Finite-time blowup and global existence for the complex Ginzburg-Landau equation

Joint work with Flávio Dickstein (UFRJ) and Fred Weissler (Paris 13)

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Introduction

Original motivation: Finite-time blowup for complex Ginzburg-Landau eq.

$$e^{-i\theta} u_t = \Delta u + |u|^\alpha u, \qquad (1)$$

on $\mathbb{R}^N$, where $\alpha > 0$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$\theta = 0$: the nonlinear heat equation $u_t - \Delta u = |u|^\alpha u$.

$\theta = \pm\pi/2$: the nonlinear Schrödinger equation $\pm i u_t + \Delta u + |u|^\alpha u = 0$.

Thus (1) is "intermediate" between the nonlinear heat and Schrödinger equations.

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$$u_t = e^{i\theta}\Delta u + e^{i\phi}|u|^{\alpha}u + \gamma u. \tag{2}$$

Local/global existence for (2) known under various boundary conditions and assumptions on the parameters. On the other hand, few blowup results when (2) is neither NLH nor NLS.

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Snoussi & Tayachi: (Convexity argument) Blowup of negative energy solutions of (1) with $N = 1, 2$, $\alpha = 2$, $|\theta| < \pi/4$. Same argument yields blowup when $N \geq 1$, $\alpha > 0$ and $\cos^2\theta > \frac{2}{\alpha+2}$.

Masmoudi & Zaag: (Ansatz technique) Blowup occurs if $|\theta|, |\phi| < \pi/2$ and $\tan^2\phi + (\alpha + 2)\tan\theta\tan\phi < \alpha + 1$. ($L^\infty$ solutions, not necessarily finite-energy.) For (1), this means $\tan^2\theta < \frac{\alpha+1}{\alpha+3}$. (In particular, $\theta < \pi/4$.)

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1. Finite-time blowup
   - Finite-time blowup
   - Behavior of the blowup time
   - GL with linear driving
   - Some open problems

2. Standing waves

A complex Ginzburg-Landau equation

Consider the equation

$$\begin{cases} e^{-i\theta} u_t = \Delta u + |u|^\alpha u, \\ u(0,x) = u_0(x), \end{cases} \tag{GL}$$

on $\mathbb{R}^N$, where $\alpha > 0$ and $\frac{\pi}{2} < \theta < \frac{\pi}{2}$. It is easy to show LWP in $C_0(\mathbb{R}^N)$ and in $C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. We call $T_{\max} = T_{\max}(u_0)$ the maximal existence time.
For the ODE $e^{-i\theta} z' = |z|^\alpha z$, the solution with $z(0) = c \neq 0$ is $z(t) = c[1 - t\alpha |c|^\alpha \cos\theta]^{-\frac{1}{\alpha}(1 + i\tan\theta)}$. It blows up at $T = \frac{1}{\alpha |c|^\alpha \cos\theta} < \infty$. (No blowup for the other sign.)

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The main feature of (GL), with respect to (2), is that its solutions satisfy energy identities. More precisely,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|u|^2 = -\cos\theta I(u(t)), \tag{3}$$

$$\frac{d}{dt}E(u(t)) = -\cos\theta\int_{\mathbb{R}^N}|u_t|^2, \tag{4}$$

where

$$E(w) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla w|^2 - \frac{1}{\alpha+2}\int_{\mathbb{R}^N}|w|^{\alpha+2},$$

$$I(w) = \int_{\mathbb{R}^N}|\nabla w|^2 - \int_{\mathbb{R}^N}|w|^{\alpha+2}.$$

Finite-time blowup

Negative energy solutions blow up in finite time.

Theorem

Let $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. If $E(u_0) < 0$, then $T_{\max} < \infty$, i.e. the corresponding solution u of (GL) blows up in finite time. (Recall that $|\theta| < \pi/2$.)

Using the energy identities, the result follows essentially from Levine's calculations.

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Behavior of the blowup time

Fix $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ such that $E(u_0) < 0$. Given $|\theta| < \pi/2$, let u^θ be the corresponding solution of (GL), so that u^θ blows up at the finite time $T^\theta_{\max}$.

- If $\alpha < 4/N$, then the solution of NLS (i.e. (GL) for $\theta = \pm\pi/2$) is global. Does $T^\theta_{\max} \to \infty$ as $\theta \to \pm\pi/2$?
- If $4/N \le \alpha < 4/(N-2)$ and if u_0 has finite variance, then the corresponding solution of NLS blows up in finite time. Does $T^\theta_{\max}$ remain bounded as $\theta \to \pm\pi/2$?

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Theorem

If $0 < \alpha < \frac{4}{N}$, then there exists $c > 0$ such that $T^{\theta}_{\max} \geq \frac{c}{\cos\theta}$ for all $|\theta| < \frac{\pi}{2}$.

Global existence for NLS is proved by using the energy identities and Gagliardo-Nirenberg's inequality. The above theorem is proved by using the same tools. (The proof of blowup shows $T^{\theta}_{\max} \leq \frac{C}{\cos\theta}$.)

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Suppose $N \geq 2$ and $\frac{4}{N} \leq \alpha \leq 4$. Fix $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, u_0 radial, and let u^θ denote the corresponding maximal solution of (GL). If $E(u_0) < 0$, then $\exists \overline{T} < \infty$ s.t. $T^\theta_{\max} \leq \overline{T}$ for all $|\theta| < \frac{\pi}{2}$.

The proof follows the "truncated variance" method used by Ogawa and Tsutsumi for NLS. The extra terms are not too difficult to control. The "unnatural" assumptions that $\alpha \leq 4$ and u_0 is radial come from the same technical reasons as in the paper of Ogawa and Tsutsumi.

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If one is willing to assume finite variance, then the standard variance argument of NLS can be used. However, the extra terms that appear involve

$$\int_{\mathbb{R}^N} \left\{ -2|x|^2|\nabla u^\theta|^2 + \frac{\alpha+4}{\alpha+2}|x|^2|u^\theta|^{\alpha+2} + 2N|u^\theta|^2 \right\}.$$

It seems the only way to control that term is by a Caffarelli-Kohn-Nirenberg inequality. Interestingly, the appropriate inequality requires **the very same assumptions** $\alpha \le 4$ and u_0 is radial as in the previous calculations.

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GL with linear driving

Consider (1) with a driving term, i.e.,

$$u_t = e^{i\theta}[\Delta u + |u|^\alpha u] + \gamma u, \tag{5}$$

with $\gamma \in \mathbb{R}$.

ODE: $z' = e^{i\theta}|z|^\alpha z + \gamma z$, solution with $z(0) = c$ is

$$z(t) = e^{\gamma t}\Big[1 - \frac{e^{\alpha\gamma t} - 1}{\gamma}|c|^\alpha \cos\theta\Big]^{-\frac{1}{\alpha}(1+i\tan\theta)} c.$$

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Some open problems

- For the nonlinear heat equation (i.e., (GL) with $\theta = 0$) there is a Fujita critical exponent: If $\alpha \leq 2/N$, then arbitrarily small initial values (in any reasonable norm) may produce solutions that blow up in finite time. (In fact, any nonzero, nonnegative initial value produces a blowing-up solution.) If $\alpha > 2/N$, then small initial values (in appropriate norms) produce global solutions. For NLS, there is no such exponent: small initial values always produce global solutions.

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- For equation (GL) with nonlinearity of other sign, i.e. $e^{-i\theta} u_t = \Delta u - |u|^\alpha u$, the factor of $|u|^{\alpha+2}$ comes with a positive sign in both I and E.

The energy identities (3) and (4) yield a control of $\|u(t)\|_{H^1} + \|u(t)\|_{L^{\alpha+2}}$ for $0 \le t < T_{\max}$.
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In view of the above observations, we emphasize the following open problems.

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Consider the equation $e^{-i\theta}u_t = \Delta u - |u|^\alpha u$ *and suppose* $N \geq 3$ *and* $\alpha \geq 4/(N-2)$. *Given any* $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ *let* u *be the corresponding solution. Is* u *global?*

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Standing waves

Look for standing waves the general complex GL equation (2) of the form $u(t,x) = e^{i\omega t} w(x)$. The equation for w is

$$e^{i\theta}\Delta w + e^{i\phi}|w|^{\alpha}w + (\gamma - i\omega)w = 0. \tag{6}$$

- If $\theta = \phi$, then this is

$$\Delta w + |w|^{\alpha}w + e^{-i\theta}(\gamma - i\omega)w = 0.$$

OK if ω chosen so that $e^{-i\theta}(\gamma - i\omega) \in \mathbb{R}$, i.e. $\omega = -\gamma \tan\theta$. Standard elliptic problem

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Nontrivial H^1 solutions if $\gamma < 0$ and α subcritical or $\gamma = 0$, α critical and $N \geq 5$.

- If $\theta \neq \phi$, not variational. Possible approaches:

– ODE method (shooting): Coupled system of two (real-valued) second order ODEs (nonautonomous if $N \geq 2$).

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Another perturbative approach: Play with α and try IFT starting from a solution of $e^{i\theta}\Delta w + e^{i\phi}w + (\gamma - i\omega)w = 0$. This involves eigenvectors of the Laplacian, so one must change the boundary conditions.
For example: Dirichlet on a bounded domain and periodic on $\mathbb{R}^N$ (on $\mathbb{T}^N$).
On $\mathbb{T}^N$, already plenty of constant or, more generally, plane wave solutions $w(x) = ce^{iy\cdot x}$. Equation is $-|y|^2 e^{i\theta} + |c|^\alpha e^{i\phi} + \gamma = i\omega$. OK if we choose $|y|^2 \cos\theta > \gamma$. (Possible, only restriction: $y_j \in 2\pi\mathbb{Z}$ for all j.) $|c|$ is determined by $|c|^\alpha \cos\phi = |y|^2 \cos\theta - \gamma$ and ω is given by $\omega = |c|^\alpha \sin\phi - |y|^2 \sin\theta$.

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Given θ, ϕ, γ, one can let Ω be a small cube, so that $\lambda_1 \cos\theta > \gamma$, and extend the solutions to $\mathbb{R}^N$ by odd (hence, periodic) extension. One obtains solutions on $\mathbb{T}^N$ (different from the trivial ones) for small α.

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Nontrivial standing waves in $H^1(\mathbb{R}^N)$ (or in $H^1(\mathbb{T}^N)$) for general α, θ, ϕ?

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