BOOTSTRAPPING *Ad*-EQUIVARIANT MAPS, DIFFEOMORPHISMS AND INVOLUTIONS

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ABSTRACT. We describe several bootstrapping techniques to construct new *Ad*-equivariant maps from old. The base case of the bootstrap provides examples of exotic difeomorphisms and involutions, and therefore these techniques produce candidates for new examples.

1. INTRODUCTION

An orientation-preserving diffeomorphism $\sigma : S^n \to S^n$ is said to be *exotic* if it is not isotopic to the identity, i.e., there is no deformation through diffeomorphisms σ_t such that $\sigma_0 = \sigma$ and $\sigma_1 = Id_{S^n}$. The existence of such diffeomorphisms was inferred by Milnor [13] in his seminal paper on spheres homeomorphic but not diffeomorphic to the Euclidean sphere, since differentiable structures on (oriented) $n+1$-dimensional spheres are essentially in a bijective correspondence with isotopy classes of orientation-preserving diffeomorphisms of S^n via the twisted sphere construction. This is done by setting Σ_σ^{n+1} to be two disks glued by their boundaries via σ ; see the preliminaries section of [1] for a quick review or [11, 12] for details.

However, no explicit formula for these diffeomorphisms existed until [4]: write

$$S^6 = \left\{ (p, w) \in \mathbb{H} \times \mathbb{H} \;\middle|\; \Re(p) = 0,\, |p|^2 + |w|^2 = 1 \right\},$$

and

$$\sigma(p,w) = \begin{cases} \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|} p \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}, \; \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|} w \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}, & w \neq 0 \\ (p, 0), & w = 0, \end{cases}$$

Then σ is an exotic diffeomorphism of S^6.

The study of the structural properties of this diffeomorphism has provided many insights into the general theme of exotic phenomena [1, 5, 6, 7]. Particularly, in [1] it is observed that $\delta = -\sigma$ is an exotic free involution, where the exoticity is reflected in the fact that the quotient S^n/δ is not diffeomorphic to the standard real projective plane; having an explicit formula allows the analysis of the geometry of such involutions [1, 2].

A natural development is then to abstract the structural principles that underlie this one example, the long-term goal being to construct exotic phenomena in a more algorithmic, less ad-hoc way. A begining was made in [9]: let G be a Lie groups acting on a manifold M and consider an Ad-equivariant map $\alpha : M \to G$ (that is $\alpha(g \cdot x) = g\alpha(x)g^{-1}$; we abuse notation by calling both the conjugation action on G and its derivative on the Lie algebra by Ad) . To such a map α we associate

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the *equivariant reentrance* $\hat{\alpha} : M \to M$ given by $\hat{\alpha}(m) = \alpha(m) \cdot m$, where the dot denotes the group action. We have that $\hat{\alpha}$ is always a diffeomorphism of M, and Ad-equivariant homotopy classes of maps $\alpha : M \to G$ correspond to isotopy classes of diffeomorphisms $\hat{\alpha} : M \to M$ (see [9] for details). Furthermore, if in addition there is an involution δ of M such that $\alpha(\delta(m)) = \alpha^{-1}(m)$, and δ commutes with the G-action (thus producing a $G \times \mathbb{Z}_2$-action on M), then the reentrance $\delta\hat{\alpha}$ is another involution of M.

A shortcoming of this result was that the only non-trivial example of the structural theorem was the one example we began with, namely the maps σ and α, and their equivariant deformations; the Ad-equivariant map in this case being the Blakers-Massey element $b : S^6 \to S^3$ given by

$$b(p, w) = \begin{cases} \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}, & w \neq 0 \\ -1 & w = 0 \,. \end{cases}$$

Constructing (differentiably) such equivariant maps by hand leads to difficult problems regarding the structure of the orbit spaces near singular orbits.

The main result of this note is to provide several "bootstrapping" methods that allow for the construction of new examples form old. In particular, the maps σ and α are the initial step in infinite families of Ad-equivariant maps in the sense described above; now there is an enormous wealth of candidates for exotic diffeomorphisms, involutions and their corresponding spaces.

2. Bootstrapping equivariant maps

Here we describe the main bootstrapping construction; the first two are inspired in traditional topological bootstrapping methods. We will postpone the differentiability issues to section 4, where we treat the join construction.

2.1. **Joins.** Recall that the join $X * Y$ of two topological spaces is the quotient $X \times Y \times [0, 1]/ \equiv$, where $(x, y_1, 0) \equiv (x, y_2, 0)$ and $(x_1, y, 1) \equiv (x_2, y, 1)$. The join $S^n * S^m$ of two Euclidean spheres is homeomorphic to the sphere S^{m+n+1}, explicitly given by $[(x, y, t)] \mapsto (\cos(t)x, \sin(t)y)$, the interval given by $t \in [0, \pi/2]$ for geometric convenience (in such a way the join is realized by geodesics joining totally geodesics S^n, S^m inside of S^{m+n+1}). Let us remark that exotic spheres can also have geodesic join structures and the equivariant geometry of such codifies some of the exoticity: see [6].

As in the equivariant reentrance construction, let " $\cdot$ " denote an action of the Lie group $SU(2)$ on S^n, and consider an Ad-equivariant map $\alpha : S^n \to SU(2)$. We could then form the equivariant reentrance $\hat{\alpha} : S^n \to S^n$, but we will *not* do that. Instead, we identify S^2 as the unit sphere of the Lie algebra of $SU(2)$, and consider the map $\alpha^* : S^2 * S^n \equiv S^{n+3} \to SU(2)$ given by

$$\alpha^*(\xi, x, t) = \alpha(x) \exp(\pi \cos(t)\xi)\alpha^{-1}(x) \,.$$

We also endow $S^n * S^2$ with an action $\bullet$ of $SU(2)$: given $\theta \in SU(2)$, set

$$\theta \bullet (\xi, x, t) = (Ad_\theta \xi, \theta \cdot x, t)$$

A moment's reflection will convince the reader that both α^* and the action $\bullet$ are well defined under the equivalence relation $\equiv$ (and for this it is essential that our Lie group is $SU(2)$, so that the exponential map from the sphere of radius π in the Lie algebra collapses to a point).

Also, we have that

$$\begin{aligned}\alpha^*(\theta \bullet (\xi, x, t)) &= \alpha^*(Ad_\theta \xi, \theta \cdot x, t) \\ &= \alpha(\theta \cdot x) \exp(\pi \cos(t) Ad_\theta \xi) \alpha^{-1}(\theta \cdot x) \\ &= \theta \alpha(x) \theta^{-1} \theta \exp(\pi \cos(t) \xi) \theta^{-1} \theta \alpha^{-1}(x) \theta^{-1} \\ &= \theta \alpha^*(\xi, x, t) \theta^{-1},\end{aligned}$$

which means that now $\alpha^* : S^{n+3} \to SU(2)$ satisfies the hypothesis of the reentrance theory. Thus we can bootstrap in this way going in steps of 3.

Another interesting remark is that the exotic maps b and σ are *not* the first step in the bootstrapping ladder: in fact, if we beging with $\alpha(x) = x$, the identity of $SU(2)$ and the conjugation action, after one step we get exactly the Blakers-Massey map $b : S^6 \to S^3$ and the corresponding reentrance $\sigma : S^6 \to S^6$.

2.2. **Smash products.** We consider now the smash product $X \wedge Y = X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y\})$. Again , the smash product $S^n \wedge S^m$ is homeomorphic to the sphere S^{n+m}. We will now use the commutator instead of conjugation: let again $\alpha : S^n \to SU(2)$ and consider $\alpha^\wedge : S^3 \vee S^n \cong S^{n+3} \to SU(2)$, given by

$$\alpha^\wedge(q, x) = [q, \alpha(x)] = q\alpha(x)q^{-1}\alpha(x)^{-1},$$

where we have used that $SU(2) \cong S^3$, considered as the unit quaternions. The map is well-defined choosing $q = 1$ and some preimage x_0 of the identity (which we assume exists) as the base points. The properties of the commutator produce the desired Ad-equivariance properties for $\alpha^\wedge$, again providing a map satisfying the hypothesis of the reentrance theorem in 3 dimensions more.

Let us also remark that the original expression for the Blakers-Massey element as a generator of $\pi_6(S^3)$ was given in this fashion: as the commutator map $c : S^3 \wedge S^3 \cong S^6 \to S^3$ [10].

2.3. **Hopf maps.** We can use the Hopf map $h : S^3 \to S^2$ to increase the bootstrap number to 4, i.e., to go from S^n to S^{n+4}. First recall the presentation of the Hopf map we use: given a unit quaternion q, $h(q) = q\mathbf{i}q^{-1}$. There are two –probably related– ways of using the Hopf maps: either by pre- or post-composition.

Pre-composing is given as follows: given $\alpha : S^n \to SU(2)$ as usual, let ${}^h\alpha : S^3 * S^n \cong S^{n+4} \to SU(2)$ by

$${}^h\alpha(x, q, t) = \alpha^*(x, h(q), t) = \alpha(x) \exp(\pi \cos(t) h(q)) \alpha^{-1}(x).$$

The equivariance condition $h(rq) = rh(q)r^{-1}$ satisfied by the Hopf map immediately gives the Ad-equivariance condition for ${}^h\alpha$, where the $SU(2)$-action on $S^3 * S^n$ is now given by $\theta \bullet (q, x) = (\theta q, \theta \cdot x)$.

Post-composition is inspired by the traditional expression for the Hopf map in complex coordinates $(z, w) \mapsto (|z|^2 - |w|^2, 2z\bar{w})$, which in join coordinates of $S^3 \cong S^1 * S^1$ is written as $(z, w, t) \mapsto (\cos(2t) + \sin(2t)z\bar{w})$. Consider $\alpha^h : S^3 * S^n \cong S^{n+4} \to SU(2)$ by

$$\alpha^h(q, x, t) = (\cos(2t) + \sin(2t)\alpha(x)q^{-1}).$$

Note that for $\alpha^h(q, x, 0) = 1$ and $\alpha^h(q, x, \pi/2) = -1$, making the map well-defined. If we let $SU(2)$ act in $S^3 * S^n$ by $\theta \bullet (q, x) = (\theta q, \theta \cdot x)$ we obtain the required equivariance properties.

2.4. **Involutions.** With the exception of the smash product, all of these constructions have easily defined maps $\delta : S^{n+k} \to SU(2)(S^{n+k})$ satisfying $\alpha\delta = \iota\alpha$, where ι is the group inverse, and therefore $\delta\hat{\alpha}$ is another (possibly exotic) involution; the map δ is simply the antipodal map on S^2 or S^3 we are joining: for the join construction, set $\delta(\xi, x, t) = (-\xi, x, t)$, and for both Hopf maps, $\delta(q, x, t) = (-q, x, t)$. Thus we immediately have examples generalizing the exotic involutions of [1] (the generalization being of the algebra, we still do not know about the exoticity of such examples).

In addition to this, just the algebraic properties of the involutions $\delta\hat{\alpha}$ provide many other involutions, whose commutation relations are interesting in themselves.

For example, we have

Proposition 2.1. *Let α, δ as above. Then for all integers m, n, $\hat{\alpha}^n\delta\hat{\alpha}^m$ are also involutions.*

Thus, there are many more involutions than the involution $\delta\hat{\alpha}$ given on [9].

Proof. Note that for all the constructions above we have $\alpha^k\delta = \iota\alpha^k$ so

$$\delta(\alpha^k(\delta m) \cdot \delta m) = \delta\delta(\alpha^k(\delta m) \cdot m) = \iota(\alpha^k(m)) \cdot m = \hat{\alpha}^{-k}(m)$$

and

$$(\hat{\alpha}^n\delta\hat{\alpha}^m)(\hat{\alpha}^n\delta\hat{\alpha}^m) = \hat{\alpha}^n(\delta\hat{\alpha}^{m+n}\delta)\alpha^m = \hat{\alpha}^n(\hat{\alpha}^{-(m+n)})\hat{\alpha}^m = \text{identity}.$$

□

In principle, all of these involutions are distinct. Note that $(\delta\hat{\alpha})^{-1} = \delta\hat{\alpha}$ on the one hand for being an involution, and on the other hand $(\delta\hat{\alpha})^{-1} = \hat{\alpha}^{-1}\delta$, and we have the commutation relation $\delta\hat{\alpha} = \hat{\alpha}^{-1}\delta$. Then we have

$$\begin{aligned} \hat{\alpha}^m\delta\hat{\alpha}^n = \hat{\alpha}^r\delta\hat{\alpha}^s \quad &\iff \quad \hat{\alpha}^{m-r}\delta\hat{\alpha}^{n-s} = \delta \\ &\iff \quad \hat{\alpha}^{m+s-r-n}\delta = \delta, \text{ using the commutation relation above} \\ &\iff \quad \hat{\alpha}^{m+s-r-n} = \text{identity}\,, \end{aligned}$$

which generically does not happen if $m + s - r - n \neq 0$ (it indeed does not happen in the primordial example given in the introduction).

In fact, we can relax the conditions; by asking not if these involutions are equal but only if they commute. As an example, we have

Proposition 2.2. *Let α, δ as above. Then $\hat{\alpha}\delta$ commutes $\delta\hat{\alpha}$ if and only if $\widehat{(\alpha\delta)}$ is a fourth root of unity.*

Proof. We can consider both the sides of the comutator:

$$(\hat{\alpha}\delta)(\delta\hat{\alpha}) = \hat{\alpha}^2 \tag{2.1}$$

$$(\delta\hat{\alpha})(\hat{\alpha}\delta) = \delta\alpha^2\delta. \tag{2.2}$$

So the comutation of these involutions is equivalent to $\delta\hat{\alpha}^2 = \hat{\alpha}^2\delta$. Now using the identity $\alpha(\delta x) = \alpha(x)^{-1}$ onde can deduce:

$$\hat{\alpha}^2 = \alpha(x)^{-2} \cdot \delta x \tag{2.3}$$

$$\delta\alpha^2\delta = \alpha^2(x) \cdot \delta x. \tag{2.4}$$

This implies that $\delta\hat{\alpha}^2 = \hat{\alpha}^2\delta \iff \alpha^4(x) \cdot (\delta x) = \delta x$. Calling $\delta x = y$ and using again that $\alpha(\delta x) = \alpha(x)^{-1}$ we have $\alpha^{-4}(y) \cdot y = y$ what, by definition, means that $\widehat{(\alpha\delta)^4} = \text{identity}$. □

3. Bundles and geometric realizations

Say we have a twisted sphere Σ^n. In order to study its geometric properties, or even computing basic differentiable invariants, one must construct some sort of geometric model of it.

One of the ways of accomplishing this is via non-cancellation phenomena: for example, it is known that for any twisted sphere Σ^7, $\Sigma^7 \times S^3$ is diffeomorphic to the standard product $S^7 \times S^3$ [3, 14]. Then on the one hand, Σ^7 will be the quotient of $\Sigma^7 \times S^3$ by the standard left action of S^3 on itself. On the other hand, we can translate this action via a diffeomorphism $\Sigma^7 \times S^3 \to S^7 \times S^3$ to an action on the standard product, and we are left with the "cross" diagram

$$\begin{tikzcd} & S^3 \arrow[d] & \\ S^3 \arrow[r] & S^7 \times S^3 \arrow[r] \arrow[d] & S^7 \\ & \Sigma^7 & \end{tikzcd}$$

where the horizontal diagram is the canonical projection onto the first coordinate, and the vertical arrows come from the translated action on $\Sigma^7 \times S^3$. This would give a geometric presentation of Σ^7 as a quotient of a free S^3 action on $S^7 \times S^3$. Sadly, no explicit formula for such a non-cancellation diffeomorphism is known.

The natural next step is then to consider principal S^3-bundles over spheres instead of products. Total spaces of such bundles can be diffeomorphic, giving a "twisted non-cancellation". The first and most well-known example is the Gromoll-Meyer presentation of the Milnor sphere $\Sigma^7_{2,-1}$ as a quotient of the Lie group $Sp(2)$, giving the non-trivial cross diagram

$$\begin{tikzcd} & S^3 \arrow[d] & \\ S^3 \arrow[r] & Sp(2) \arrow[r] \arrow[d] & S^7 \\ & \Sigma^7 & \end{tikzcd}$$

Here the horizontal projection is simply the projection to the first column of the given matrix in $Sp(2)$. The geometry of this cross diagram has been extensively analyzed [1, 4, 5, 6, 7]; and has also been generalized to include all exotic 7-spheres [8].

We shall presently see that all twisted spheres coming from the equivariant reentrance construction for $G = SU(2)$ allow such twisted non-cancellation: let $\hat{\alpha}$ be an equivariant reentrance corresponding to the equivariant map $\alpha : S^n \to SU(2)$. Let D^{n+1} be the disk of radius π with polar coordinates $(t, x) \in [0, \pi] \times S^n$ and the functions $f, g : (D^{n+1} - \{0\}) \times S^3 \to (D^{n+1} - \{0\}) \times S^3$ defined by

$$f(x, t, q) = (\delta\hat{\alpha}(x), \pi - t, q(\alpha(x))^{-1}); \quad g(x, t, q) = (\delta(x), \pi - t, q\alpha(x)),$$

and the spaces $\Sigma p = D^{n+1} \times S^3 \cup_f D^{n+1} \times S^3$ and $\mathbb{S}p = D^{n+1} \times S^3 \cup_g D^{n+1} \times S^3$.

Then we have

Theorem 3.1. *The total spaces Σp and $\mathbb{S}p$ are diffeomorphic.*

Proof. Consider the map $F : D^{n+1} \times S^3 \to D^{n+1} \times S^3$ defined by $(t, x, q) \mapsto (t, q \cdot x, \bar{q})$. It is straightforward to see that $fF = Fg$ and $gF = Ff$ so it extends to well-defined function $F_\Sigma : \Sigma p \to \mathbb{S}p$ and $F_\mathbb{S} : \mathbb{S}p \to \Sigma p$ wich are inverse to each other since F is an involution (compare §6 of [1]).

□

Then for all these examples, and in particular the bootstrapping examples of section 2, we have the corresponding cross diagram

$$\begin{array}{ccccc} & & S^3 & & \\ & & \downarrow & & \\ S^3 & \longrightarrow & Sp & \longrightarrow & S^{n+1} \\ & & \downarrow & & \\ & & \Sigma^{n+1}_{\hat{\alpha}} & & \end{array}$$

which can be used to compute differentiable invariants of $\Sigma_{\hat{\alpha}}$. However, in order to take geometric advantage of this result, the geometry of principal S^3-bundles over *Euclidean* spheres should be thoroughly understood; the example to follow being $Sp(2)$ as the generator of S^3-bundles over the 7-sphere.

4. Smoothness and analiticity

As the main concern of these procedures is explicit realization of differentiable exotic phenomena it is essential to study the differentiability of our constructions. We have the following:

Proposition 4.1. *Let $\alpha : S^n \to SU(2)$ be as above and with continuous first derivative, then $\alpha^*, {}^h\alpha$ and α^h are of class C^1.*

Proof. Since the Hopf map is analytic it will be enough to prove the result for α^*. We start by considering the map α^* with a direct identification of the join with S^{n+3}, and of $SU(2)$ as the unit quaternions:

$$\begin{array}{rcl} \alpha^* : S^{n+3} & \to & S^3 \\ \begin{pmatrix} \xi \\ w \end{pmatrix} & \mapsto & \alpha\left(\dfrac{w}{|w|}\right) e^{\pi\xi} \alpha\left(\dfrac{w}{|w|}\right)^{-1}, \end{array}$$

where ξ is a pure quaternion, $w \in \mathbb{R}^{n+1}$ and $|\xi|^2 + |w|^2 = 1$. It is clear that the critical set is when $w = 0$. Spelling out the exponential we have

$$\alpha^*(\xi, w) = \cos(\pi|\xi|) + \sin(\pi|\xi|)\alpha\left(\frac{w}{|w|}\right)\frac{\xi}{|\xi|}\alpha\left(\frac{w}{|w|}\right)^{-1}.$$

It follows that the real part is also well behaved at a neighborhood of this set, so we only need to deal with the imaginary part. Now consider $x = (u, v) \in \mathbb{R}^3 \times \mathbb{R}^n$ and the stereographic projection to S^{n+3} given by:

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \xi(u,v) \\ w(u,v) \end{pmatrix} = \begin{pmatrix} \frac{2u}{1+||x||^2} \\ \frac{||x||^2+2v-1}{1+||x||^2} \end{pmatrix}.$$

Let $\theta : \mathbb{R}^3 \times \mathbb{R}^n - \{0\} \to \mathbb{R}^3 \times \mathbb{R}^n - \{0\}$ be the map which normalizes the vectors, then we have from the imaginary part:

$$\mathrm{Im}(\alpha^*(\xi, w)) = \sin \frac{2\pi|u|}{1+||x||^2}\alpha(\theta(w))\theta(u)\alpha(\theta(w))^{-1}.$$

As the derivative of the sine function is also good when $|w| \neq 0$, we can just worry about the other part. For the existence of the derivative at $|w| = 0$ it is enough to prove, for any given u with $|u| = 1$ and $\xi(t) = (\xi_u(t), \xi_v(t)) \in \mathbb{R}^3 \times \mathbb{R}^n$, a smooth curve with $\xi(0) = 0$, that

$$\lim_{t\to 0} \frac{1}{|\xi(t)|}\left|\sin \frac{2\pi|u+\xi_u(t)|}{1+|x+\xi(t)|^2}\alpha(\theta(w(x+\xi(t))))\theta(u+\xi_u(t))\alpha(\theta(w(x+\xi(t))))^{-1}\right| = 0,$$

since in fact we are proving that the derivative is zero at those points.

As $|\alpha(\theta(w)\theta(u)\alpha(\theta(w))^{-1}| = 1$ for any u and w the above limit is equal to

$$\lim_{t\to 0} \frac{1}{|\xi(t)|}\left|\sin \frac{2\pi|u+\xi_u(t)|}{1+|x+\xi(t)|^2}\right| = 0.$$

Now we have

$$\begin{aligned}\frac{\pi}{|\xi|}\left(1 - \frac{2|u+\xi_u|}{1+|x+\xi|^2}\right) &= \frac{\pi}{|\xi|} \cdot \frac{(|u+\xi_u|-1)^2 + |\xi_v|^2}{1+|x+\xi|^2} \leq \\ \leq \frac{\pi}{|\xi|} \cdot \frac{(|u|+|\xi_u|-1)^2 + |\xi_v|^2}{1+|x+\xi|^2} &= \frac{\pi}{1+|x+\xi|^2} \cdot \frac{|\xi|^2}{|\xi|}.\end{aligned}$$

With the first term going to $\pi/2$ and the second to zero. Knowing that $\sin(\xi) = \sin(\pi - \xi)$ a standard Taylor-expansion argument shows that the desired limit vanishes.

Letting now $(u(t), v(t))$ be a smooth curve with $(u(0), v(0)) = (u, v)$ and $w' = Dw_{(u,v)}(u'(0), v'(0))$, we set $w(t) = (u(t), v(t))$ and $w(0) = w$. Taking derivatives we have:

$$\begin{aligned}\frac{d}{dt}Ad_\alpha(\theta(w(t)))\theta(u(t))\Big|_{t=0} &= \left[D\alpha_{\theta(w)} \circ D\theta_w(w')\alpha(\theta(w))^{-1}, Ad_{\alpha(\theta(w))}\theta(u)\right] \\ &\quad + Ad_{\alpha(\theta(w))}(\theta(u(t)))'(0).\end{aligned} \tag{4.1}$$

The second term in the right-hand side of the expression is well behaved at $|w| \to 0$, for the first term we have

$$\sin \pi|\xi|D\theta_w = \frac{\sin \pi(1-|\xi|)}{1-|\xi|}\frac{1}{1+|\xi|}\left\{\delta^i_j|w| - \frac{w_i w_j}{|w|}\right\}_{ij}.$$

As $D\alpha$ has bounded norm as anmap from S^n to the linear operators algebra and $|\alpha(\theta(w))^{-1}| = 1$, the expression above goes to zero when $|w| \to 0$ wich makes clear that 4.1 goes to zero. Now a quick calculation shows that the derivative of $\sin \pi|\xi|$ vanishes as well in the desired set, and therefore α^* is C^1. □

For a restricted class of maps we can proceed in another way. Consider the following composition

$$\begin{array}{ccccc} S^2 * S^n & \to & S^2 * S^3 & \to & S^3 \\ \begin{pmatrix} \xi \\ w \end{pmatrix} & \mapsto & \begin{pmatrix} \xi \\ |w|\alpha\left(\frac{w}{|w|}\right) \end{pmatrix} & \mapsto & b\left(\xi, |w|\alpha\left(\frac{w}{|w|}\right)\right) \end{array}$$

Noting that the last term is exactly α^* we prove the following result:

Proposition 4.2. *If for $w \in D^{n+1}$ the map $w \mapsto |w|\alpha(w|w|^{-1})$ is analytic then α^* is also analytic.*

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