NLS in the partially periodic case

Nicola Visciglia (University of Pisa)

(Joint work with Z. Hani, B. Pausader,N. Tzvetkov)

University of Pisa

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Our problem

We consider the following NLS

$$\begin{cases} i\partial_t u - \Delta_{x,y} u + u|u|^\alpha = 0, & (t, x, y) \in \mathbf{R} \times \mathbf{R}^d \times \mathbf{R}/(2\pi\mathbf{Z}), \\ u(0, x, y) = f(x, y) \in H^1_{x,y}, & \end{cases}$$

in the partially periodic case. We are interested in:

- Local and Global Cauchy Theory;
- Long-time behavior (Scattering)

The following facts are classical (Lin-Strauss, Ginibre-Velo, Nakanishi, Planchon-Vega, Colliander-Grillakis-Tzirakis,Cazenave book).
Consider

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then:

- For every $0 < \alpha < 4/(d-1)$ the Cauchy Problem is GWP in $H^1_{x,y}$ (hence G.W.P. in the energy subcritical regime).
- For every $4/(d+1) < \alpha < 4/(d-1)$ and for every $f \in H^1_{x,y}$, there exist $f_\pm \in H^1_{x,y}$ such that

$$\lim_{t\to\pm\infty} \|u(t,x,y) - e^{-it\Delta_{\mathbf{R}^{d+1}}} f_\pm\|_{H^1_{x,y}} = 0$$

(hence Scattering in the L^2 supercritical and energy subcritical regime).

Results on $\mathbf{R}^{d+1}$

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  (hence Scattering in the L^2 supercritical and energy subcritical regime).

What about $H^1_{x,y}$-scattering in the L^2-subcritical regime?

In the FOCUSING case scattering is FALSE also for initial data small in $H^1_{x,y}$.

Think about solitary waves $e^{i\lambda^2 t}Q(\lambda x)\lambda^{\frac{2}{\alpha}}$ where $-\Delta Q + Q = Q|Q|^\alpha$

I don't know any result that prevents possibility of scattering in the L^2-subcritical regime for DEFOCUSING NLS on $\mathbf{R}^{d+1}$!!!!

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Recall that in the case $4/(d+1) < \alpha < 4/(d-1)$ the we have the following equivalence

$$\text{Scattering} \quad \textit{iff} \quad \lim_{t\to\pm\infty} \|u(t,x,y)\|_{L^p_{x,y}} = 0, \forall 2 < p < 2(d+1)/(d-1)$$

(for the focusing and defocusing NLS)

The basic result by Ginibre-Velo is

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The following property occurs:

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for every $2 < p < 2(d+1)/(d-1)$ *and for* $u(t,x,y)$ *the unique global solution to DEFOCUSING NLS on* $\mathbf{R}^{d+1}$ *and* α *energy subcritical*

No need to get L^2-supercritical nonlinearity!

Theorem above is FALSE in the L^2-subcritical regime for the FOCUSING NLS with initial data small in $H^1_{x,y}$!

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Our Results

There is a huge literature about Cauchy theory for NLS on manifolds: Bourgain, Burq-Gérard-Tzvetkov, Ionescu-Staffilani, Ionescu-Staffilani-Pausader, Tzvetkov-Visciglia etc. etc. However those results do not cover the energy subcritical NLS in the partially periodic case $\mathbf{R}^d \times \mathbf{R}/(2\pi\mathbf{Z})$.

Theorem

Let NLS on $\mathbf{R}^d \times \mathbf{R}/(2\pi\mathbf{Z})$, with energy subcritical nonlinearity, then:

- *for any initial datum $f \in H^1_{x,y}$, there exists a unique local solution*

$$u(t,x,y) \in \mathcal{C}((-T,T); H^1_{x,y}), \text{ where } T = T(\|f\|_{H^1_{x,y}}) > 0;$$

- *the map $B_r(0) \ni f \to u(t,x,y) \in \mathcal{C}((-T,T); H^1_{x,y})$ is Lipschitz continuous;*
- *the solution $u(t,x,y)$ can be extended globally in time.*

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The Strichartz estimates

$$\|e^{it\Delta_{\mathbf{R}^{d+1}}}\|_{L^p_t L^q_{x,y}} \lesssim \|f\|_{L^2_{x,y}}$$

are TRUE for any $(p,q) \in [1,\infty] \times [1,\infty]$ such that $2/p + (d+1)/q = (d+1)/2$, $(p,d) \neq (2,1)$

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The following type of Strichartz estimates (see Tzvetkov-Visciglia 2012)

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Theorem

Let $d \geq 1$, $4/d < \alpha < 4/(d-1)$ and $f(x,y) \in H^1_{x,y}$. Then there exist $f_\pm \in H^1_{x,y}$ such that

$$\lim_{t \to \pm\infty} \|u(t,x,y) - e^{-it\Delta_{\mathbb{R}^d \times \mathbb{R}/(2\pi\mathbb{Z})}} f_\pm\|_{H^1_{x,y}} = 0$$

As far as we know Theorem above is new also in the case of initial data $f(x,y)$ that are small in $H^1_{x,y}$.

Despite to the case of $\mathbb{R}^{d+1}$ (where we require $\alpha > 4/(d+1)$ beside the energy sub criticality) here we require the extra restriction $\alpha > 4/d$. On the other hand this restriction is quite natural since for $\alpha < 4/d$, the Cauchy problem in the partially periodic case reduces to L^2-subcritical NLS in $\mathbb{R}^d$, if one chooses initial data $f(x,y) = f(x)$!

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Notice that it is meaningful to consider the H^1-scattering theory for NLS on product spaces $\mathbf{R}^d \times M^k$ only in the case $k = 1$ and $k = 2$. In fact on one hand the nonlinearity α needs to be L^2-supercritical w.r.t. to $\mathbf{R}^d$, i.e. $\alpha \geq 4/d$. On the other hand, in order to be well-posed the Cauchy problem in $H^1(\mathbf{R}^d \times M^k)$ we need $\alpha \leq 4/(d + k - 2)$. It is easy to check that there is compatibility between the above conditions only for $k = 1$ and $k = 2$. In particular in the case $k = 2$ they collapse to the energy critical nonlinearity $\alpha = 4/d$.

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On the Proof

The proof of the local existence is splitted in two steps: first we construct via a fixed point argument a solution in an auxiliary space and a-posteriori we show the unconditional uniqueness in $\mathcal{C}((-T,T);H^1_{x,y})$. The key ingredient is a suitable version of Strichartz estimates of the type

$$\|e^{it\Delta_{x,y}}f\|_{L^l_t L^p_x L^2_y} \leq C\|f\|_{L^2_{x,y}},$$

where (l,p) are such that $\frac{2}{l}+\frac{d}{p}=\frac{d}{2}, \quad p\geq 2, \quad (d,l)\neq(2,2)$. Observe that the estimate above roughly speaking is a mixture of the (non dispersive) L^2-conservation w.r.t. the compact y variable and the classical Strichartz estimates w.r.t. the dispersive directions $\mathbf{R}^d$. Of course along with the estimate above one can also consider similar ones for the Duhamel operator.

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Armed with those estimates (and its version with derivatives) one can perform a fixed point argument is the spaces $L_t^l L_x^p H_y^{1/2+\delta}$, where we have denoted by H_y^γ the usual Sobolev space on $\mathbf{R}/(2\pi\mathbf{Z})$. The main advantage of working in those spaces is that one can consider the $\mathbf{C}$-valued solution $u(t,x,y)$ as functions dependent on the (t,x) variables and valued in the algebra $H_y^{1/2+\delta}$. Hence one can perform a fixed point argument by exploiting in the partially periodic setting the same numerology involved in the analysis of NLS posed in the euclidean space $\mathbf{R}^d$ via admissible Strichartz norms $L_t^l L_x^p$. On the other hand it is well-known (by classical Euclidean theory) that the best nonlinearity that can be reached with this technique is the L^2-critical nonlinearity in $\mathbf{R}^d$, i.e. $0 < \alpha < 4/d$.

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Therefore the main new contribution are the cases $4/d \leq \alpha < 4/(d-1)$ (i.e. nonlinearites which are L^2- supercritical and $H^{1/2}$-subcritical in $\mathbf{R}^d$). The main difficulty in the transposition of the analysis above in this larger regime of nonlinearities is that, in analogy with the analysis of L^2-supercritical NLS in $\mathbf{R}^d$, it seems to be necessary to work with Strichartz estimates involving derivatives w.r.t. x variable! To overcome this obstacle we exploit a class of inhomogeneous Strichartz estimates with respect the x variable true in a range of Lebesgue exponents larger than the one given in the usual homogeneous estimates context.

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Observe also that following our discussion above we are interested on one hand to work with a functional space which is an algebra w.r.t. to y variable, i.e. H_y^γ with $\gamma > 1/2$, on the other hand we need to consider at most $1/2$ derivatives w.r.t to x variable since our threshold nonlinearity $\alpha = 4/(d-1)$ is $H^{1/2}$ subcritical in $\mathbf{R}^d$. Summarizing we need globally at most one derivative w.r.t. to the full set of variables (x, y). This is the main reason that allows us to treat initial data in $H^1_{x,y}$.

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$$\int_{\mathbf{R}} \sup_{x_0\in\mathbf{R}^d} \Big(\int\int_{(2Q^d(x_0))^2\times(0,2\pi)^2} |u(t,x_2,y_2)|^{2+\alpha}|u(t,x_1,y_1)|^2\, dx_1\, dx_2\, dy_1\, dy_2\Big)\, dt$$

$$\leq C\|\varphi\|^4_{H^1_{x,y}}$$

where $Q^d(x_0)$ is the unit cube in $\mathbf{R}^d$ centered in x_0 and $A^2 = A\times A$ for any general set A.

THANK YOU
FOR YOUR ATTENTION!