Dispersive perturbations of Burgers and hyperbolic equations

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Our motivation is to study the influence of dispersion on the space of resolution, on the lifespan and on the dynamics of solutions to the Cauchy problem for “weak” dispersive perturbations of hyperbolic quasilinear equations or systems, as for instance the Boussinesq systems for surface water waves.

Outline

- Motivation
- Main Result
- Ingredients
- Further Comments

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Motivation

In this talk we will focus on the model equation (the so-called Whitham equation):

$$u_t + uu_x + \int_{-\infty}^{\infty} k(x-y)u_x(y,t)dy = 0. \qquad (1)$$

This equation can also be written on the form

$$u_t + uu_x - Lu_x = 0, \qquad (2)$$

where the Fourier multiplier operator L is defined by

$$\widehat{Lf}(\xi) = p(\xi)\hat{f}(\xi),$$

where $p = \hat{k}$.

In the original Whitham equation, the kernel k was given by

$$k(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\tanh \xi}{\xi} \right)^{1/2} e^{ix\xi} d\xi, \tag{3}$$

that is $p(\xi) = \left(\frac{\tanh \xi}{\xi} \right)^{1/2}$.

The dispersion is in this case that of the finite depth surface water waves without surface tension.

The general idea is to investigate the "fight" between nonlinearity and dispersion. Usually we attack this problem by fixing the dispersion (*eg* that of the KdV equation) and varying the nonlinearity (say $u^p u_x$ in the context of generalized KdV).

Our viewpoint is to fix the quadratic, nonlinearity (*eg* uu_x) and to vary (lower) the dispersion. In fact in many problems arising from Physics or Continuum Mechanics the nonlinearity is quadratic, with terms like $(u \cdot \nabla)u$ and the dispersion is in some sense weak. In particular the dispersion is not strong enough for yielding the dispersive estimates that allows to solve the Cauchy problem in relatively large functional classes (like the KdV or Benjamin-Ono equation in particular), down to the energy level for instance.

Two basic issues can be addressed

1. Which amount of dispersion prevents the hyperbolic (*ie* by shock formation) blow-up of the underlying hyperbolic quasilinear equation or system. This question has been apparently raised for the first time by Whitham for the Whitham equation (1).

A typical result suggests that for *not too dispersive Whitham type equations* that is for instance when

$$p(\xi) = |\xi|^\alpha, \quad -1 < \alpha \leq 0,$$

(1) still presents a blow-up of Burgers type. This has been proved for Whitham type equations, with a regular kernel k satisfying $k \in C(\mathbb{R}) \cap L^1(\mathbb{R})$, symmetric and monotonically decreasing on $\mathbb{R}_+$, by Naumkin and Shishmarev and by Constantin and Escher. The blow-up is obtained for initial data which are sufficiently asymmetric.

More precisely :

Theorem 1 (Constantin and Escher)**.** *Let* $u_0 \in H^\infty(\mathbb{R})$ *be such that*

$$\inf_{x\in\mathbb{R}} |u_0'(x)| + \sup_{x\in\mathbb{R}} |u_0'(x)| \leq -2k(0).$$

Then the corresponding solution of (1) *undergoes a wave breaking phenomena, that is there exists* $T = T(u_0) > 0$ *with*

$$\sup_{(x,t)\in[0,T)\times\mathbb{R}} |u(x,t)| < \infty, \textit{ while } \sup_{x\in\mathbb{R}} |u_x(x,t)| \to \infty \textit{ as } t \to T.$$

The previous result does not include the case of the Whitham equation (1) with kernel given by (3) since then $k(0) = \infty$. However the method of proof adapts to more general kernels. This has been proven recently by Castro, Cordoba and Gancedo for the equation

$$u_t + uu_x + D^\beta \mathcal{H} u = 0, \tag{4}$$

where $\mathcal{H}$ is the Hilbert transform and D^β is defined via Fourier transform by

$$\widehat{D^\beta f}(\xi) = |\xi|^\beta \hat{f}(\xi), \tag{5}$$

for any $\beta \in \mathbb{R}$.

They established that for $0 \leq \beta < 1$, there exist initial data $u_0 \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R}),\ 0 < \delta < 1$, and $T(u_0)$ such that the corresponding solution u of (4) satisfies

$$\lim_{t \to T} \|u(\cdot, t)\|_{C^{1+\delta}(\mathbb{R})} = +\infty.$$

This rules out the case $-1 < \alpha < 0$ in our notation. It would be interesting to extend this result to a non pure power dispersion, for instance (3).

The case $0 < \alpha < 1$ is much more delicate.

2. Investigate the influence of the dispersive term on the theory of the local well-posedness of the Cauchy problem associated to the general "dispersive nonlinear hyperbolic system"

$$\partial_t U + \epsilon\, \mathcal{A}(U, \nabla U) + \epsilon \mathcal{L} U = 0. \tag{6}$$

Recall that, for the underlying hyperbolic system (that is when $\mathcal{L} = 0$ in (6)) assumed to be symmetrizable, the Cauchy problem is locally well-posed for data in the Sobolev space $H^s(\mathbb{R}^n)$ for any $s > \frac{n}{2} + 1$. The question is then to look to which extent the presence of $\mathcal{L}$ can lower the value of s. This issue is well understood for *scalar* equations with a relatively high dispersion, as the KdV, BO equations, much less for equations or systems with a *weak* dispersive part.

Again, we will focus on the scalar equation (1) on its form (2) that is

$$\partial_t u - D^\alpha \partial_x u + u \partial_x u = 0, \tag{7}$$

where $x,\ t \in \mathbb{R}$, ϵ is small positive number and D^α is the Riesz potential of order $-\alpha$ defined in (5). When $\alpha = 1$, respectively $\alpha = 2$, equation (7) corresponds to the well-known Benjamin-Ono and respectively Korteweg -de Vries equations. This equation has been extensively studied for $1 \le \alpha \le 2$. In the following we will consider the less dispersive case $0 < \alpha < 1$.

Remark 1. *The case* $\alpha = \frac{1}{2}$ *is somewhat reminiscent of the linear dispersion of finite depth water waves with surface tension that have phase velocity (in dimension one and two, where* $\hat{\mathbf{k}}$ *is a unit vector) which writes in dimension one or two*

$$\mathbf{c}(\mathbf{k}) = \frac{\omega(\mathbf{k})}{|\mathbf{k}|}\hat{\mathbf{k}} = g^{\frac{1}{2}} \left(\frac{\tanh(|\mathbf{k}|h_0)}{|\mathbf{k}|} \right)^{\frac{1}{2}} \left(1 + \frac{T}{\rho g}|\mathbf{k}|^2 \right)^{\frac{1}{2}} \hat{\mathbf{k}}, \tag{8}$$

In the case $\alpha = 0$, equation (7) becomes the original Burgers equation

$$\partial_t \widetilde{u} = \widetilde{u} \partial_x \widetilde{u}, \tag{9}$$

by performing the natural change of variable $\widetilde{u}(x,t) = u(x - \epsilon t, t)$, while the case $\alpha = -1$ corresponds to the Burgers-Hilbert equation

$$\partial_t u + \mathcal{H} u = u \partial_x u, \tag{10}$$

where $\mathcal{H}$ denotes the Hilbert transform. Equation (10) has been studied by Hunter and Ifrim, Castro, Córdoba and Gancedo.

The following quantities are conserved by the flow associated to (7),

$$M(u) = \int_{\mathbb{R}} u^2(x)dx, \tag{11}$$

and

$$H(u) = \int_{\mathbb{R}} (\frac{1}{2}|D^{\frac{\alpha}{2}}u|^2 - \frac{1}{6}u^3)dx. \tag{12}$$

Moreover, equation (7) is invariant under the scaling transformation

$$u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^{\alpha+1}t),$$

for any positive number λ. A straightforward computation shows that $\|u_\lambda\|_{\dot{H}^s} = \lambda^{s+\alpha-\frac{1}{2}}\|u_\lambda\|_{\dot{H}^s}$, and thus the critical index corresponding to (7) is $s_\alpha = \frac{1}{2} - \alpha$. In particular, equation (7) is L^2-critical for $\alpha = \frac{1}{2}$.

By using standard compactness methods, one can prove that the Cauchy problem associated to (7) is well-posed in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$. Moreover, interpolation arguments or the following Gagliardo-Nirenberg inequality,

$$\|u\|_{L^3} \lesssim \|u\|_{L^2}^{\frac{3\alpha-1}{3\alpha}} \|D^{\frac{\alpha}{2}} u\|_{L^2}^{\frac{1}{3\alpha}}, \quad \alpha \geq \frac{1}{3},$$

combined with the conserved quantities M and H defined in (11) and (12) implies the existence of global weak solution in the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$ as soon as $\alpha > \frac{1}{2}$ and for small data in $H^{\frac{1}{4}}(\mathbb{R})$ when $\alpha = \frac{1}{2}$. More precisely, We recall that we excludes the value $\alpha = 1$ which corresponds to the Benjamin-Ono equation for which much more complete results are known:

Theorem 2. *Let* $\frac{1}{2} < \alpha < 1$ *and* $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R})$*. Then* (7) *possesses a global weak solution in* $L^\infty([0,T]; H^{\frac{\alpha}{2}}(\mathbb{R}))$ *with initial data* u_0*. The same result holds when* $\alpha = \frac{1}{2}$ *provided* $\|u_0\|_{L^2}$ *is small enough.*

Moreover, Ginibre and Velo using that a Kato type local smoothing property holds, showed the global existence of weak L^2 solutions :

Theorem 3. *Let* $\frac{1}{2} < \alpha < 1$ *and* $u_0 \in L^2(\mathbb{R})$*. Then* (7) *possesses a global weak solution in* $L^\infty([0,\infty); L^2(\mathbb{R})) \cap L^2_{loc}(\mathbb{R}; H^{\frac{\alpha}{2}}_{loc}(\mathbb{R}))$ *with initial data* u_0.

However, the case $0 < \alpha < \frac{1}{2}$ is more delicate and the previous results are not known to hold. In particular the Hamiltonian H together with the L^2 norm do not control the $H^{\frac{\alpha}{2}}(\mathbb{R})$ norm anymore. Note that the Hamiltonian does not make sense when $0 < \alpha < \frac{1}{3}$.

Molinet, Tzvetkov and Saut (2001) proved that, for $0 < \alpha < 2$ the Cauchy problem is C^2- ill-posed, that is, the flow map cannot be C^2 for initial data in any Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$, and in particular that the Cauchy problem cannot be solved by a Picard iterative scheme implemented on the Duhamel formulation.

On the one hand, it is well-known that one can still prove local well-posedness (assuming only the continuity of the flow) for equation (7) below $H^{\frac{3}{2}+}(\mathbb{R})$ when $\alpha \geq 1$. Actually, the Benjamin-Ono equation (corresponding to $\alpha = 1$) is well-posed in $L^2(\mathbb{R})$ (Ionescu-Kenig, Molinet-Pilod) as well as equation (7) when $1 < \alpha < 2$ Herr-Ionescu-Kenig-Koch. On the other hand the question to know whether the same occurs in the case $0 < \alpha < 1$ seems to be still open.

Main Result

The space of resolution of the local Cauchy problem enlarges with α.

Theorem 4. *Let $0 < \alpha < 1$. Define $s(\alpha) = \frac{3}{2} - \frac{3\alpha}{8}$ and assume that $s > s(\alpha)$. Then, for every $u_0 \in H^s(\mathbb{R})$, there exist a positive time $T = T(\|u_0\|_{H^s})$ (which can be chosen as a nonincreasing function of its argument), and a unique solution u to (7) satisfying $u(\cdot, 0) = u_0$ such that*

$$u \in C([0,T] : H^s(\mathbb{R})) \quad \text{and} \quad \partial_x u \in L^1([0,T] : L^\infty(\mathbb{R})). \tag{13}$$

Moreover, for any $0 < T' < T$, there exists a neighborhood $\mathcal{U}$ of u_0 in $H^s(\mathbb{R})$ such that the flow map data-solution

$$S^s_{T'} : \mathcal{U} \longrightarrow C([0,T']; H^s(\mathbb{R})), \; u_0 \longmapsto u, \tag{14}$$

is continuous.

Remark 2. *In the case* $\alpha = 1$ *in Theorem 4, we get* $s(1) = \frac{9}{8}$*, which corresponds to Kenig, Koenig's result for BO.*

Remark 3. *Of course, the problem to prove well-posedness in* $H^{\frac{\alpha}{2}}(\mathbb{R})$ *in the case* $\frac{1}{2} \leq \alpha < 1$*, which would imply global well-posedness by using the conserved quantities (11) is still open. This conjecture is supported by the numerical simulations. The use of the techniques in Herr et al might be useful to lower the value of* s*. Observe that the value* $\alpha = 1/2$ *is the* L^2 *critical exponent.*

Remark 4. *Theorem 4 can very likely be extended for non pure power dispersions like*

$$\mathbf{c}(\mathbf{k}) = \frac{\omega(\mathbf{k})}{|\mathbf{k}|}\hat{\mathbf{k}} = g^{\frac{1}{2}} \left(\frac{\tanh(|\mathbf{k}|h_0)}{|\mathbf{k}|}\right)^{\frac{1}{2}} \left(1 + \frac{T}{\rho g}|\mathbf{k}|^2\right)^{\frac{1}{2}} \hat{\mathbf{k}}. \tag{15}$$

Remark 5. *One could wonder about the existence of global solutions with small initial data. This was solved by Sidi, Sulem and Sulem when $\alpha \geq 1$ but the case $\alpha < 1$ seems to be open.*

Main Ingredients

Since we cannot prove Theorem 4 by a contraction method as explained above, we use a compactness argument. Standard energy estimates, the Kato-Ponce commutator estimate:

Let $s > 0$, $p,\ p_2,\ p_3 \in (1,\infty)$ and $p_1,\ p_4 \in (1,\infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Then,

$$\|[J^s, f]g\|_{L^p} \lesssim \|\partial_x f\|_{L^{p_1}} \|J^{s-1}g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \tag{16}$$

and Gronwall's inequality provide the following bound for smooth solutions

$$\|u\|_{L^\infty_T H^s_x} \leq c\|u_0\|_{H^s_x} \exp\Big(c\int_0^T \|\partial_x u\|_{L^\infty_x} dt\Big).$$

Therefore, it is enough to control $\|\partial_x u\|_{L^1_T L^\infty_x}$ at the H^s-level to obtain our *a priori* estimates.

Linear estimates

Next we consider the linear IVP associated to (7)

$$\begin{cases} \partial_t u - D^\alpha \partial_x u = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{17}$$

whose solution is given by the unitary group $e^{tD^\alpha \partial_x}$, defined by

$$e^{tD^\alpha \partial_x} u_0 = \mathcal{F}^{-1}\left(e^{it|\xi|^\alpha \xi} \mathcal{F}(u_0)\right) . \tag{18}$$

We will study the properties of $e^{tD^\alpha \partial_x}$ in the case where $0 < \alpha < 1$.

Strichartz estimates

The following estimate is obtained as an application of a general result proved by Kenig, Ponce and Vega '91

Proposition 1. *Assume that* $0 < \alpha < 1$*. Let* q *and* r *satisfy* $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ *with* $2 \le q, r \le +\infty$*. Then*

$$\|e^{tD^\alpha \partial_x} D^{\frac{\alpha-1}{q}} u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2} \ , \tag{19}$$

for all $u_0 \in L^2(\mathbb{R})$*.*

Remark 6. *In particular, if we choose* $(q, r) = (4, \infty)$*, then we obtain from* (19) *a Strichartz estimate with a lost of* $(1-\alpha)/4$ *derivatives*

$$\|e^{tD^\alpha \partial_x} u_0\|_{L_t^4 L_x^\infty} \lesssim \|D^{\frac{1-\alpha}{4}} u_0\|_{L^2} \, .$$

Next, we derive a refined Strichartz estimate for solutions of the nonhomogeneous linear equation

$$\partial_t u - D^\alpha \partial_x u = F \; . \tag{20}$$

This estimate generalizes the one derived by Kenig and Koenig in the Benjamin-Ono case $\alpha = 1$.

Proposition 2. *Assume that $0 < \alpha < 1$, $T > 0$ and $\delta \geq 0$. Let u be a smooth solution to (20) defined on the time interval $[0, T]$. Then, there exist $0 < \kappa_1, \; \kappa_2 < \frac{1}{2}$ such that*

$$\|\partial_x u\|_{L^2_T L^\infty_x} \lesssim T^{\kappa_1} \|J^{1+\frac{\delta}{4}+\frac{1-\alpha}{4}+\theta} u\|_{L^\infty_T L^2_x} + T^{\kappa_2} \|J^{1-\frac{3\delta}{4}+\frac{1-\alpha}{4}+\theta} F\|_{L^2_{T,x}} \; , \tag{21}$$

for any $\theta > 0$.

Remark 7. *In our analysis, the optimal choice in estimate* (21) *corresponds to* $\delta = 1 - \frac{\alpha}{2}$. *Indeed, if we denote* $a = 1 + \frac{\delta}{4} + \frac{1-\alpha}{4} + \theta$ *and* $b = 1 - \frac{3\delta}{4} + \frac{1-\alpha}{4} + \theta$, *we should adapt* δ *to get* $a = b + 1 - \frac{\alpha}{2}$, *since we need to absorb* 1 *derivative appearing in the nonlinear part of* (7) *and we are able to recover* $\frac{\alpha}{2}$ *derivatives by using the smoothing effect associated with solutions of* (20). *The use of* $\delta = 1 - \frac{\alpha}{2}$ *in estimate* (21) *provides the optimal regularity* $s > s(\alpha) = \frac{3}{2} - \frac{3\alpha}{8}$ *in Theorem* 4.

Maximal Function estimates

We also use a maximal function estimate for $e^{tD^\alpha\partial_x}$ in the case $0 < \alpha < 1$, which follows directly from the arguments of Kenig, Ponce and Vega (91). We get the following maximal function estimate in L^2 for the group $e^{tD^\alpha\partial_x}$.

Proposition 3. *Assume that $0 < \alpha < 1$. Let $s > \frac{1}{2}$. Then, we have that*

$$\|e^{tD^\alpha\partial_x}u_0\|_{L^2_x L^\infty_{[-1,1]}} \leq \Big(\sum_{j=-\infty}^{+\infty} \sup_{|t|\leq 1} \sup_{j\leq x<j+1} |e^{tD^\alpha\partial_x}u_0(x)|^2 \Big)^{\frac{1}{2}} \lesssim \|u_0\|_{H^s}\ , \quad (22)$$

for any $u_0 \in H^s(\mathbb{R})$.

Corollary 1. *Assume that* $0 < \alpha < 1$. *Let* $s > \frac{1}{2}$, $\beta > \frac{1}{2}$ *and* $T > 0$. *Then, we have that*

$$\Big(\sum_{j=-\infty}^{+\infty} \sup_{|t| \leq T} \sup_{j \leq x < j+1} |e^{tD^{\alpha}\partial_x} u_0(x)|^2 \Big)^{\frac{1}{2}} \lesssim (1+T)^{\beta} \|u_0\|_{H^s} \,, \tag{23}$$

for any $u_0 \in H^s(\mathbb{R})$.

To complete our argument, we need a local smoothing effect for the solutions of the nonlinear equation (7), which is based on series expansions and remainder estimates for commutator of the type $[D^\alpha \partial_x, u]$ derived by Ginibre and Velo ('89).
We see that the solutions of the linear equation (17) recover $\alpha/2$ spatial derivatives locally in space (Kenig, Ponce, Vega '91).

Proposition 4. *Assume that* $0 < \alpha < 1$*. Then, we have that*

$$\|D^{\frac{\alpha}{2}} e^{tD^\alpha \partial_x} u_0\|_{L^\infty_x L^2_T} \lesssim \|u_0\|_{L^2}, \tag{24}$$

for any $u_0 \in L^2(\mathbb{R})$*.*

However in our analysis, we will need a nonlinear version of Proposition 4, whose proof uses the original ideas of Kato '83.

Proposition 5. *Let* χ *denote a nondecreasing smooth function such that supp* $\chi' \subset (-1,2)$ *and* $\chi_{|_{[0,1]}} = 1$. *For* $j \in \mathbb{Z}$, *we define* $\chi_j(\cdot) = \chi(\cdot - j)$. *Let* $u \in C([0,T] : H^\infty(\mathbb{R}))$ *be a smooth solution of* (7) *satisfying* $u(\cdot,0) = u_0$ *with* $0 < \alpha < 1$. *Assume also that* $s \geq 0$ *and* $l > \frac{1}{2}$. *Then,*

$$\begin{aligned} \Big(\int_0^T \int_{\mathbb{R}} \big(|D^{s+\frac{\alpha}{2}} u(x,t)|^2 + |D^{s+\frac{\alpha}{2}} \mathcal{H} u(x,t)|^2\big) \chi_j'(x) dx dt \Big)^{\frac{1}{2}} \\ \lesssim \big(1 + T + \|\partial_x u\|_{L^1_T L^\infty_x} + T \|u\|_{L^\infty_T H^l_x}\big)^{\frac{1}{2}} \|u\|_{L^\infty_T H^s_x}. \end{aligned} \tag{25}$$

The proof of Proposition 5 is based on the following identity.

Lemma 1. *Assume* $0 < \alpha < 1$. *Let* $h \in C^\infty(\mathbb{R})$ *with* h' *having compact support. Then,*

$$\int_{\mathbb{R}} f(D^\alpha \partial_x f) h\, dx = \frac{\alpha+1}{2} \int_{\mathbb{R}} \left(|D^{\frac{\alpha}{2}} f|^2 + |D^{\frac{\alpha}{2}} \mathcal{H} f|^2\right) h' dx + \int_{\mathbb{R}} f R_\alpha(h) f,$$

where $\|R_\alpha(h) f\|_{L^2} \le c_\alpha \|\mathcal{F}(D^\alpha h')\|_{L^1} \|f\|_{L^2}$, *for any* $f \in L^2(\mathbb{R})$.

A priori estimates for smooth solutions

Theorem 5. *Let $0 < \alpha < 1$ and $s > \frac{3}{2}$. For any $u_0 \in H^s(\mathbb{R})$, there exist a positive time $T = T(\|u_0\|_{H^s})$ and a unique solution to (7) $u \in C([0,T] : H^s(\mathbb{R}))$ satisfying $u(\cdot,0) = u_0$.*
Moreover, the map:

$$u_0 \in H^s(\mathbb{R}) \mapsto u \in C([0,T] : H^s(\mathbb{R}))$$

is continuous.

Proposition 6. *Assume* $0 < \alpha < 1$ *and* $s > \frac{3}{2} - \frac{3\alpha}{8}$. *For any* $M > 0$, *there exists a positive time* $\widetilde{T} = \widetilde{T}(M)$ *such that for any initial data* $u_0 \in H^\infty(\mathbb{R})$ *satisfying* $\|u_0\|_{H^s} \leq M$, *the solution* u *obtained in Theorem 5 is defined on* $[0, \widetilde{T}]$ *and satisfies*

$$\Lambda_T^s(u) \leq C_s(\widetilde{T})\|u_0\|_{H^s}, \tag{26}$$

for all $T \in (0, \widetilde{T}]$, *where*

$$\Lambda_T^s(u) := \max\left\{\|u\|_{L_T^\infty H_x^s}, \|\partial_x u\|_{L_T^2 L_x^\infty}, (1+T)^{-\rho}\Big(\sum_{j=-\infty}^{+\infty} \|u\|^2_{L^\infty([j,j+1)\times[0,T])}\Big)^{\frac{1}{2}}\right\},$$

$\rho > \frac{1}{2}$ *and* $C_s(\widetilde{T})$ *is a positive constant depending only on* s *and* $\widetilde{T}$.

All those estimates allow us to obtain the desired *a priori* bound for $\|\partial_x u\|_{L^1_T L^\infty_x}$ at the H^s-level, when $s > s(\alpha) = \frac{3}{2} - \frac{3\alpha}{8}$, *via* a recursive argument. Finally, we conclude the proof of Theorem 4, by applying the same method to the differences of two solutions of (7) and by using the Bona-Smith argument.

Further Comments

- Solitary wave solutions (Frank and Lenzmann)

$$D^\alpha Q_c + c\,Q_c - \frac{1}{2}Q_c^2 = 0. \tag{27}$$

Theorem 6. *Assume that* $0 < \alpha < \frac{1}{3}$*. Then* (27) *does not possesses any nontrivial solution* Q_c *in the class* $H^{\frac{\alpha}{2}}(\mathbb{R}) \cap L^3(\mathbb{R})$*.*

- Long time behavior

  Germain-Masmudi-Shatah

- “Nonlinear-Dispersive” blow-up.

  This blow-up phenomenum, due to the competition between nonlinearity and dispersion is expected to occur for L^2 *critical or supercritical* equations such as the generalized Korteweg-de Vries equation (GKdV)

$$\partial_t u + u^p \partial_x u + \partial_x^3 u = 0, \tag{28}$$

  when $p \geq 4$. The only known result for GKdV is that of the critical case $p = 4$. The supercritical case $p > 4$ is still open but the numerical simulations suggest that blow-up occurs in this case too.

Recently, Kenig, Martel and Robbiano proved that the same type of blow-up occurs for the critical equation

$$\partial_t u - D^\alpha \partial_x u + |u|^{2\alpha} \partial_x u = 0 \tag{29}$$

when α is closed to 2, *i.e.* near the GKdV equation with critical nonlinearity. Recall that for the dispersive Burgers equation (7), the critical case corresponds to $\alpha = \frac{1}{2}$ (or $\alpha = \frac{1}{2}$ for equation (29)).

Things are a bit different for the dispersive Burgers equation (7) equation since in addition to the L^2 critical exponent $\alpha = 1/2$, one has the *energy critical* exponent $\alpha = 1/3$ which has no equivalent for the generalized KdV equations. As this stage one could conjecture that the Cauchy problem for the dispersive Burgers equation (7) is globally well-posed (in a suitable functional setting) when $\alpha > \frac{1}{2}$, that a blow-up similar to the critical GKdV case, occurs when $\alpha = \frac{1}{2}$, that a *supercritical* blow-up occurs when $\frac{1}{3} \leq \alpha < \frac{1}{2}$, and that a blow-up of a totally nature occurs in the *energy supercritical* case, that is when $0 < \alpha < \frac{1}{3}$.This is supported by numerical simulations but should be difficult to prove.

- BBM version of the dispersive Burgers equation,

$$\partial_t u + \partial_x u + u\partial_x u + D^\alpha \partial_t u = 0. \tag{30}$$

Theorem 7. *Let* $0 < \alpha < 1$. *Then the Cauchy problem for* (30) *is locally well-posed for initial data in* $H^r(\mathbb{R}),\ r > r_\alpha = \frac{3}{2} - \alpha.$