Lecture 1. Matrix-Vector Multiplication

You already know the formula for matrix-vector multiplication. Nevertheless, the purpose of this first lecture is to describe a way of interpreting such products that may be less familiar. If $b = Ax$, then *b is a linear combination of the columns of A.*

Familiar Definitions

Let x be an n-dimensional column vector and let A be an $m \times n$ matrix (m rows, n columns). Then the matrix-vector product $b = Ax$ is an m-dimensional column vector defined as follows:

$$b_i = \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \ldots, m. \tag{1.1}$$

Here b_i denotes the ith entry of b, a_{ij} denotes the i,j entry of A (ith row, jth column), and x_j denotes the jth entry of x. For simplicity, we assume in all but a few lectures of this book that quantities such as these belong to $\mathbb{C}$, the field of complex numbers. The space of m-vectors is $\mathbb{C}^m$, and the space of $m \times n$ matrices is $\mathbb{C}^{m \times n}$.

The map $x \mapsto Ax$ is *linear*, which means that, for any $x, y \in \mathbb{C}^n$ and any $\alpha \in \mathbb{C}$,

$$\begin{aligned} A(x+y) &= Ax + Ay, \\ A(\alpha x) &= \alpha A x. \end{aligned}$$

Conversely, every linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ can be expressed as multiplication by an $m \times n$ matrix.

A Matrix Times a Vector

Let a_j denote the jth column of A, an m-vector. Then (1.1) can be rewritten

$$b = Ax = \sum_{j=1}^{n} x_j a_j. \tag{1.2}$$

This equation can be displayed schematically as follows:

$$\left[\begin{array}{c} \\ \\ b \\ \\ \\ \end{array}\right] = \left[\begin{array}{c|c|c|c} & & & \\ & & & \\ a_1 & a_2 & \cdots & a_n \\ & & & \\ & & & \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \left[\begin{array}{c} \\ \\ a_1 \\ \\ \\ \end{array}\right] + x_2 \left[\begin{array}{c} \\ \\ a_2 \\ \\ \\ \end{array}\right] + \cdots + x_n \left[\begin{array}{c} \\ \\ a_n \\ \\ \\ \end{array}\right].$$

In (1.2), b is expressed as a linear combination of the columns a_j. Nothing but a slight change of notation has occurred in going from (1.1) to (1.2). Yet thinking of Ax in terms of the form (1.2) is essential for a proper understanding of the algorithms of numerical linear algebra.

One way to summarize these different ways of viewing matrix-vector products is like this. As mathematicians, we are used to viewing the formula $Ax = b$ as a statement that A acts on x to produce b. The formula (1.2), by contrast, suggests the interpretation that x acts on A to produce b.

Example 1.1. Fix a sequence of numbers $\{x_1, \ldots, x_m\}$. If p and q are polynomials of degree $< n$ and α is a scalar, then $p+q$ and αp are also polynomials of degree $< n$. Moreover, the values of these polynomials at the points x_i satisfy the following linearity properties:

$$\begin{aligned} (p+q)(x_i) &= p(x_i) + q(x_i), \\ (\alpha p)(x_i) &= \alpha(p(x_i)). \end{aligned}$$

Thus the map from vectors of coefficients of polynomials p of degree $< n$ to vectors $(p(x_1), p(x_2), \ldots, p(x_m))$ of sampled polynomial values is linear. Any linear map can be expressed as multiplication by a matrix; this is an example. In fact, it is expressed by an $m \times n$ *Vandermonde matrix*

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix}.$$

If c is the column vector of coefficients of p,

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}, \qquad p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1},$$

then the product Ac gives the sampled polynomial values. That is, for each i from 1 to m, we have

$$(Ac)_i \;=\; c_0 + c_1 x_i + c_2 x_i^2 + \cdots + c_{n-1} x_i^{n-1} \;=\; p(x_i). \tag{1.3}$$

In this example, it is clear that the matrix-vector product Ac need not be thought of as m distinct scalar summations, each giving a different linear combination of the entries of c, as (1.1) might suggest. Instead, A can be viewed as a matrix of columns, each giving sampled values of a monomial,

$$A = \left[\begin{array}{c|c|c|c|c} & & & & \\ 1 & x & x^2 & \cdots & x^{n-1} \\ & & & & \end{array}\right], \tag{1.4}$$

and the product Ac should be understood as a single vector summation in the form of (1.2) that at once gives a linear combination of these monomials,

$$Ac \;=\; c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} \;=\; p(x).$$

□

The remainder of this lecture will review some fundamental concepts in linear algebra from the point of view of (1.2).

A Matrix Times a Matrix

For the matrix-matrix product $B = AC$, *each column of B is a linear combination of the columns of A*. To derive this fact, we begin with the usual formula for matrix products. If A is $\ell \times m$ and C is $m \times n$, then B is $\ell \times n$, with entries defined by

$$b_{ij} = \sum_{k=1}^{m} a_{ik} c_{kj}. \tag{1.5}$$

Here b_{ij}, a_{ik}, and c_{kj} are entries of B, A, and C, respectively. Written in terms of columns, the product is

$$\left[\begin{array}{c|c|c|c} & & & \\ b_1 & b_2 & \cdots & b_n \\ & & & \end{array}\right] = \left[\begin{array}{c|c|c|c} & & & \\ a_1 & a_2 & \cdots & a_m \\ & & & \end{array}\right] \left[\begin{array}{c|c|c|c} c_1 & c_2 & \cdots & c_n \\ & & & \end{array}\right],$$

and (1.5) becomes

$$b_j = Ac_j = \sum_{k=1}^{m} c_{kj} a_k. \tag{1.6}$$

Thus b_j is a linear combination of the columns a_k with coefficients c_{kj}.

Example 1.2. A simple example of a matrix-matrix product is the *outer product*. This is the product of an m-dimensional column vector u with an n-dimensional row vector v; the result is an $m \times n$ matrix of rank 1. The outer product can be written

$$\left[\begin{array}{c} \\ u \\ \\ \end{array}\right] \left[\begin{array}{cccc} v_1 & v_2 & \cdots & v_n \end{array}\right] = \left[\begin{array}{c|c|c|c} & & & \\ v_1 u & v_2 u & \cdots & v_n u \\ & & & \end{array}\right] = \left[\begin{array}{ccc} v_1 u_1 & \cdots & v_n u_1 \\ \vdots & & \vdots \\ v_1 u_m & \cdots & v_n u_m \end{array}\right].$$

The columns are all multiples of the same vector u, and similarly, the rows are all multiples of the same vector v. □

Example 1.3. As a second illustration, consider $B = AR$, where R is the upper-triangular $n \times n$ matrix with entries $r_{ij} = 1$ for $i \le j$ and $r_{ij} = 0$ for $i > j$. This product can be written

$$\left[\begin{array}{c|c|c} & & \\ b_1 & \cdots & b_n \\ & & \end{array}\right] = \left[\begin{array}{c|c|c} & & \\ a_1 & \cdots & a_n \\ & & \end{array}\right] \left[\begin{array}{ccc} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{array}\right].$$

The column formula (1.6) now gives

$$b_j \;=\; Ar_j \;=\; \sum_{k=1}^{j} a_k. \tag{1.7}$$

That is, the jth column of B is the sum of the first j columns of A. The matrix R is a discrete analogue of an indefinite integral operator. □

Range and Nullspace

The *range* of a matrix A, written range(A), is the set of vectors that can be expressed as Ax for some x. The formula (1.2) leads naturally to the following characterization of range(A):

Theorem 1.1. range(A) *is the space spanned by the columns of* A.

Proof. By (1.2), any Ax is a linear combination of the columns of A. Conversely, any vector y in the space spanned by the columns of A can be written as a linear combination of the columns, $y = \sum_{j=1}^{n} x_j a_j$. Forming a vector x out of the coefficients x_j, we have $y = Ax$, and thus y is in the range of A. $\square$

In view of Theorem 1.1, the range of a matrix A is also called the *column space* of A.

The *nullspace* of $A \in \mathbb{C}^{m\times n}$, written $\operatorname{null}(A)$, is the set of vectors x that satisfy $Ax = 0$, where 0 is the 0-vector in $\mathbb{C}^m$. The entries of each vector $x \in \operatorname{null}(A)$ give the coefficients of an expansion of zero as a linear combination of columns of A: $0 = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$.

Rank

The *column rank* of a matrix is the dimension of its column space. Similarly, the *row rank* of a matrix is the dimension of the space spanned by its rows. Row rank always equals column rank (among other proofs, this is a corollary of the singular value decomposition, discussed in Lectures 4 and 5), so we refer to this number simply as the *rank* of a matrix.

An $m\times n$ matrix of *full rank* is one that has the maximal possible rank (the lesser of m and n). This means that a matrix of full rank with $m \geq n$ must have n linearly independent columns. Such a matrix can also be characterized by the property that the map it defines is one-to-one:

Theorem 1.2. *A matrix $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ has full rank if and only if it maps no two distinct vectors to the same vector.*

Proof. ($\Longrightarrow$) If A is of full rank, its columns are linearly independent, so they form a basis for $\operatorname{range}(A)$. This means that every $b \in \operatorname{range}(A)$ has a unique linear expansion in terms of the columns of A, and therefore, by (1.2), every $b \in \operatorname{range}(A)$ has a unique x such that $b = Ax$. ($\Longleftarrow$) Conversely, if A is not of full rank, its columns a_j are dependent, and there is a nontrivial linear combination such that $\sum_{j=1}^{n} c_j a_j = 0$. The nonzero vector c formed from the coefficients c_j satisfies $Ac = 0$. But then A maps distinct vectors to the same vector since, for any x, $Ax = A(x + c)$. $\square$

Inverse

A *nonsingular* or *invertible* matrix is a square matrix of full rank. Note that the m columns of a nonsingular $m \times m$ matrix A form a basis for the whole space $\mathbb{C}^m$. Therefore, we can uniquely express any vector as a linear combination of them. In particular, the canonical unit vector with 1 in the jth entry and zeros elsewhere, written e_j, can be expanded:

$$e_j = \sum_{i=1}^{m} z_{ij} a_i. \tag{1.8}$$

Let Z be the matrix with entries z_{ij}, and let z_j denote the jth column of Z. Then (1.8) can be written $e_j = Az_j$. This equation has the form (1.6); it can be written again, most concisely, as

$$\left[\begin{array}{c|c|c} \\ e_1 & \cdots & e_m \\ \\ \end{array}\right] = I = AZ.$$

The matrix Z is the *inverse* of A. Any square nonsingular matrix A has a unique inverse, written A^{-1}, that satisfies $AA^{-1} = A^{-1}A = I$.

The following theorem records a number of equivalent statements that hold when a square matrix is nonsingular. These conditions appear in linear algebra texts, and we shall not give a proof here. Concerning *(f)*, see Lecture 5.

Theorem 1.3. *For $A \in \mathbb{C}^{m\times m}$, the following conditions are equivalent:*
(a) A has an inverse A^{-1},
(b) $\operatorname{rank}(A) = m$,
(c) $\operatorname{range}(A) = \mathbb{C}^m$,
(d) $\operatorname{null}(A) = \{0\}$,
(e) 0 is not an eigenvalue of A,
(f) 0 is not a singular value of A,
(g) $\det(A) \neq 0$.

Concerning *(g)*, we mention that the determinant, though a convenient notion theoretically, rarely finds a useful role in numerical algorithms.

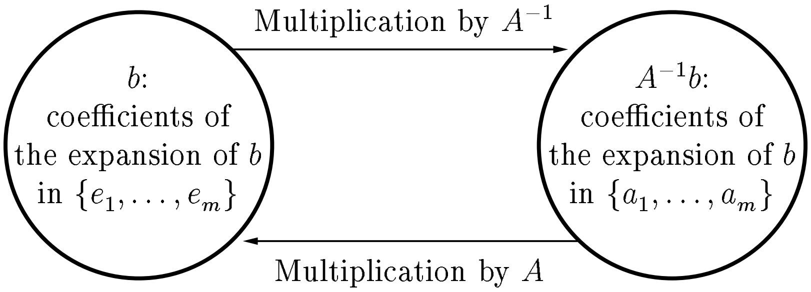
A Matrix Inverse Times a Vector

When writing the product $x = A^{-1}b$, it is important not to let the inverse-matrix notation obscure what is really going on! Rather than thinking of x as the result of applying A^{-1} to b, we should understand it as the unique vector that satisfies the equation $Ax = b$. By (1.2), this means that x is the vector of coefficients of the unique linear expansion of b in the basis of columns of A.

This point cannot be emphasized too much, so we repeat:

$A^{-1}b$ is the vector of coefficients of the expansion of b in the basis of columns of A.

Multiplication by A^{-1} is a *change of basis* operation:



In this description we are being casual with terminology, using "b" in one instance to denote an m-tuple of numbers and in another as a point in an abstract vector space. The reader should think about these matters until he or she is comfortable with the distinction.

A Note on m and n

Throughout numerical linear algebra, it is customary to take a rectangular matrix to have dimensions $m \times n$. We follow this convention in this book.

What if the matrix is square? The usual convention is to give it dimensions $n \times n$, but in this book we shall generally take the other choice, $m \times m$. Many of our algorithms require us to look at rectangular submatrices formed by taking a subset of the columns of a square matrix. If the submatrix is to be $m \times n$, the original matrix had better be $m \times m$.

Exercises

1. Let B be a 4×4 matrix to which we apply the following operations:
   1. double column 1,
   2. halve row 3,
   3. add row 3 to row 1,
   4. interchange columns 1 and 4,
   5. subtract row 2 from each of the other rows,
   6. replace column 4 by column 3,
   7. delete column 1 (so that the column dimension is reduced by 1).

   (a) Write the result as a product of eight matrices.

   (b) Write it again as a product ABC (same B) of three matrices.

2. Suppose masses m_1, m_2, m_3, m_4 are located at positions x_1, x_2, x_3, x_4 in a line and connected by springs with spring constants k_{12}, k_{23}, k_{34} whose natural

lengths of extension are $\ell_{12}, \ell_{23}, \ell_{34}$. Let f_1, f_2, f_3, f_4 denote the rightward forces on the masses, e.g., $f_1 = k_{12}(x_2 - x_1 - \ell_{12})$.

(a) Write the 4×4 matrix equation relating the column vectors f and x. Let K denote the matrix in this equation.

(b) What are the dimensions of the entries of K in the physics sense (e.g., mass times time, distance divided by mass, etc.)?

(c) What are the dimensions of $\det(K)$, again in the physics sense?

(d) Suppose K is given numerical values based on the units meters, kilograms, and seconds. Now the system is rewritten with a matrix K' based on centimeters, grams, and seconds. What is the relationship of K' to K? What is the relationship of $\det(K')$ to $\det(K)$?

3. Generalizing Example 1.3, we say that a square or rectangular matrix R with entries r_{ij} is *upper-triangular* if $r_{ij} = 0$ for $i > j$. By considering what space is spanned by the first n columns of R and using (1.8), show that if R is a nonsingular $m \times m$ upper-triangular matrix, then R^{-1} is also upper-triangular. (The analogous result also holds for lower-triangular matrices.)

4. Let $f_1, \ldots, f_8$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1, \ldots, d_8$, there exists a set of coefficients $c_1, \ldots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \qquad i = 1, \ldots, 8.$$

(a) Show by appealing to the theorems of this lecture that $d_1, \ldots, d_8$ determine $c_1, \ldots, c_8$ uniquely.

(b) Let A be the 8×8 matrix representing the linear mapping from data $d_1, \ldots, d_8$ to coefficients $c_1, \ldots, c_8$. What is the i, j entry of A^{-1}?