Instability of cnoidal-peak solutions for the NLS equation with a periodic δ–interaction

Jaime Angulo Pava

Dept. of Mathematics/USP
São Paulo/Brazil

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Goal of this talk

To show the existence and nonlinear instability of a family of periodic cnoidal standing-wave solutions for the Schrödinger equation (NLS-δ equation henceforth):

$$iu_t + u_{xx} + Z\delta(x)u + |u|^2 u = 0, \qquad (1)$$

where $u = u(x,t) \in \mathbb{C}$, $(x,t) \in \mathbb{T} \times \mathbb{R}$, δ is the Dirac distribution at the origin, which we see as the linear functional

$$(\delta, v) = v(0), \quad \text{for} \;\; v \in H^1_{per},$$

and $Z \in \mathbb{R}$.

Physical Relevance: $iu_t + u_{xx} + Z\delta(x)u + |u|^2u = 0$, (1)

Equation in (1), $Z \neq 0$, has been arisen in physical model with a point defect: In nonlinear optics and Bose-Einstein condensates. Indeed, the Dirac distribution is used to model an impurity, or defect, localized at the origin.

The Mathematical Model

- Caudrelier&Mintchev&Ragoucy, *The quantum non-linear Schrödinger equation with point-like defect*, J. Physics A: Mathematical and General. 37 (30) (2004).

It is described by the following boundary problem

$$\begin{cases} iu_t(x,t) + u_{xx}(x,t) = |u(x,t)|^p\, u(x,t), \quad x \neq 0,\ t \in \mathbb{R} \\ \lim\limits_{x\to 0^+} [u(x,t) - u(-x,t)] = 0, \\ \lim\limits_{x\to 0^+} [\partial_x u(x,t) - \partial_x u(-x,t)] = -Zu(0,t), \quad \lim\limits_{x\to\pm\infty} u(x,t) = 0, \end{cases} \tag{2}$$

hence $u(x,t)$ must be solution of the non-linear Schrödinger equation on $\mathbb{R}^-$ and $\mathbb{R}^+$, continuous at $x = 0$ and satisfy a "jump condition" at the origin .

NLS-δ/ Non-periodic case results: Standing-Wave

$$iu_t + u_{xx} + Z\delta(x)u + |u|^p u = 0,$$

$u = u(x,t) \in \mathbb{C}$, $(x,t) \in \mathbb{R} \times \mathbb{R}$, $p > 0$.

- Standing-wave:

$$u(x,t) = e^{i\omega t}\phi(x)$$

where $\omega \in \mathbb{R}$, $\phi : \mathbb{R} \to \mathbb{R}$ and $\lim_{|x|\to+\infty} \phi(x) = 0$. Then,

$$\Big(-\frac{d^2}{dx^2} - Z\delta(x)\Big)\phi + \omega\phi - |\phi|^p\phi = 0. \qquad (3)$$

Standing-Wave: $(-\frac{d^2}{dx^2} - Z\delta(x))\phi + \omega\phi - |\phi|^p\phi = 0$

- The solution ϕ needs to satisfy $\phi \in D(-\frac{d^2}{dx^2} - Z\delta)$.

⋆ From the von Neumann theory of self-adjoint extensions for symmetric operators,

$$-\Delta_Z\zeta = -\frac{d^2}{dx^2}\zeta, \qquad \zeta \in D(-\Delta_Z),$$

$$D(-\Delta_Z) = \{\zeta \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}) : \zeta'(0+) - \zeta'(0-) = -Z\zeta(0)\}$$

represents the formal linear diff. oper. : $-\frac{d^2}{dx^2} - Z\delta$, where for $\zeta \in D(-\Delta_Z)$

$$-\Delta_Z\zeta(x) = -\frac{d^2}{dx^2}\zeta(x), \quad x \neq 0$$

Standing-Wave: $-\phi'' - Z\delta(x)\phi + \omega\phi - |\phi|^p\phi = 0$ (3)

- Goodman&Holmes&Weinstein, Ohta&Ozawa and Fukuizumi&Jeanjean: There exists a unique positive even solution of (3) (modulo rotations):

$$\phi_{\omega,Z,p}(x) = \Big[\frac{(p+2)\omega}{2} sech^2\Big(\frac{p\sqrt{\omega}}{2}|x| + \tanh^{-1}\Big(\frac{Z}{2\sqrt{\omega}}\Big)\Big)\Big]^{\frac{1}{p}}. \quad (4)$$

if $\omega > Z^2/4$.

Profile of $\phi_{\omega,Z,p}$

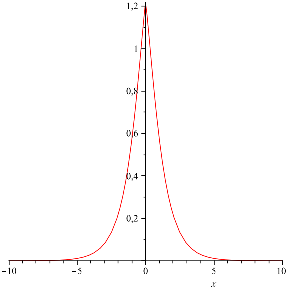

Figure : for $Z > 0$

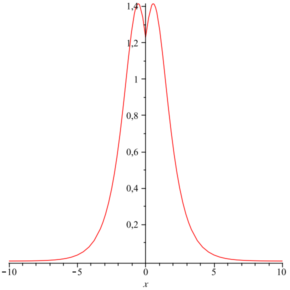

Figure : for $Z < 0$

Construction: $(-\frac{d^2}{dx^2} - Z\delta(x))\phi + \omega\phi - |\phi|^p\phi = 0$ (3)

- Solution

$$\phi_{\omega,Z,p}(x) = \left[\frac{(p+2)\omega}{2} sech^2\left(\frac{p\sqrt{\omega}}{2}|x| + \tanh^{-1}\left(\frac{Z}{2\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p}}$$

is constructed from the solution with $Z = 0$ in (3):

1. $-\phi''(\xi) + \omega\phi(\xi) - \phi^{p+1}(\xi) = 0$, for $\xi \neq 0$.
2. $\phi \in \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}) : f'(0+) - f'(0-) = -Zf(0)\}$.

Orbital Stability

$$iu_t + u_{xx} + Z\delta(x)u + |u|^p u = 0 \qquad (NLS - \delta)$$

- The basic symmetry associated to NLS-δ is the phase-invariance:

$$\text{if } u(x,t) \text{ is solution } \rightarrow e^{i\theta}u(x,t) \text{ is solution for all } \theta \in \mathbb{R}.$$

- The translation invariance: $u(x,t) \rightarrow u(x+y,t)$, $y \in \mathbb{R}$;

it is not true because of the defect.

- $\phi_{\omega,Z}$-orbit: $\Omega_{\phi_{\omega,Z}} = \{e^{i\theta}\phi_{\omega,Z} : \theta \in [0,2\pi)\}$.

Orbital Stability

Definition

For $\eta > 0$ *and* $\phi = \phi_{\omega,Z}$, *we put*

$$U_\eta(\phi) = \{v \in X : \inf_{\theta \in \mathbb{R}} \|v - e^{i\theta}\phi\|_X < \eta\}.$$

The standing wave $e^{i\omega t}\phi$ *is (orbitally) stable in* X *if for* $\epsilon > 0$ *there exists* $\eta > 0$ *s. t. for* $u_0 \in U_\eta(\phi)$, *the solution* $u(t)$ *of the NLS-*δ *with* $u(0) = u_0$ *satisfies* $u(t) \in U_\epsilon(\phi)$ *for all* $t \in \mathbb{R}$.

Otherwise, $e^{i\omega t}\phi$ *is said to be (orbitally) unstable in* X.

Stability Results: $iu_t + u_{xx} + Z\delta(x)u + |u|^p u = 0$

From Goodman&Holmes&Weinstein, Fukuizumi&Ohta&Ozawa, Fukuizumi&Jeanjean, Le Coz&Fukuizumi&Fibich&Ksherim&Sivan:

- Let $Z > 0$ and $\omega > Z^2/4$:
  1. $0 < p \leqq 4$, $e^{i\omega t}\phi_{\omega,Z,p}$ is stable in $H^1(\mathbb{R})$ for $\omega \in (Z^2/4, +\infty)$.
  2. If $p > 4$, there exists a unique $\omega_1 > Z^2/4$ such that:
     1. $e^{i\omega t}\phi_{\omega,Z,p}$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (Z^2/4, \omega_1)$,
     2. $e^{i\omega t}\phi_{\omega,Z,p}$ is unstable in $H^1(\mathbb{R})$ for any $\omega \in (\omega_1, +\infty)$.

Stability Results: $iu_t + u_{xx} + Z\delta(x)u + |u|^p u = 0$

- Let $Z < 0$ and $\omega > Z^2/4$ [In general, they are unstable in $H^1(\mathbb{R})$]
  1. If $0 < p \leqq 2$ and $\omega > Z^2/4$
     1. $e^{i\omega t}\phi_{\omega,Z,p}$ is stable in $H^1_{even}(\mathbb{R})$.
     2. $e^{i\omega t}\phi_{\omega,Z,p}$ is unstable in $H^1(\mathbb{R})$.
  2. If $2 < p < 4$, there exists a $\omega_2 > Z^2/4$ such that:
     1. $e^{i\omega t}\phi_{\omega,Z,p}$ is unstable in $H^1(\mathbb{R})$, for $\omega \in (Z^2/4, \omega_2) \cup (\omega_2, +\infty)$
     2. $e^{i\omega t}\phi_{\omega,Z,p}$ is stable in $H^1_{even}(\mathbb{R})$, for $\omega \in (\omega_2, +\infty)$.
  3. if $p \geqq 4$, then $e^{i\omega t}\phi_{\omega,Z,p}$ is unstable in $H^1(\mathbb{R})$.

NLS-δ/ periodic case results: standing-wave

$$iu_t + u_{xx} + Z\delta(x)u + |u|^2 u = 0, \tag{5}$$

$u = u(x,t) \in \mathbb{C}$, $(x,t) \in \mathbb{T} \times \mathbb{R}$.

- Periodic standing-wave:

$$u(x,t) = e^{i\omega t}\phi(x) \tag{6}$$

where $\omega \in \mathbb{R}$, $\phi : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function satisfying,

$$-\phi'' - Z\delta(x)\phi + \omega\phi - \phi^3 = 0.$$

Conditions: Existence of 2π-periodic peak

- The solution ϕ needs to satisfy $\phi \in D(-\frac{d^2}{dx^2} - Z\delta)$.

⋆ From the von Neumann theory of self-adjoint extensions for symmetric operators,

$$-\Delta_Z \zeta = -\frac{d^2}{dx^2}\zeta, \qquad \zeta \in D(-\Delta_Z)$$

$$D(-\Delta_Z) = \{\zeta \in H^1_{\text{per}}([-\pi,\pi]) \cap H^2(0,2\pi) : \zeta'(0+) - \zeta'(0-) = -Z\zeta(0)\}$$

represents the formal linear diff. oper. : $-\frac{d^2}{dx^2} - Z\delta$, where for $\zeta \in D(-\Delta_Z)$

$$-\Delta_Z \zeta(x) = -\frac{d^2}{dx^2}\zeta(x), \quad x \neq 2\pi n, \; n \in \mathbb{Z}.$$

Conditions: Existence of 2π-periodic peakon

$$D(-\Delta_Z) = \{\zeta \in H^1_{\text{per}}([-\pi,\pi]) \cap H^2(0,2\pi) : \zeta'(0+) - \zeta'(0-) = -Z\zeta(0)\}$$

- Periodic-peak solutions of

$$-\phi'' - Z\delta(x)\phi + \omega\phi = \phi^3 \tag{7}$$

will satisfy:

1. $\phi(x+2\pi) = \phi(x)$, for all $x \in \mathbb{R}$.
2. $\phi \in C^j(\mathbb{R} - \{2n\pi : n \in \mathbb{Z}\}) \cap C(\mathbb{R})$, $j = 1, 2$.
3. $-\phi''(x) + \omega\phi(x) = \phi^3(x)$, for $x \in (-\pi, 0) \cup (0, \pi)$.
4. $\phi'(0+) - \phi'(0-) = -Z\phi(0)$.

Jacobi Elliptic Functions

- For $-1 \leqq y \leqq 1$, $k \in (0,1)$ (fixed), we consider the elliptic integral (strictly increasing)

$$u(y;k) \equiv u = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

We denote its inverse function (odd) by

$$y = sn(u;k), \qquad \textbf{(snoidal)}$$

- **Cnoidal**: $cn(u;k) = \sqrt{1 - sn^2(u;k)}$
- **Dnoidal**: $dn(u;k) = \sqrt{1 - k^2 sn^2(u;k)}$
- $sn(u + 4K(k);k) = sn(u;k)$, $cn(u + 4K(k);k) = cn(u;k)$, $dn(u + 2K(k);k) = dn(u;k)$,

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Periodic wave: $-\phi'' - Z\delta(x)\phi + \omega\phi - \phi^3 = 0 \quad (7)$

For $Z = 0$ / Angulo (2007)

- **Smooth Dnoidal Solutions**: There exists a family of smooth positive even solution of (7), $\omega \to \phi_{\omega,0} \in H^1_{per}([-\pi,\pi])$, $\omega > \frac{1}{2}$:

$$\phi_{\omega,0}(\xi) = \eta_0 dn\Big(\frac{\eta_0}{\sqrt{2}}\xi; k\Big), \quad \xi \in [-\pi,\pi]$$

$\eta_0 = \eta_0(\omega) \in (\omega, \sqrt{2\omega})$, $k = k(\omega) \in (0,1)$.

Profile of Dnoidal Solution: $\phi_{\omega,0}(\xi) = \eta_0 dn\left(\frac{\eta_0}{\sqrt{2}}\xi; k\right)$

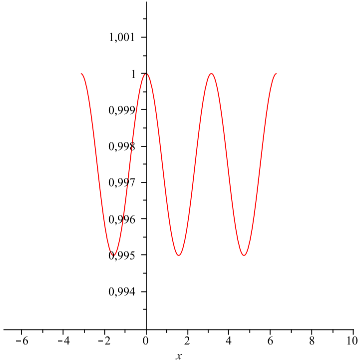

Figure : Profile with $k = 0.1$

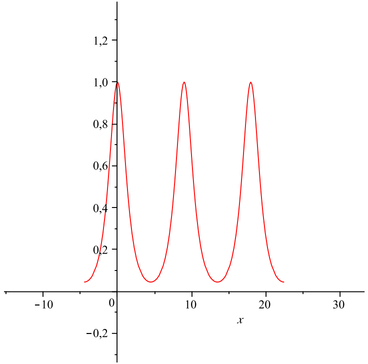

Figure : Profile with $k = 0.99$

Asymptotic limit of the dnoidal smooth solution

- Since $dn(y;1) = sech(y)$, we obtain the convergence (uniformly on compact-set) for $k \to 1^+$,

$$\phi_{\omega,0}(\xi) = \eta_0 dn\left(\frac{\eta_0}{\sqrt{2}}\xi; k\right) \to \sqrt{2\omega} sech\left(\sqrt{\omega}\xi\right)$$

- $\varphi(\xi) = e^{i\omega t}\sqrt{2\omega} sech\left(\sqrt{\omega}\xi\right)$, $\omega > 0$, it is the well-known soliton-solution for the cubic não-linear Schrödinger equation,

$$iu_t + u_{xx} + |u|^2 u = 0.$$

- The standing wave $e^{i\omega t}\phi_{\omega,0}$ is orbitally stable in $H^1_{per}([-\pi,\pi])$.

Cnoidal Solution for: $-\phi'' - Z\delta(x)\phi + \omega\phi = \phi^3$ (7)

For $Z = 0$ / Angulo (2007)

- **Smooth Cnoidal Solutions:** There exists a family of smooth sign changing even solution of (7),
  $\omega \in (0, +\infty) \to \varphi_{\omega,0} \in H^n_{per}([-\pi, \pi])$,

$$\varphi_{\omega,0}(\xi) = b_0 cn\Big(\sqrt{b_0^2 - \omega}\, \xi; k\Big).$$

Here, $b_0 = b_0(\omega) \in (\sqrt{2\omega}, +\infty)$ and $k(\omega) \in (0, 1)$.

Profile of Cnoidal Solution: $\varphi(\xi) = cn(\xi; k)$

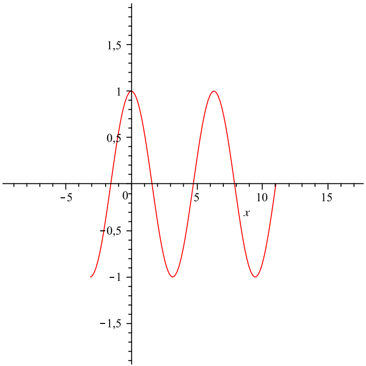

Figure : Profile with $k = 0.1$

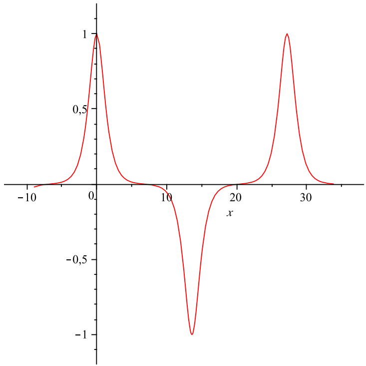

Figure : Profile with $k = 0.99$

Cnoidal picture: $\varphi(x) = cn(x; k)$

Figure : A cnoidal wave, characterized by sharper crests and flatter troughs than in a sine wave. For the shown case, the elliptic parameter is $k = 0.9$.

Cnoidal picture:

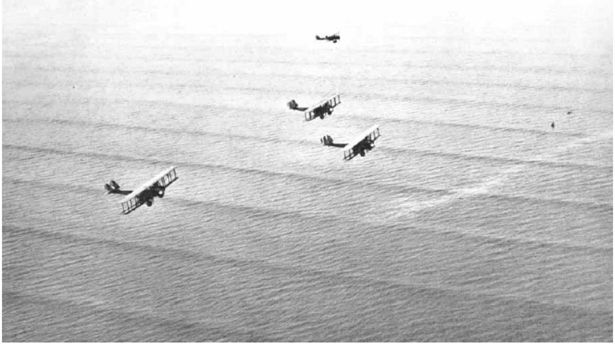

Figure : Aircrafts flying over near-periodic swell in shallow water, close to the Panama coast (1933). The sharp crests and very flat troughs are characteristic for cnoidal waves.

Comments about cnoidal profile's stability

- **Open problem**: Is the cnoidal-orbit:

$$\Omega_{\varphi_{\omega,0}} = \{e^{i\theta}\varphi_{\omega,0}(\cdot + y) : y \in \mathbb{R},\ \theta \in [0, 2\pi)\}$$

  stable by the periodic flow of the cubic-NLS?

- **In the case of KdV models they are unstable**: For $k \approx 1^+$ the orbit $\Omega_{\varphi_{\omega,0}} = \{\varphi_{\omega,0}(\cdot + y) : y \in \mathbb{R}\}$ is unstable, for instance, by the periodic flow of the

$$u_t + u_{xxx} + (u^3)_x = 0, \quad u_t + u_x + (u^3)_x - u_{xxt} = 0$$

  (Angulo&Natali/2012)

Dnoidal-Peak: $-\phi'' - Z\delta(x)\phi + \omega\phi - \phi^3 = 0 \quad (7)$

Theorem

- *Angulo&Ponce (2012): There exists a peakon-family of positive even solution of* (7), $(\omega, Z) \to \phi_{\omega,Z} \in H^1_{per}([-\pi,\pi])$, $\omega > Z^2/4$:

$$\phi_{\omega,Z}(\xi) = \eta dn\Big(\frac{\eta}{\sqrt{2}}|\xi| \pm a; k\Big), \quad \xi \in [-\pi,\pi] \qquad (8)$$

$$\eta = \eta(\omega, Z) \in \Big(\frac{\sqrt{2}|Z|}{4} + \sqrt{\frac{8\omega - Z^2}{8}}, \sqrt{2\omega}\Big), \; k = k(Z,\omega) \in (0,1).$$

- $\lim_{Z\to 0} a(\omega, Z) = 0.$

Profile of $\phi_{\omega,Z}$, $Z \neq 0$: $\phi_{\omega,Z}(\xi) = \eta dn(\frac{\eta}{\sqrt{2}}|\xi| \pm a; k)$

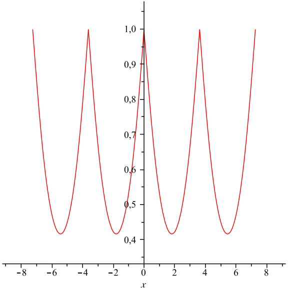

Figure : The periodic dnoidal-peak $\phi_{\omega,Z}$, $Z > 0$.

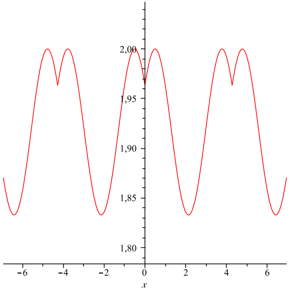

Figure : The periodic dnoidal-peak $\phi_{\omega,Z}$, $Z < 0$.

Asymptotic limit of the dnoidal-peak

- For

$$\phi_{\omega,Z}(\xi) = \eta dn\Big(\frac{\eta}{\sqrt{2}}|\xi| \pm a; k\Big), \quad \xi \in [-\pi, \pi],$$

since $\lim_{Z\to 0} a(\omega, Z) = 0$, we obtain for $\omega > \frac{1}{2}$

$$lim_{Z\to 0}\phi_{\omega,Z}(\xi) = \phi_{\omega,0}(\xi) = \eta_0 \, \mathrm{dn}\Big(\frac{\eta_0}{\sqrt{2}}\xi; k\Big).$$

Dnoidal-peak stability results

From Angulo&Ponce (2012): Let ω be large. Then,

1. For $Z > 0$, the dnoidal-peak standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is stable in $H^1_{per}([-\pi,\pi])$.
2. For $Z < 0$, the dnoidal-peak standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is unstable in $H^1_{per}([-\pi,\pi])$.
3. For $Z < 0$, the dnoidal-peak standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is stable in $H^1_{per,even}([-\pi,\pi])$.

Problems: Existence and stability of cnoidal-peak

- Existence: $\omega \to \varphi_{\omega,Z}$ of **sign changing** 2π-periodic solutions for

$$-\varphi'' - Z\delta(x)\varphi + \omega\varphi = \varphi^3, \qquad (9)$$

s.t. $\varphi_{\omega,Z} \in D(-\frac{d^2}{dx^2} - Z\delta) = D(-\Delta_Z)$, $Z \neq 0$, and

$$\begin{cases} -\varphi''_{\omega,Z}(x)+\omega\varphi_{\omega,Z}(x) = \varphi^3_{\omega,Z}(x), & \text{for } x \neq \pm 2n\pi,\ n \in \mathbb{N} \\ \lim_{Z\to 0} \varphi_{\omega,Z} = \varphi_{\omega,0} \end{cases}$$

where $\varphi_{\omega,0}$ is the cnoidal solution for (9) with $Z = 0$.

Problems: Existence and stability of cnoidal-peak

- Stability of the orbit:

$$\Omega_{\varphi_{\omega,Z}} = \{e^{i\theta}\varphi_{\omega,Z} : \theta \in [0, 2\pi)\}.$$

Cnoidal-peak family

Theorem

- $\omega \in (Z^2/4, +\infty) \to \varphi_{\omega,Z} \in H^1_{per}([-\pi,\pi])$

$$\varphi_{\omega,Z}(\xi) = bcn\Big(\sqrt{b^2-\omega}\,|\xi| \pm \theta_{\omega,Z}; k\Big), \quad \xi \in [-\pi,\pi]$$

$$b = b(\omega, Z) \in (\sqrt{2\omega}, +\infty),\ k \in (\tfrac{1}{2}, 1).$$

- $\varphi_{\omega,Z} \in D(-\frac{d^2}{dx^2} - Z\delta)$.
- *For* $\omega > Z^2/4$ *and* $\xi \in [-\pi,\pi]$

$$lim_{Z\to 0}\varphi_{\omega,Z}(\xi) = \varphi_{\omega,0}(\xi) = b_0\ cn\Big(\sqrt{b_0^2-\omega}\,\xi; k\Big).$$

Cnoidal-peak family

Theorem

The shift $\theta_{\omega,Z} = \theta(\omega, Z)$ *is defined by*

$$\theta(\omega, Z) = cn^{-1}\left(\frac{\rho(\omega, Z)}{b_{\omega,Z}}; k\right)$$

where $\rho(\omega, Z)$ *is defined by*

$$\rho^2(\omega, Z) = \frac{(2\omega - \frac{Z^2}{2}) + \sqrt{(2\omega - \frac{Z^2}{2})^2 + 4b^2_{\omega,Z}(b^2_{\omega,Z} - 2\omega)}}{2}.$$

Moreover,

$$\lim_{Z\to 0} \theta(\omega, Z) = 0.$$

Profile of cnoidal-peak family

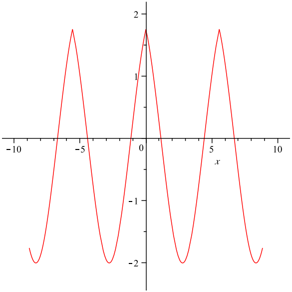

Figure : The periodic cnoidal-peak $\varphi_{\omega,Z}$, $Z > 0$.

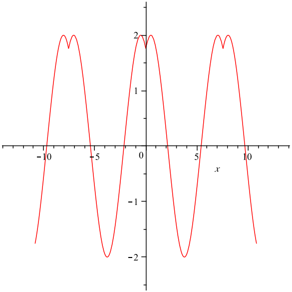

Figure : The periodic cnoidal-peak $\varphi_{\omega,Z}$, $Z < 0$.

Instability Theorem

Theorem

Let ω be large. Then for $Z < 0$ and small the cnoidal-peak standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is unstable in $H^1_{per}([-\pi,\pi])$.

Remarks:

1. For $Z > 0$ our approach does not give information about the stability of the cnoidal-peak $e^{i\omega t}\varphi_{\omega,Z}$ in $H^1_{per}([-\pi,\pi])$.
2. For $Z < 0$ our approach does not give information about the stability of the cnoidal-peak $e^{i\omega t}\varphi_{\omega,Z}$ in $H^1_{per,even}([-\pi,\pi])$.

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Grillakis&Shatah&Strauss's framework

- For a Hamiltonian system which is invariant under a one-parameter unitary group of operators (rotations), three main informations are required for a stability study:

(1) The *Cauchy problem*: The initial value problem associated to the NLS-δ equation is global well-posedness in $H^1_{per}([0, 2\pi])$.

$$E(u) = \frac{1}{2}\int |u'(x)|^2 \, dx - \frac{1}{4}\int |u(x)|^4 \, dx - \frac{Z}{2}|u(0)|^2,$$
$$Q(u) = \frac{1}{2}\int |u(x)|^2 \, dx.$$

Local/Global W.P.: $iu_t + u_{xx} + Z\delta(x)u + |u|^2 u = 0$

Theorem

For $u_0 \in H^1_{per}([0,2\pi])$, $\exists\ T = T(\|u_0\|_1) > 0$ and a unique solution $u \in C([-T,T]; H^1_{per}([0,2\pi])) \cap C^1([-T,T]; H^{-1}_{per}([0,2\pi]))$ of NLS-δ, such that $u(0) = u_0$. For each $T_0 \in (0,T)$ the mapping

$$u_0 \in H^1_{per}([0,2\pi]) \to u \in C([-T_0,T_0]; H^1_{per}([0,2\pi]))$$

is continuous. Moreover, since u satisfies the conservation of the energy and the charge, namely,

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0),$$

for all $t \in [0,T)$, we can choose $T = +\infty$.

Local W. P.: $iu_t + u_{xx} + Z\delta(x)u + |u|^2u = 0$

Proof.

1. For $-\Delta_Z$ $(=-\frac{d^2}{dx^2} - Z\delta)$, $-\Delta_Z \geqq -\beta$, where $\beta = k_Z^2$, if $Z > 0$ and $\beta = 0$ if $Z < 0$. $k_Z > 0$ with $Z = 2k_Z tanh(k_Z\pi)$.

$$-\beta = inf\{\|v_x\|^2 - Z|v(0)|^2 : \|v\| = 1, v \in H^1_{per}\}.$$

2. $\mathcal{A} \equiv \Delta_Z - \beta$ is a self-adjoint operator on $X = L^2_{per}$ and $\mathcal{A} \leqq 0$.
3. $X_{\mathcal{A}} = (H^1_{per}, \|\cdot\|_{X_{\mathcal{A}}})$ has an equivalent-norm to H^1_{per}-norm,

$$\|u\|^2_{X_{\mathcal{A}}} = \|u_x\|^2 + (\beta+1)\|u\|^2 - Z|u(0)|^2.$$

- ⋆ From Theorem 3.7.1 of Cazenave's book (Semi-linear Schrödinger equation, AMS), we obtain the l.w.p result in H^1_{per}.

Grillakis&Shatah&Strauss's framework

$$\Big(-\frac{d^2}{dx^2}-Z\delta(x)\Big)\varphi_{\omega,Z}+\omega\varphi_{\omega,Z}-\varphi^3_{\omega,Z}=0$$

(2) The *spectral study*: For ζ real-valued:

1. The self-adjoint operator $\mathcal{L}_{2,Z}$ with domain $D(-\Delta_Z)$:
$$\mathcal{L}_{2,Z}\zeta\equiv(-\frac{d^2}{dx^2}+\omega-\varphi^2_{\omega,Z})\zeta$$
has $Ker(\mathcal{L}_{2,Z})=[\varphi_{\omega,Z}]$.
2. The self-adjoint operator $\mathcal{L}_{1,Z}$ with domain $D(-\Delta_Z)$,
$$\mathcal{L}_{1,Z}\zeta\equiv(-\frac{d^2}{dx^2}+\omega-3\varphi^2_{\omega,Z})\zeta$$
has $Ker(\mathcal{L}_{1,Z})=\{0\}$ for all $Z\in\mathbb{R}-\{0\}$.
3. The number of negative eigenvalues of $\mathcal{L}_{1,Z}$ and $\mathcal{L}_{2,Z}$.

Grillakis&Shatah&Strauss's framework

(3) The *slope condition*: The sign of

$$\partial_\omega \|\varphi_{\omega,Z}\|^2 = \partial_\omega \int_{-\pi}^{\pi} \varphi^2_{\omega,Z}(\xi) d\xi.$$

Stability/Instability-Criterium

- Define $\mathcal{H}_{\omega,Z} = \begin{pmatrix} \mathcal{L}_{1,Z} & 0 \\ 0 & \mathcal{L}_{2,Z} \end{pmatrix}$
- Let $n(\mathcal{H}_{\omega,Z})$ be the number of negative eigenvalues of $\mathcal{H}_{\omega,Z}$.
- Define $p_z(\omega_0) = 1$, if $\partial_\omega \|\varphi_{\omega,Z}\|^2 > 0$ at $\omega = \omega_0$, and $p_z(\omega_0) = 0$ if $\partial_\omega \|\varphi_{\omega,Z}\|^2 < 0$ at $\omega = \omega_0$.

Theorem (Grillakis&Shatah&Strauss&Weinstein)

Suppose $Ker(\mathcal{L}_{2,Z}) = [\varphi_{\omega,Z}]$ and $Ker(\mathcal{L}_{1,Z}) = \{0\}$. Then,

1. *The cnoidal-peak standing wave $e^{i\omega_0 t}\varphi_{\omega_0,Z}$ is stable in $H^1_{per}([-\pi,\pi])$ if we have $n(\mathcal{H}_{\omega,Z}) = p_Z(\omega_0)$.*
2. *The cnoidal-peak standing wave $e^{i\omega_0 t}\varphi_{\omega_0,Z}$ is unstable in $H^1_{per}([-\pi,\pi])$ if we have $n(\mathcal{H}_{\omega,Z}) - p_Z(\omega_0)$ is odd.*

Trivial kernel for $\mathcal{L}_{1,Z}$

Theorem

Let $Z \in \mathbb{R} - \{0\}$ *and* $\omega > Z^2/4$ *and large. Then*

$$\mathcal{L}_{1,Z} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,Z}$$

has a trivial kernel on $\mathcal{D}(\mathcal{L}_{1,Z}) = D(-\Delta_Z)$.

Proof.

Follows from Floquet theory, from theory of elliptic functions and from $\frac{d}{dx}\varphi_{\omega,Z} \notin D(-\Delta_Z)$.

□

negative eigenvalues for $\mathcal{L}_{1,Z} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,Z}$

Theorem

Let $\omega > \frac{Z^2}{4}$ and ω large. Then we have,

1. *For $Z > 0$, $n(\mathcal{L}_{1,Z}) = 2$.*
2. *For $Z < 0$, $n(\mathcal{L}_{1,Z}) = 3$.*

$\star$ **Strategy**: for Z small, $\mathcal{L}_{1,Z}$ can be seen as a *real-holomorphic perturbation* of

$$\mathcal{L}_{1,0} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,0}, \quad D(\mathcal{L}_{1,0}) = H^2_{per},$$

$\varphi_{\omega,0}$ being the smooth cnoidal solution to the NLS. So, the spectrum of $\mathcal{L}_{1,Z}$ depends holomorphically on the spectrum of $\mathcal{L}_{1,0}$.

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negative eigenvalues for $\mathcal{L}_{1,Z} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,Z}$

Lemma (Spectrum of $\mathcal{L}_{1,0}$/ Angulo (2007))

For $Z = 0$, the Hill operator $\mathcal{L}_{1,0}$ defined on $L^2_{per}([0, 2\pi])$ by

$$\mathcal{L}_{1,0} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,0},$$

with domain $H^2_{per}([0, 2\pi])$ and $\omega > 0$, has exactly two simple negative eigenv. with associated even eigenf. The eigenv. zero is the third one, which is simple with eigenf. $\frac{d}{dx}\varphi_{\omega,0}$. The rest of the spectrum is positive, discrete and converging to infinity.

Proof.

Floquet theory. □

negative eigenvalues for $\mathcal{L}_{1,Z} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,Z}$

Lemma (Analyticity)

As a function of Z, $(\mathcal{L}_{1,Z})$ and $(\mathcal{L}_{2,Z})$ are a real-analytic family of self-adjoint operators in the sense of Kato.

Proof.

$$\mathcal{Q}^1_{\omega,Z}(f,g) = \int f_x g_x dx + \omega \int fg dx - Zf(0)g(0) - \int 3\varphi^2_{\omega,Z} fg dx$$
$$\mathcal{Q}^2_{\omega,Z}(f,g) = \int f_x g_x dx + \omega \int fg dx - Zf(0)g(0) - \int \varphi^2_{\omega,Z} fg dx$$

1. $D(\mathcal{Q}^i_{\omega,Z}) = H^1_{per}([-\pi,\pi])$, for all Z and $i = 1, 2$.
2. They are symmetric, bounded from below and closed.
3. $Z \to \mathcal{Q}^i_{\omega,Z}(f,f)$ is analytic for every $f \in H^1_{per}([-\pi,\pi])$.
4. The self-adjoint operators induced by $\mathcal{Q}^i_{\omega,Z}$ are $\mathcal{L}_{i,Z}$.

The number of negative eigenvalues for $\mathcal{L}_{1,Z}$ with Z small

$\star$ The spectrum of $\mathcal{L}_{1,Z}$ depends holomorphically on the spectrum of $\mathcal{L}_{1,0}$. In fact, from analytic perturb. and from the Kato-Rellich Theorem:

Lemma

There exist $Z_0 > 0$ and analytic functions $\Pi : (-Z_0, Z_0) \to \mathbb{R}$ and $\Omega : (-Z_0, Z_0) \to L^2_{per}$ such that

(i) $(\Pi(0), \Omega(0)) = (0, \frac{d}{dx}\varphi_{\omega,0})$.

(ii) *$\Pi(Z)$ is the simple isolated third eigenvalue of $\mathcal{L}_{1,Z}$ and $\Omega(Z)$ is an eigenvector for $\Pi(Z)$, with $Z \in (-Z_0, Z_0)$.*

(iii) *For Z_0 small enough, except the three first eigenvalues (simple), the spectrum of $\mathcal{L}_{1,Z}$ is positive.*

The sign of $\Pi(Z)$ for Z small

Lemma

For $Z \in (-Z_0, 0)$, $\Pi(Z) < 0$. For $Z \in (0, Z_0)$, $\Pi(Z) > 0$.

Proof.

- Since $\Pi(0) = 0$, from Taylor's theorem for $Z \in (-Z_0, Z_0)$

$$\Pi(Z) = \beta Z + O(Z^2),$$

  where $\beta \in \mathbb{R}$, $\beta = \Pi'(0)$.

$\star$ Idea $\beta > 0$: $\beta = -\frac{\omega\varphi^2_{\omega,0}(0) - \varphi^4_{\omega,0}(0)}{\|\frac{d}{dx}\varphi_{\omega,0}\|^2} + O(Z)$.

- $b_0 = \varphi_{\omega,0}(0) > \sqrt{2\omega}$: $\omega\varphi^2_{\omega,0}(0) - \varphi^4_{\omega,0}(0) = \omega b_0^2 - b_0^4 < 0$.

□

negative eigenvalues for $\mathcal{L}_{1,Z}$ for Z small

Theorem

Let $\omega > \frac{Z^2}{4}$. Then we have,

1. *For $Z > 0$ and small, $n(\mathcal{L}_{1,Z}) = 2$.*
2. *For $Z < 0$ and small, $n(\mathcal{L}_{1,Z}) = 3$.*

negative eigenv. for $\mathcal{L}_{1,Z} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,Z}$

Proof: For $Z < 0$, $n(\mathcal{L}_{1,Z}) = 3$.

- Let Z_∞ be defined by

$$Z_\infty = \inf\{r < 0 : \mathcal{L}_{1,Z} \text{ has exactly three negative eigenvalues for all } Z \in (r, 0)\}.$$

- The last Theorem implies $\mathcal{L}_{1,Z}$ has exactly three negative eigenvalues for all $Z \in (-Z_0, 0)$, so $Z_\infty \in [-\infty, 0)$.
- $Z_\infty = -\infty$. Suppose $Z_\infty > -\infty$.

□

negative eigenv. for $\mathcal{L}_{1,Z} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,Z}$

For ω large, $Ker(\mathcal{L}_{1,Z}) = \{0\}$.

- Let Γ a closed curve with $0 \in \Gamma \subset \rho(\mathcal{L}_{1,Z_\infty})$ and all the negatives eigenv. of $\mathcal{L}_{1,Z_\infty}$ belong to the inner domain of Γ.
- Since for $\xi \in \Gamma$, $Z \to (\mathcal{L}_{1,Z} - \xi)^{-1}$ is analytic, the existence of the analytic family of Riesz-projections

$$Z \to P(Z) = -\frac{1}{2\pi i}\int_\Gamma (\mathcal{L}_{1,Z} - \xi)^{-1} d\xi$$

implies for δ small, that for $Z \in [Z_\infty - \delta, Z_\infty + \delta]$

$$\dim(Rank\ P(Z)) = \dim(Rank\ P(Z_\infty)),$$

because of $\|P(Z) - P(Z_\infty)\| < 1$.

negative eigenvalues for $\mathcal{L}_{1,Z} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,Z}$

$\star$ $Z \in [Z_\infty - \delta, Z_\infty + \delta]$: $\dim(Rank\ P(Z)) = \dim(Rank\ P(Z_\infty))$

For ω large, $Ker(\mathcal{L}_{1,Z}) = \{0\}$.

- By definition of Z_∞ we obtain that $n(\mathcal{L}_{1,Z_\infty+\delta}) = 3$.
- Therefore $n(\mathcal{L}_{1,Z}) = 3$ for $Z \in (Z_\infty - \delta, 0)$, contradicting the definition of Z_∞.

$\square$

negative eigenvalues for $\mathcal{L}_{1,Z} = -\frac{d^2}{dx^2} + \omega - 3\varphi^2_{\omega,Z}$

Proof: for $Z > 0$ and ω large, $n(\mathcal{L}_{1,Z}) = 2$.

The proof is similar to the case $Z < 0$. □

Spectral analysis for $\mathcal{L}_{2,Z} = -\frac{d^2}{dx^2} + \omega - \varphi^2_{\omega,Z}$

Theorem

Let $Z \in \mathbb{R}$ small and $\omega > Z^2/4$. Then,

1. $Ker(\mathcal{L}_{2,Z}) = [\varphi_{\omega,Z}]$.
2. $n(\mathcal{L}_{2,Z}) = 1$.

Remark: $\varphi_{\omega,Z} \in D(-\Delta_Z) = \mathcal{D}(\mathcal{L}_{2,Z})$ and

$$\mathcal{L}_{2,Z}(\varphi_{\omega,Z}) = -\frac{d^2}{dx^2}\varphi_{\omega,Z} + \omega\varphi_{\omega,Z} - \varphi^3_{\omega,Z} = 0,$$

then zero is an eigenvalue for $\mathcal{L}_{2,Z}$.

Spectral analysis for $\mathcal{L}_{2,Z} = -\frac{d^2}{dx^2} + \omega - \varphi^2_{\omega,Z}$

Theorem

Let $Z \in \mathbb{R}$ small and $\omega > Z^2/4$. Then,

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Spectral analysis for $\mathcal{L}_{2,Z} = -\frac{d^2}{dx^2} + \omega - \varphi^2_{\omega,Z}$

Lemma (Spectrum of $\mathcal{L}_{2,0}$/ Angulo-2007)

For $Z = 0$, the Hill operator $\mathcal{L}_{2,0}$ defined on $L^2_{per}([0, 2\pi])$ by

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with domain $H^2_{per}([0, 2\pi])$ and $\omega > 0$, has exactly one negative eigenvalue which is simple. The eigenvalue zero is also simple with eigenfunction $\varphi_{\omega,0}$. The rest of the spectrum is positive, discrete and converging to infinity

Proof.

Floquet theory. □

$Ker(\mathcal{L}_{2,z}) = [\varphi_{\omega,Z}]$ and $n(\mathcal{L}_{2,z}) = 1$, Z small

Proof.

- It follows from the last Lemma and from that $\mathcal{L}_{2,Z}$ can be seen as a *real-holomorphic perturbation* of $\mathcal{L}_{2,0}$.

□

Slope Condition

Theorem

Let $Z \in \mathbb{R} - \{0\}$ and ω large. Then for the cnoidal-peak smooth curve $\omega \to \varphi_{\omega,Z}$ we have

$$\partial_\omega \|\varphi_{\omega,Z}\|^2 = \partial_\omega \int_{-\pi}^{\pi} \varphi^2_{\omega,Z}(\xi)d\xi > 0.$$

Therefore, $p_Z(\omega) = 1$.

Proof.

By using the theory of elliptic integrals and Jacobian elliptic functions. □

Proof of Instability Theorem:

Proof.

- $p_Z(\omega) = 1$, for all $Z \in \mathbb{R} - \{0\}$ and ω large,
- For $Z < 0$ and small, $n(\mathcal{L}_{1,Z}) = 3$ and $n(\mathcal{L}_{2,Z}) = 1$. Then

$$n(\mathcal{H}_{\omega,Z}) - p_Z(\omega) = 4 - 1 = 3.$$

Therefore we obtain **instability of cnoidal-peak in** H^1_{per} for $Z < 0$ and small, and ω large.

□

Remark: Grillakis *et al.* theory can not be applied to $Z > 0$:

- $n(\mathcal{L}_{1,Z}) = 2$ and $n(\mathcal{L}_{2,Z}) = 1$, then

$$n(\mathcal{H}_{\omega,Z}) - p_Z(\omega) = 3 - 1 = 2$$

Proof of Instability Theorem:

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THANKS!!!!!