On the persistence properties of solutions of a fifth order KdV type equation in weighted Sobolev spaces

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A fifth order KdV type equation

We consider the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x^5 u + N(u, \partial_x u) = 0, & x, t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where

- $N_1 = u\partial_x u$, $N_2 = u^2\partial_x u$ (In this talk)
- $u_0 \in H^s(\mathbb{R}) \cap L^2(p(x)dx)$ (Weighted Sobolev space)
- p is a non-negative function.

Our aim is to study the Cauchy problem (1) in the weighted Sobolev spaces

$$\mathcal{Z}_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx)$$

- Well-posedness results in $\mathcal{Z}_{s,r}$
- Relation between decay and regularity for the solutions of the Cauchy problem (1)

The KdV equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0. \tag{2}$$

- Kato (1983) Well-posedness results in $\mathcal{Z}_{s,r}$, with $s \geq 2r$ and $r = 1, 2, 3...$
  Global well-posedness in $\mathcal{S}(\mathbb{R})$.

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- Nahas and Ponce (2011-2012) Well-posedness results in $\mathcal{Z}_{s,r}$, $s \geq 2r$ and $s \geq 1$.
  For $s \in [0, 1)$, well-posedness in $\mathcal{Z}_{s,r-\epsilon}$, for any $\epsilon > 0$

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- Isaza, Linares and Ponce (2013) If $u \in C(\mathbb{R}; L^2(\mathbb{R}))$ is a global solution of the KdV equation and there exist $\alpha > 0$ such that in two different times $t_0, t_1 \in \mathbb{R}$

$$|x|^\alpha u(t_0), |x|^\alpha u(t_1) \in L^2(\mathbb{R}),$$

then $u \in C(\mathbb{R}, H^{2\alpha}(\mathbb{R}))$.

Benjamin-Ono equation

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It is necessary an additional condition on the Fourier transform of the initial data

$$\dot{\mathcal{Z}}_{s,r} = \{f \in \mathcal{Z}_{s,r} : \; \widehat{f}(0) = 0\}.$$

- Iorio (1989) and (2003) Local well-posedness in $\mathcal{Z}_{2,2}$ and $\dot{\mathcal{Z}}_{3,3}$.
  If $u \in C([0,T]; H^2(\mathbb{R}))$ is a solution of the Cauchy problem associated to the Benjamin-Ono equation and there exist three different times t_1, t_2, t_3, such that

$$u(t_i) \in \dot{\mathcal{Z}}_{4,4} \;\text{ for }\; i = 1, 2, 3, \tag{4}$$

  then $u \equiv 0$.

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- Unique continuation result in $\dot{\mathcal{Z}}_{7/2,7/2}$

NDE in weighted Sobolev spaces

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- The condition on the Fourier transform of the initial data is necessary for $r \geq 5/2$.
- Fonseca, Linares and Ponce (2012) The condition involving three different times cannot be reduced to two different times (in contrast with some unique continuation principles for KdV type equations)

Other NDE in weighted Sobolev spaces

- $p(x)$ of exponencial type.

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- KdV equation: Kato (1983) Kenig, Ponce Vega and Escauriaza (2002-2007), Isaza, Linares and Ponce (2013)
- For higher order dispersive models

$$\partial_t u + (-1)^{k+1}\partial_x^n + N(u, \partial_x u, \cdots, \partial_x^{n-2} u) = 0. \tag{5}$$

Dawson (2007) and Isaza (2013).

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  Dawson (2007) and Isaza (2013).
- For $k = \frac{n-1}{2}$, if u is a sufficiently smooth solution of (5) and there exist two times t_1, t_2, and $a >> 1$ such that

$$u(t_i) \in L^2(e^{ax^{\frac{4}{3}+\epsilon}}), \quad \text{for} \quad i = 1, 2 \tag{6}$$

  then $u \equiv 0$.

Theorem (1)

Let $\frac{5}{16} < r < \frac{1}{2}$ *and* $u_0 \in Z_{4r,r}$*. Then there exist* $T = T(\|u_0\|_{Z_{4r,r}}) > 0$ *and a unique* u*, solution of the Cauchy problem (1) with* $N = N_1$*, satisfying*

$$u \in C([0,T]; Z_{4r,r}) \,, \tag{7}$$

$$\partial_x u \in L^4([0,T]; L^\infty(\mathbb{R})) \,, \tag{8}$$

$$\|D_x^{4r} \frac{\partial u}{\partial x}\|_{L_x^\infty L_T^2} < \infty \,, \quad \textit{and} \tag{9}$$

$$\|u\|_{L_x^2 L_T^\infty} < \infty \,. \tag{10}$$

Moreover, for any $T' \in (0,T)$ *there exists a neighborhood* V *of* u_0 *in* $Z_{4r,r}$ *such that the map datum-solution* $\tilde{u}_0 \mapsto \tilde{u}$ *from* V *into the class defined by* (7)-(10) *with* T' *instead of* T *is Lipschitz.*

Main tools of the proof.

- Contracting principle (Kenig, Ponce and Vega (1993) for the KdV equation)
- A pointwise formula for "fractional weights" (Fonseca, Linares and Ponce) , which can be applied, without significant changes, to the group $\{W(t)\}_{t\in\mathbb{R}}$ associated to the linear fifth order KdV equation. More precisely, for $r \in (0,1)$ and $u_0 \in Z_{4r,r}$

$$|x|^r[W(t)u_0](x) = W(t)(|x|^r u_0)(x) + W(t)\{\Phi_{t,r}(\hat{u_0})\}^{\vee}(x), \quad (11)$$

where,

$$\|(\Phi_{t,r}(\hat{u_0})(\xi))^{\vee}\|_{L^2} \le C_r(1+|t|)(\|u_0\|_{L^2} + \|D_x^{4r}u_0\|_{L^2}). \quad (12)$$

- In this case $s = 4r$ in order to preserve the decay. (KdV $s = 2r$, BO $s = r$).

Theorem (2)

Let $r \geq 1/2$ *and* $u_0 \in Z_{4r,r}$. *Then there exist* $T = T(\|u_0\|_{H^{4r}}) > 0$ *and a unique* $u \in C([0,T]; Z_{4r,r})$, *solution of the Cauchy problem* (1), *belonging to the class defined by the conditions*

$$u \in C([0,T]; H^s(\mathbb{R}))\,, \tag{13}$$

$$\partial_x u \in L^4([0,T]; L^\infty(\mathbb{R}))\,, \tag{14}$$

$$\|D_x^s \frac{\partial u}{\partial x}\|_{L_x^\infty L_T^2} < \infty\,, \quad \textit{and} \tag{15}$$

$$\|u\|_{L_x^2 L_T^\infty} < \infty\,. \tag{16}$$

with $s = 4r$.
Besides, for any $T' \in (0,T)$ *there exists a neighborhood* V *of* u_0 *in* $Z_{4r,r}$ *such that the map data-solution* $\tilde{u_0} \mapsto \tilde{u}$ *from* V *to* $C([0,T']; Z_{4r,r})$ *is continuous.*

Main tools of the proof.

- Local well-posedness in $H^s(\mathbb{R})$.
- Interpolation inequalities. Let $a > 0$ and $b > 0$. Assume that $J^a f := (1-\partial_x^2)^{\frac{a}{2}} f \in L^2(\mathbb{R})$ and $\langle x \rangle^b f := (1+x^2)^{\frac{b}{2}} f \in L^2(\mathbb{R})$. Then for any $\theta \in (0,1)$

$$\| J^{\theta a}(\langle x \rangle^{(1-\theta)b} f)\|_{L^2} \leq C \| \langle x \rangle^b f\|_{L^2}^{1-\theta} \| J^a f\|_{L^2}^{\theta} \,. \tag{17}$$

- Gronwall inequalities.
- The solution can be extended to any interval $[0,T]$, $T>0$. (Conservation laws)
- A similar result is valid for the nonlinearity $N_2 = u^2 \partial_x u$

Following the method used by Isaza, Linares and Ponce (2013) we prove the next result for the solutions of the fifth order modified KdV type equation.

Theorem (3)

For $T > 0$, *let* $u \in C([0,T]; Z_{2,1/2})$ *the solution of the fifth order KdV type equation, obtained in Theorem (2), with* $N = N_2$ *Let us suppose that for* $\alpha \in (0, 1/8]$ *there exist two different times* $t_0, t_1 \in [0,T]$, *with* $t_0 < t_1$, *such that*

$$|x|^{1/2+\alpha} u(t_0), |x|^{1/2+\alpha} u(t_1) \in L^2(\mathbb{R}), \tag{18}$$

then $u \in C([0,T]; H^{2+4\alpha}(\mathbb{R}))$.

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