for $k_\beta = \max|\widetilde{\beta}(\eta)|$. For the cases under study, we find that $k_\beta = k_\beta(p)$ increases with p. This means that, for stability, the CFL numbers should be smaller for higher orders of the interpolation scheme. The next table shows the values of the CFL for different orders. Since the maximum value of CFL, for the staggered grid model, is 1 (for p=1) [6], then from the Tab. 2 it is possible to conclude that the stability factor is less limiting if we use the staggered grid model. In particular for $p = 1$ the CFL is two times greater than the CFL for the standard FDTD. The plot of $\bar{v}$ as a function of the number of cells per wavelength is show in the Fig. 1, for p=2 and CFL numbers varying within the stability range of each scheme.

Table 2 – CFL values for different orders

Model	CFL
p=1	2
p=2	1.4575
p=3	1.2712
p=4	1.16972

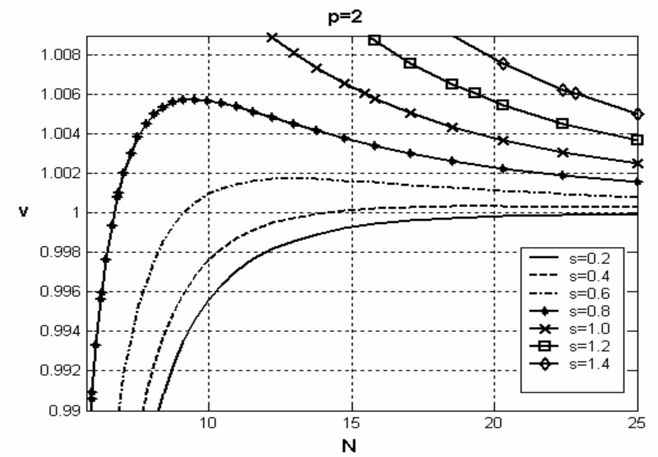

Figure 1- Dispersion analysis in the plane $N \times \bar{v}$ for the non-staggered grid scheme: CFL effect.

3. SIMULATED RESULTS

In order to demonstrate the advantages of interpolating wavelets for the solution of Maxwell's equations, using a non staggered grid we considered an example involving the homogeneous medium illustrated in the Fig. 2.

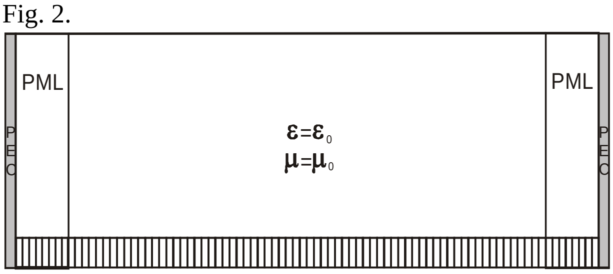

Figure 2- Homogeneous medium.

We consider an example where the fields at $t = 0$, are assumed to be $E(x,0) = e^{-150(x-0.5)^2}$ and $H(x,0) = 0$ for $0 \le x \le 1$. As boundary conditions we consider the PML and the usual PEC. The simulation parameters are CFL=1.4575 and p=2, the SPR grid has 20 points in the coarsest level and 5 levels of resolution are used. The algorithm starts by converting the fields at t=0 to sparse representations on non staggered grid, using recursive cubic polynomial interpolation and thresholding, leading to N_s no uniformly sampled points. When the time evolves the number of samples N_s varies with the time as it is possible to see in the Fig. 4. For the discretization of the spatial derivatives, at each point of the sparse grids, a fourth order finite difference scheme is applied with uniform step size corresponding to the finest level. If some required stencil point is not present, it is obtained by using the recursive interpolation scheme. The results obtained are shown in Fig. 3. The plots refer to the electric field at three different time instants.

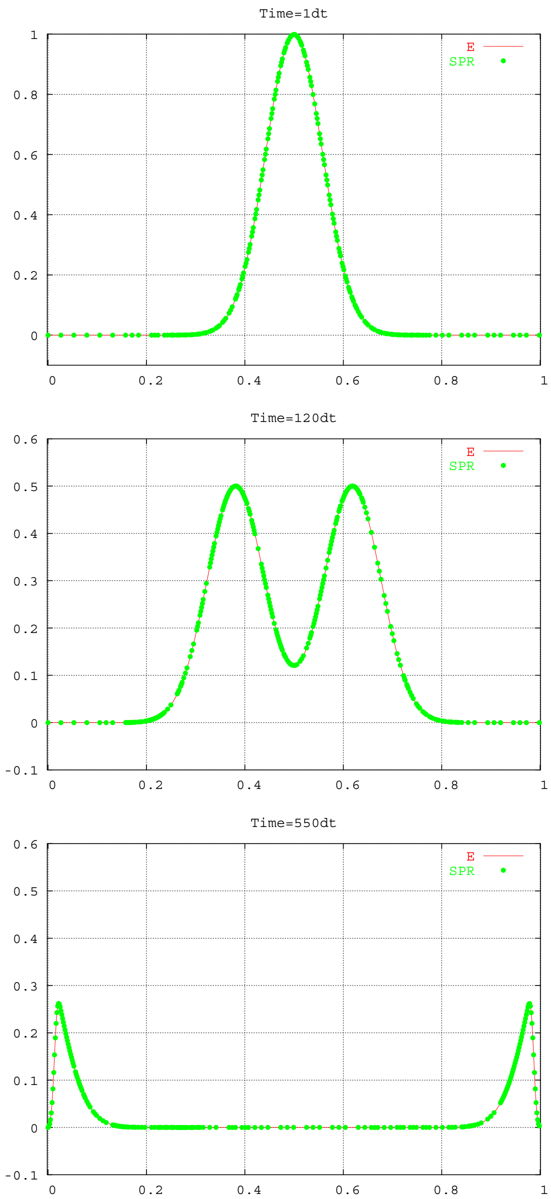

Figure 3 Time evolution of SPR representation

In Fig. 4 it is possible to see the evolution with time of the percentage of points used in the simulation with respect to the total number of points in the finest grid. It demonstrates that with this method there is an economy in terms of the number of points and consequently a faster algorithm is achieved.

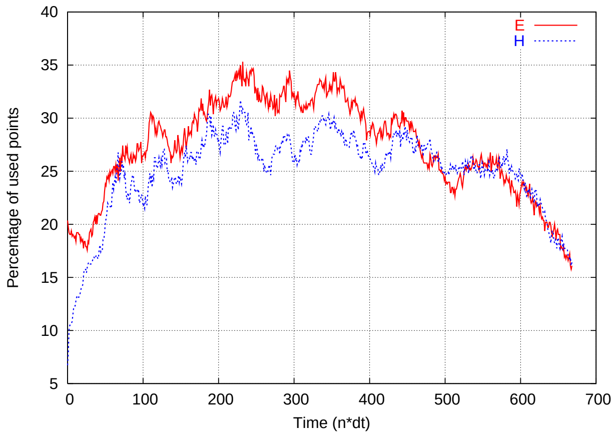

Figure 4 Evolution of the number of grid points in the SPR grids for the E and H fields.

Finally, for comparison purposes, Fig. 5 shows the E field distributions obtained after the same computational time, using two different schemes: staggered and non-staggered. From this figure it is possible to conclude that after the same simulation time, with the non-staggered scheme we are closer of the final distribution. That means that the non-staggered scheme is faster than the staggered one. This is due to the fact that the CFL parameter is less restrictive for the non-staggered scheme.

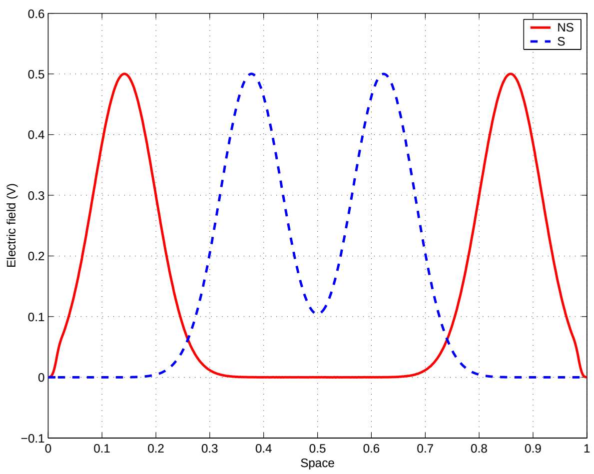

Figure 5 Comparison, between the E field distributions, after the same simulation time, using staggered and non-staggered grids schemes.

4. CONCLUSIONS

This paper describes an application of interpolating wavelets and recursive interpolation schemes with thresholding, aiming the representation of the electric and magnetic fields in nonuniform, adaptive grids. Applied to Maxwell's equations, the method leads to sparse grids that adapt in space to the local smoothness of the fields, and at the same time track the evolution of the fields over time. In general, the number of points in the grid, N_s, is well below the maximum number of points, N. It is possible to control N_s, by trading off representation accuracy and data compression, and therefore speed. When $N_s << N$ there are substantial gains in memory and speed, two important advantages over FDTD scheme, which deals with dense grids that, as a rule, over sample the fields in space.

5. REFERENCES

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