ON A PERTURBATION OF THE BENJAMIN ONO EQUATION

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If T can be taken arbitrarily large, the Caucy problem (E) is **globally well-posed in Y.**

$$(PBO)\begin{cases} u_t + uu_x + \beta\mathcal{H}u_{xx} + \eta(\mathcal{H}u_x - u_{xx}) = 0, & x \in \mathbb{R}, \quad t \geq 0, \\ u(x,0) = \phi(x), \end{cases}$$

where β, $\eta > 0$ and $\mathcal{H}$ denotes the usual Hilbert transform given by

$$\mathcal{H}f(x) = \frac{1}{\pi}\, p.v. \int_{-\infty}^{\infty} \frac{f(y)}{y-x}\, dy\,,$$

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This equation was introduced by H. H. Chen and Y. C. Lee (1982) to describe fluid and plasma turbulence.

The Benjamin-Ono-Burgers equation which was studied by M. Otani (2005,06) as a particular case of the initial value problem for the generalized Benjamin-Ono-Burgers (gBOB) equations when $a = 0$ and $\alpha = 1$. That Cauchy problem is

$$\begin{cases} u_t + uu_x - \partial_x |D_x|^{1+a} u + |D_x|^{2\alpha} u = 0 \qquad x \in \mathbb{R}, \quad t \geq 0, \\ \qquad\qquad\qquad\qquad u(x,0) = u_0(x), \end{cases}$$

where $|D_x|^k$ is the Fourier multiplier operator with symbol $|\xi|^k$.

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Otani proved that these equations are globally well-posed in Sobolev spaces $H^s(\mathbb{R})$ for $s > -(a + 2\alpha - 1)/2$, with $a + 2\alpha \leq 3$ and $\alpha > (3 - a)/4 \geq 1/2$.

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$$\begin{cases} u_t + uu_x + \mathcal{H}u_{xx} + |D_x|^{\alpha} u = 0 \qquad x \in \mathbb{R}, \quad t \geq 0, \quad 0 \leq \alpha \leq 2, \\ u(x,0) = u_0(x), \end{cases}$$

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When $1 \leq \alpha \leq 2$, and $s < -\alpha/4$, there is not $T > 0$ such that this problem admits a unique local solution defined on the interval $[0, T]$ and such that the flow map $u_0 \mapsto u$ is of class C^3 in a neighborhood of the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.

Since the linear symbol of equation PBO is

$$i(\tau - q(\xi)) + p(\xi),$$

where $q(\xi) = \beta\xi|\xi|$ and $p(\xi) = \eta\left(\xi^2 - |\xi|\right)$, we denote by

$$E(\xi, t) = e^{iq(\xi)\, t - p(\xi)\, t},$$
$$S(t)\phi = e^{-(\beta\mathcal{H}\partial_x^2 + \eta(\mathcal{H}\partial_x - \partial_x^2))t}\phi = \left(E(\xi, t)\widehat{\phi}\right)^{\vee},$$

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Proposition

Let $\phi \in H^s(\mathbb{R})$. Then, $u(t) = S(t)\phi \in C([0, \infty), H^s(\mathbb{R}))$ is the unique solution of the linear problem. Moreover, $u \in C((0, \infty), H^\infty(\mathbb{R}))$.

We denote by U the unitary group in $H^s(\mathbb{R})$,

$$U(t) = e^{iq(\partial_x)\,t}, \qquad U(t)\phi = (e^{iq(\xi)\,t}\widehat{\phi}\,)^{\vee},$$

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Next, for given s, $b \in \mathbb{R}$ we introduce the function space $X^{s,b}_{\tau=q(\xi)}$ to be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ on $\mathbb{R}^2$ endowed with

$$\|u\|_{X^{s,b}_{\tau=q(\xi)}} = \left\| \langle\xi\rangle^s \langle\tau - q(\xi)\rangle^b \widehat{u}(\xi,\tau) \right\|_{L^2_\xi L^2_\tau}. \tag{1}$$

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By the identity,

$$(U(-t)u)^{\wedge}(\xi,\tau) = \widehat{u}(\xi,\tau + q(\xi)), \tag{2}$$

the norm $X^{s,b}_{\tau=q(\xi)}$ is written equivalently as

$$\|u\|_{X^{s,b}_{\tau=q(\xi)}} = \|U(-t)u\|_{H^{s,b}}, \qquad s,b \in \mathbb{R},$$

$$\|u\|^2_{H^{s,b}} = \int_{\mathbb{R}^2} \langle\tau\rangle^{2b} \langle\xi\rangle^{2s} |\widehat{u}(\xi,\tau)|^2 \, d\xi \, d\tau.$$

Lemma ($X^{s,b}_{\tau=q(\xi)} \hookrightarrow L^\infty_t H^s_x$)

Let $s \in \mathbb{R}$, $b > 1/2$. There exists $C > 0$, depending only on b, such that

$$\|u\|_{H^s_x} \leq C \|u\|_{X^{s,b}_{\tau=q(\xi)}}.$$

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By analogy with (1), we define the space $X^{s,b}$ provided with the norm

$$\|u\|_{X^{s,b}} = \left\| \langle \xi \rangle^s \langle i(\tau - q(\xi)) + p(\xi) \rangle^b \widehat{u}(\xi, \tau) \right\|_{L^2_\xi L^2_\tau}.$$

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From (2), we can rewrite the norm of $X^{s,b}$ as

$$\begin{aligned} \|u\|_{X^{s,b}} &= \left\| \langle \xi \rangle^s \langle i\tau + p(\xi) \rangle^b (U(-t)u)^\wedge(\xi,\tau) \right\|_{L^2_\xi L^2_\tau} \\ &\sim \|U(-t)u\|_{H^{s,b}} + \left\| \langle \xi \rangle^s \langle p(\xi) \rangle^b \widehat{u}(\xi,t) \right\|_{L^2_\xi L^2_t} \end{aligned}$$

and this shows that $X^{s,b} \hookrightarrow X^{s,b}_{\tau=q(\xi)}$.

We extended $S(t)$ to all $t \in \mathbb{R}$ by setting

$$S(t)\phi = (e^{i\,q(\xi)\,t - p(\xi)\,|t|}\widehat{\phi})^{\vee} \quad \text{for} \quad \phi \in H^s(\mathbb{R}),\ t \in \mathbb{R}.$$

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For $T > 0$, we define $X_T^{s,b}$ to be the restriction of $X^{s,b}$ on $\mathbb{R} \times [0, T]$, i.e., $X_T^{s,b}$ consists of functions $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ such that there exists $v \in X^{s,b}$ such that $v\,|_{\mathbb{R}\times[0,T]} = u$, with the norm

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We will mainly work on the integral formulation of the equation PBO,

$$u(t) = S(t)\phi - \int_0^t S(t-t')[u(t')u_x(t')]\,dt' \quad t \geq 0. \tag{3}$$

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We will apply a fixed point argument to the following truncated version:

$$u(t) = \Psi(t)\Big[S(t)\phi - \frac{\chi_{\mathbb{R}^+}(t)}{2}\int_0^t S(t-t')\partial_x(\Psi_T^2(t')\,u^2(t'))\,dt'\Big]. \tag{4}$$

Proposition

Let $s \in \mathbb{R}$ and $b \in [1/2, 1]$. There exist $C > 0$ such that

$$\|\Psi(t)\, S(t)\, \phi\|_{X^{s,b}} \leq C\, \|\phi\|_{H^{s+2(b-\frac{1}{2})}(\mathbb{R})}, \qquad \forall \phi \in H^{s+2(b-\frac{1}{2})}(\mathbb{R}).$$

Proposition

Let $s \in \mathbb{R}$, $\frac{1}{2} < b \leq 1$. *Then,*

(a.) There exists $C > 0$ *such that, for all* $\nu \in \mathcal{S}(\mathbb{R}^2)$,

$$\left\| \chi_{\mathbb{R}^+}(t)\, \Psi(t) \int_0^t S(t-t')\, \nu(t')\, dt' \right\|_{X^{s,b}} \leq$$

$$C \left[\|\nu\|_{X^{s,b-1}} + \left(\int_{\mathbb{R}} \langle \xi \rangle^{2s} |p(\xi)|^{2b-1} \left(\int_{\mathbb{R}} \frac{|(U(-t)\nu)^\wedge(\xi,\tau)|}{\langle i\tau + p(\xi) \rangle}\, d\tau \right)^2 d\xi \right)^{1/2} \right].$$

(b.) For any $0 < \delta < 1/2$ *there exists* C_δ *such that, for all* $\nu \in X^{s,b-1+\delta}$,

$$\left\| \chi_{\mathbb{R}^+}(t)\, \Psi(t) \int_0^t S(t-t')\, \nu(t')\, dt' \right\|_{X^{s,b}} \leq C_\delta \|\nu\|_{X^{s,b-1+\delta}}.$$

Proposition

Let $s \in \mathbb{R}$, $0 < \delta < \frac{1}{2}$ *and* $\frac{1}{2} \leq b \leq 1-\delta$. *Then, for all* $f \in X^{s,b-1+\delta}$,

$$t \longmapsto \int_0^t S(t-t')\,f(t')\,dt' \;\in C(\mathbb{R}^+, H^{s+2\delta}(\mathbb{R})).$$

Moreover,

$$\left\| \chi_{\mathbb{R}^+}(t)\,\Psi(t) \int_0^t S(t-t')\,f(t')\,dt' \right\|_{L^\infty(\mathbb{R}^+, H^{s+2\delta})} \leq C\,\|f\|_{X^{s,b-1+\delta}}.$$

Theorem

Let $s > -\frac{1}{2}$. There exists $b > \frac{1}{2}$, $\theta > 0$ and $\delta > 0$ such that for any $u,\ v \in X^{s,b}$ with compact support in $[-T, T]$, we have

$$\|(u\,v)_x\|_{X^{s,b-1+\delta}} \leq C\,T^{\theta}\,\|u\|_{X^{s,b}}\,\|v\|_{X^{s,b}}.$$

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Sketch of the proof:

The bilinear estimate is equivalent to show that $\forall w \in X^{-s,1-b-\delta}$

$$|\langle (uv)_x, w\rangle| \leq C\,T^{\theta}\,\|u\|_{X^{s,b}}\,\|v\|_{X^{s,b}}\,\|w\|_{X^{-s,1-b-\delta}}. \qquad (5)$$

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Setting $\tau_2 = \tau - \tau_1$, $\xi_2 = \xi - \xi_1$,

$$\begin{aligned}\sigma &= \tau - \beta\,\xi\,|\xi|,\\ \sigma_1 &= \tau_1 - \beta\,\xi_1\,|\xi_1|,\\ \sigma_2 &= \tau_2 - \beta\,\xi_2\,|\xi_2|,\end{aligned}$$

$$\widehat{f}(\xi_2,\tau_2) = \langle \xi_2\rangle^s \langle i\sigma_2 + p(\xi_2)\rangle^b \,\widehat{u}(\xi_2,\tau_2),$$
$$\widehat{g}(\xi_1,\tau_1) = \langle \xi_1\rangle^s \langle i\sigma_1 + p(\xi_1)\rangle^b \,\widehat{v}(\xi_1,\tau_1),$$
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We see that (5) is equivalent to

$$|I| \leq C\, T^\theta \, \|f\|_{L^2_\xi L^2_\tau} \, \|g\|_{L^2_\xi L^2_\tau} \, \|h\|_{L^2_\xi L^2_\tau},$$

where

$$I = \langle (uv)_x, w\rangle = C \int_{\mathbb{R}^2} \xi \, \widehat{u} * \widehat{v}(\xi,\tau) \, \bar{\widehat{w}}(\xi,\tau) \, d\xi, d\tau$$
$$= \int_{\mathbb{R}^4} \frac{\xi \, \langle \xi\rangle^s \, \bar{\widehat{h}}(\xi,\tau)}{\langle i\sigma + p(\xi)\rangle^{1-b-\delta}} \frac{\langle \xi_1\rangle^{-s} \, \widehat{g}(\xi_1,\tau_1)}{\langle i\sigma_1 + p(\xi_1)\rangle^b} \frac{\langle \xi_2\rangle^{-s} \, \widehat{f}(\xi_2,\tau_2)}{\langle i\sigma_2 + p(\xi_2)\rangle^b} \, d\xi \, d\tau \, d\xi_1 \, d\tau_1.$$

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For $0 < \epsilon \ll 1$, take $\delta = \frac{\epsilon}{2}$ and $b = \frac{1}{2} + \epsilon$, we rewritten I as

$$I = \int_{\mathbb{R}^4} \frac{\xi \langle \xi \rangle^s \overline{\widehat{h}}(\xi, \tau)}{\langle i\sigma + p(\xi) \rangle^{\frac{1}{2} - \frac{3}{2}\epsilon}} \frac{\langle \xi_1 \rangle^{-s} \widehat{g}(\xi_1, \tau_1)}{\langle i\sigma_1 + p(\xi_1) \rangle^{\frac{1}{2}+\epsilon}} \frac{\langle \xi_2 \rangle^{-s} \widehat{f}(\xi_2, \tau_2)}{\langle i\sigma_2 + p(\xi_2) \rangle^{\frac{1}{2}+\epsilon}} \, d\xi \, d\tau \, d\xi_1 \, d\tau_1.$$

Theorem (Local well-posedness)

Let $s > -1/2$. Then for any $\phi \in H^s(\mathbb{R})$ there exist $T = T(\|\phi\|_{H^s}) > 0$, $\frac{1}{2} < b < 1$, and a unique solution u of the Cauchy problem PBO satisfying

$$u \in C([0,T], H^s(\mathbb{R})) \cap C((0,T), H^\infty(\mathbb{R})),$$
$$u \in X^{s-2(b-\frac{1}{2}),b},$$
$$uu_x \in X^{s-2(b-\frac{1}{2}),b-1},$$
$$\partial_t u \in X^{s-2(b-\frac{1}{2}),b-1}.$$

Moreover, the flow map $\phi \mapsto u(t)$ is locally Lipschitz from $H^s(\mathbb{R})$ to $C([0,T], H^s(\mathbb{R})) \cap C((0,T], H^\infty(\mathbb{R})) \cap X^{s-2(b-\frac{1}{2}),b}$.

We assume $0 < T < 1$. Let $\phi \in H^s(\mathbb{R})$ with $s > -\frac{1}{2}$.

We take $0 < \epsilon \ll 1 \,:\, 0 < 3\epsilon \leq s + \dfrac{1}{2}$, and $b > \dfrac{1}{2}$ satisfying $2b - 1 = 2\epsilon$.

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Define

$$(\mathcal{A}u)(t) = \Psi(t)S(t)\phi - \frac{1}{2}\chi_{\mathbb{R}^+}(t)\Psi(t)\int_0^t S(t-t')\partial_x(\Psi_T(t')\,u(t'))^2\,dt'.$$

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Suppose u is in the ball

$$\mathbf{B}_R = \left\{ u \in X^{s-2(b-\frac{1}{2}),b} : \|u\|_{X^{s-2(b-\frac{1}{2}),b}} \leq R = 2C_0\|\phi\|_{H^s} \right\}.$$

We assume $0 < T < 1$. Let $\phi \in H^s(\mathbb{R})$ with $s > -\frac{1}{2}$.

We take $0 < \epsilon \ll 1 \, : \, 0 < 3\epsilon \leq s + \dfrac{1}{2}$, and $b > \dfrac{1}{2}$ satisfying $2b - 1 = 2\epsilon$.

Define

$$(\mathcal{A}u)(t) = \Psi(t)S(t)\phi - \frac{1}{2}\chi_{\mathbb{R}^+}(t)\Psi(t)\int_0^t S(t-t')\partial_x(\Psi_T(t')\,u(t'))^2\,dt'.$$

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Lemma

Let $s \in \mathbb{R}$ and $b > \frac{1}{2}$. For any $T \in (0,1]$, we have

$$\|\Psi_T u\|_{X^{s,b}} \leq C\, T^{\frac{1-2b}{2}} \|u\|_{X^{s,b}}.$$

Since $s - 2(b - \frac{1}{2}) \geq -\frac{1}{2} + \epsilon > -\frac{1}{2}$, it follows that for $u \in \mathbf{B}_R$

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$$
\begin{aligned}
&\|(\mathcal{A}u)(t)\|_{X^{s-2(b-\frac{1}{2}),b}} \leq \\
&\|\Psi(t)S(t)\phi\|_{X^{s-2\epsilon,b}} + \left\| \chi_{\mathbb{R}^+}(t)\frac{\Psi(t)}{2}\int_0^t S(t-t')\partial_x(\Psi_T(t')\,u(t'))^2\,dt' \right\|_{X^{s-2\epsilon,b}} \\
&\leq C_0\|\phi\|_{H^s} + C_\delta \left\|\partial_x(\Psi_T(t')\,u(t'))^2\right\|_{X^{s-2(b-\frac{1}{2}),b-1+\delta}} \\
&\leq C_0\|\phi\|_{H^s} + C_\delta\, T^\theta\, \|\Psi_T u\|^2_{X^{s-2(b-\frac{1}{2}),b}} \\
&\leq C_0\|\phi\|_{H^s} + C_1\, T^{\theta-2\epsilon}\, \|u\|^2_{X^{s-2(b-\frac{1}{2}),b}}.
\end{aligned}
$$

Therefore, for $u \in \mathbf{B}_R$, we have

$$
\|\mathcal{A}u\|_{X^{s-2(b-\frac{1}{2}),b}} \leq \frac{R}{2} + C_1\, T^{\theta-2\epsilon} R^2.
$$

Hence it follows that for $0 < T < (4RC_1)^{-\frac{1}{\theta-2\epsilon}}$, $\mathcal{A}u \in \mathbf{B}_R$.

Since $s-2(b-\frac{1}{2})\geq -\frac{1}{2}+\epsilon > -\frac{1}{2}$, it follows that for $u\in \mathbf{B}_R$

$$\begin{aligned}
&\|(\mathcal{A}u)(t)\|_{X^{s-2(b-\frac{1}{2}),b}} \leq \\
&\|\Psi(t)S(t)\phi\|_{X^{s-2\epsilon,b}} + \left\|\chi_{\mathbb{R}^+}(t)\frac{\Psi(t)}{2}\int_0^t S(t-t')\partial_x(\Psi_T(t')\,u(t'))^2\,dt'\right\|_{X^{s-2\epsilon,b}} \\
&\leq C_0\|\phi\|_{H^s} + C_\delta\left\|\partial_x(\Psi_T(t')\,u(t'))^2\right\|_{X^{s-2(b-\frac{1}{2}),b-1+\delta}} \\
&\leq C_0\|\phi\|_{H^s} + C_\delta\;T^{\theta}\;\|\Psi_T u\|^2_{X^{s-2(b-\frac{1}{2}),b}} \\
&\leq C_0\|\phi\|_{H^s} + C_1\;T^{\theta-2\epsilon}\;\|u\|^2_{X^{s-2(b-\frac{1}{2}),b}}.
\end{aligned}$$

Since $s - 2(b - \frac{1}{2}) \geq -\frac{1}{2} + \epsilon > -\frac{1}{2}$, it follows that for $u \in \mathbf{B}_R$

$$\|(\mathcal{A}u)(t)\|_{X^{s-2(b-\frac{1}{2}),b}} \leq$$

$$\|\Psi(t)S(t)\phi\|_{X^{s-2\epsilon,b}} + \left\|\chi_{\mathbb{R}^+}(t)\frac{\Psi(t)}{2}\int_0^t S(t-t')\partial_x(\Psi_T(t')\,u(t'))^2\,dt'\right\|_{X^{s-2\epsilon,b}}$$

$$\leq C_0\|\phi\|_{H^s} + C_\delta\left\|\partial_x(\Psi_T(t')\,u(t'))^2\right\|_{X^{s-2(b-\frac{1}{2}),b-1+\delta}}$$

$$\leq C_0\|\phi\|_{H^s} + C_\delta\; T^\theta\;\|\Psi_T u\|^2_{X^{s-2(b-\frac{1}{2}),b}}$$

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Therefore, for $u \in \mathbf{B}_R$, we have

$$\|\mathcal{A}u\|_{X^{s-2(b-\frac{1}{2}),b}} \leq \frac{R}{2} + C_1\; T^{\theta-2\epsilon}R^2.$$

Hence it follows that for $0 < T < (4RC_1)^{-\frac{1}{\theta-2\epsilon}}$, $\mathcal{A}u \in \mathbf{B}_R$.

Similarly, it follows for $u,\ v \in \mathbf{B}_R$

$$\begin{aligned}\|\mathcal{A}u - \mathcal{A}v\|_{X^{s-2\epsilon,b}} &\leq C_1\ T^{\theta-2\epsilon}\Big(\|u\|_{X^{s-2\epsilon,b}} + \|v\|_{X^{s-2\epsilon,b}}\Big)\|u-v\|_{X^{s-2\epsilon,b}} \\ &\leq 2C_1 R T^{\theta-2\epsilon}\|u-v\|_{X^{s-2(b-\frac{1}{2}),b}} \\ &\leq \frac{1}{2}\|u-v\|_{X^{s-2(b-\frac{1}{2}),b}},\end{aligned}$$

from which $\mathcal{A}$ is a contraction on $\mathbf{B}_R$.

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&\leq 2C_1RT^{\theta-2\epsilon}\|u-v\|_{X^{s-2(b-\frac{1}{2}),b}}\\
&\leq \frac{1}{2}\|u-v\|_{X^{s-2(b-\frac{1}{2}),b}},
\end{aligned}
$$

from which $\mathcal{A}$ is a contraction on $\mathbf{B}_R$.

Therefore there exists a unique solution $u(t)$ in $\mathbf{B}_R$ for

$$
0<T<(4RC_1)^{-\frac{1}{\theta-2\epsilon}}
$$

satisfying

$$
u(t)=\Psi(t)S(t)\phi-\frac{1}{2}\chi_{\mathbb{R}^+}(t)\Psi(t)\int_0^t S(t-t')\partial_x(\Psi_T(t')\,u(t'))^2\,dt'.
$$

It is known that

$$S(\cdot)\phi \in C([0,\infty), H^s(\mathbb{R})) \cap C((0,\infty), H^\infty(\mathbb{R}))$$

and

$$t \longmapsto \int_0^t S(t-t')\,\partial_x(u^2(t'))\,dt' \;\in C([0,T], H^{s+2\delta}(\mathbb{R})),$$

where $u \in X_T^{s-2(b-\frac{1}{2}),b}$ is the solution to (3) that we have already got.

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So we conclude that

$$u \in C([0,T], H^s(\mathbb{R})) \cap C((0,T], H^{s+2\delta}(\mathbb{R})).$$

We can deduce by induction that

$$u \in C([0,T], H^s(\mathbb{R})) \cap C((0,T], H^\infty(\mathbb{R})).$$

Theorem (Global well-posedness)

Let $s \geq 0$ and $\phi \in H^s(\mathbb{R})$. Then the supremum of all $T > 0$ for which all the assertions of Theorem above hold is infinity.

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Let $s \geq 0$ and $\phi \in H^s(\mathbb{R})$. Define $T^* = T^*(\|\phi\|_{H^s})$ by

$$T^* = \sup\Big\{ T > 0 : \exists! \text{ solution of (3) in } C([0,T], H^s(\mathbb{R})) \cap X_T^{s-2(b-\frac{1}{2}),b} \Big\}.$$

Let $u \in C([0,T^*), H^s(\mathbb{R})) \cap C((0,T^*), H^\infty(\mathbb{R}))$ be the local solution of (3) in the maximal time interval $[0, T^*)$.

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 &= (u, u_t)_0 \\
&= -(u, uu_x)_0 - \beta(u, \mathcal{H}u_{xx})_0 - \eta(u, \mathcal{H}u_x)_0 - \eta(u, u_{xx})_0 \\
&= \eta \int_{\mathbb{R}} (|\xi| - \xi^2)|\hat{u}(\xi)|^2 \, d\xi \\
&= \eta \Big(\int_{|\xi|\le 1} (|\xi| - \xi^2)|\hat{u}(\xi)|^2 \, d\xi + \int_{|\xi|>1} (|\xi| - \xi^2)|\hat{u}(\xi)|^2 \, d\xi \Big) \\
&\le \eta \int_{|\xi|\le 1} (|\xi| - \xi^2)|\hat{u}(\xi)|^2 \, d\xi \\
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\end{aligned}$$

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&\le \eta \int_{|\xi|\le 1} (|\xi| - \xi^2)|\hat{u}(\xi)|^2 \, d\xi \\
&\le \eta \int_{|\xi|\le 1} |\hat{u}(\xi)|^2 \, d\xi \le \eta \|u(t)\|_{L^2}^2 .
\end{aligned}
$$

Integrating the last relation between 0 and t and using the Gronwall's inequality we obtain a priori estimate

$$
\|u(t)\|_{L^2} \le \|\phi\|_{L^2} \, e^{\eta T^*} \equiv M, \quad \forall t \in (0, T^*).
$$

Theorem

Fix $s < -1$. Then there does not exist a $T > 0$ such that PBO admits a unique local solution defined on the interval $[0, T]$ and such that the flow-map data-solution

$$\phi \longmapsto u(t), \qquad t \in [0, T],$$

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Corollary

The flow map in the existing results for the equation PBO is not C^2 from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$, if $s < -1$.

Lemma

Let $s < -1$ and $T > 0$. Then there does not exist a space X_T continuously embedded in $C([0,T], H^s(\mathbb{R}))$ such that there exists $C > 0$ with

$$\|S(t)\phi\|_{X_T} \leq C\,\|\phi\|_{H^s(\mathbb{R})}; \qquad \phi \in H^s(\mathbb{R}), \tag{6}$$

and

$$\left\| \int_0^t S(t-t')[u(t')u_x(t')]\,dt' \right\|_{X_T} \leq C\,\|u\|_{X_T}^2; \qquad u \in X_T. \tag{7}$$

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Suppose that there exists a space X_T such that (6) and (7) hold. Take $u = S(t)\phi$ in (7). Then

$$\left\|\int_0^t S(t-t')[(S(t')\phi)(S(t')\phi_x)]\,dt'\right\|_{X_T} \leq C\,\|S(t)\phi\|_{X_T}^2.$$

Now using (6) and that X_T is continuously embedded in $C([0,T],H^s(\mathbb{R}))$ we obtain for any $t \in [0,T]$ that

$$\left\| \int_0^t S(t-t')[(S(t')\phi)(S(t')\phi_x)]\,dt' \right\|_{H^s(\mathbb{R})} \leq C\,\|\phi\|^2_{H^s(\mathbb{R})}. \tag{8}$$

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Take ϕ defined by its Fourier transform as

$$\widehat{\phi}(\xi) = N^{-s}\,\gamma^{-1/2}\,(\chi_I(\xi) + \chi_I(-\xi))$$

where I is the interval $[N, N+2\gamma]$ and $\gamma \ll N$. Note that $\|\phi\|_{H^s} \sim 1$.

Now using (6) and that X_T is continuously embedded in $C([0,T],H^s(\mathbb{R}))$ we obtain for any $t \in [0,T]$ that

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Taking $\gamma = O(1)$ it infers for $N \gg \gamma$ and any $T > 0$ that

$$\sup_{t\in[0,T]} \left\| \int_0^t S(t-t')[(S(t')\phi)(S(t')\phi_x)]\,dt' \right\|_{H^s} \gtrsim N^{-2s-2}.$$

This contradicts (8) for N large enough, since $\|\phi\|_{H^s} \sim 1$ and $-2s-2 > 0$ when $s < -1$.

Now, the purpose is to discuss the asymptotic behavior (as $|x| \to \infty$) of the solutions of the initial value problem PBO.

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Asymptotic properties of the solutions will be obtained by solving the equation in weighted Sobolev spaces.

$$\mathcal{F}_{s,r} = H^s(\mathbb{R}) \cap L^2_r(\mathbb{R}), \qquad s, r = 0, 1, 2, \ldots \quad \text{and}$$
$$\|f\|^2_{\mathcal{F}_{s,r}} = \|f\|^2_{H^s} + \|f\|^2_{L^2_r}. \tag{9}$$

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Here $L^2_r(\mathbb{R})$, $r \in \mathbb{R}$ is the collection of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|^2_{L^2_r} = \int_{\mathbb{R}} (1 + x^2)^r \, |f(x)|^2 \, dx < \infty. \tag{10}$$

We prove certain properties of the semigroup associated to the problem PBO.

Proposition

Let $\lambda \geq 0$ and $s \in \mathbb{R}$. Then,

(a.) $S(t) \in \mathbf{B}(H^s(\mathbb{R}), H^{s+\lambda}(\mathbb{R}))$ for all $t > 0$ and satisfies,

$$\|S(t)\phi\|_{s+\lambda} \leq C_\lambda \left(e^{\eta t} + (\eta t)^{-\lambda/2}\right) \|\phi\|_s \,, \tag{11}$$

where $\phi \in H^s(\mathbb{R})$ and C_λ is a constant depending only on λ. Moreover, the map $t \to S(t)\phi$ belongs to $C((0,\infty), H^{s+\lambda}(\mathbb{R}))$.

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(b.) $S : [0,\infty) \longrightarrow \mathbf{B}(H^s(\mathbb{R}))$ *is a* C^0*-semigroup in* $H^s(\mathbb{R})$. *Moreover, for every* $t \geq 0$,

$$\|S(t)\|_{\mathbf{B}(H^s)} \leq e^{\eta t}. \tag{12}$$

Lemma

Let $E(\xi,t)=e^{i\,q(\xi)\,t-p(\xi)\,t}$ *where* $p(\xi)=\eta(\xi^2-|\xi|)$ *and* $q(\xi)=\beta\xi|\xi|$. *Then,*

$$\partial_\xi E(\xi,t)=t\Big[(\eta+2i\beta\xi)\,sgn(\xi)-2\eta\xi\Big]E(\xi,t) \tag{13}$$

$$\partial_\xi^2 E(\xi,t)=2\eta t\delta+2t\Big[i\beta\,sgn(\xi)-\eta\Big]E(\xi,t)+ \\ +t^2\Big[(\eta+2i\beta\xi)\,sgn(\xi)-2\eta\xi\Big]^2E(\xi,t) \tag{14}$$

$$\partial_\xi^3 E(\xi,t)=2\eta t\delta'+4i\beta t\delta+3t^2\Big[(-2\eta^2-8i\beta\eta\xi)\,sgn(\xi)+2i\beta\eta+ \\ +4(\eta^2-\beta^2)\xi\Big]E(\xi,t)+t^3\Big[(\eta+2i\beta\xi)\,sgn(\xi)-2\eta\xi\Big]^3E(\xi,t) \tag{15}$$

Moreover, for $j \geq 4$ we have that

$$\partial_\xi^j E(\xi,t) = 2\eta t\delta^{(j-2)} + 4i\beta t\delta^{(j-3)} + \sum_{k=0}^{j-4} p_k(t)\delta^{(k)} + \\ + \sum_{k=0}^{j-1} t^k \left[r_k(\xi)\, sgn(\xi) + s_k(\xi)\right] E(\xi,t) + t^j \left[(\eta + 2i\beta\xi)\, sgn(\xi) - 2\eta\xi\right]^j E(\xi,t), \tag{16}$$

where δ is the Dirac delta function and $p_k(t)$, $r_k(\xi)$ and $s_k(\xi)$ are polynomials satisfying $deg(p_k(t)) \leq j-1$, $deg(r_k(\xi)) \leq j-2$ and $deg(s_k(\xi)) \leq j-2$.

Lemma

Suppose that $\eta > 0$, $t > 0$ and $\phi \in L_j^2$ or $S(t)\phi \in H^j$ as necessary, where $j \in \mathbb{N}$.

$$\left\| \partial_\xi^j \widehat{\phi}(\xi) \right\|_0 \leq C_j \|\phi\|_{L_j^2} \tag{17}$$

$$\left\| \xi^j E(\xi,t) \widehat{\phi}(\xi) \right\|_0 \leq \|S(t)\phi\|_{H^j} \tag{18}$$

$$\left\| \xi^k E(\xi,t) \partial_\xi^j \widehat{\phi}(\xi) \right\|_0 \leq C_k (e^{\eta t} + (\eta t)^{-k/2}) \|\phi\|_{L_j^2}; \quad k \geq 0 \tag{19}$$

$$\left\| \partial_\xi^k E(\xi,t) \partial_\xi^j \widehat{\phi}(\xi) \right\|_0 \leq \left(p_k(t) e^{\eta t} + \sum_{l=0}^{3k-2} C_{l,\eta} t^{(l-k+2)/2} \right) \|\phi\|_{L_j^2} \tag{20}$$

$k \geq 2$ and $(\partial_\xi^j \widehat{\phi})(0) = 0$ for $j = 0, 1, 2, \cdots$ it is a sufficient condition to obtain (20).

Proposition

Let $\eta > 0$ and $\beta > 0$ be fixed. Then,
(a.) $S : [0, +\infty) \longrightarrow \mathbf{B}(\mathcal{F}_{r,r})$, $r = 0, 1$, is a C^0-semigroup and satisfies the estimate,

$$\|S(t)\phi\|_{\mathcal{F}_{r,r}} \leq \left(e^{\eta t}\Theta_r(t) + C_{\eta,\beta} t^{r/2}\right) \|\phi\|_{\mathcal{F}_{r,r}}, \tag{21}$$

for all $\phi \in \mathcal{F}_{r,r}$, where $\Theta_r(t)$ is a polynomial of degree r with positive coefficients that depend only on η, β and r.

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for all $\phi \in \mathcal{F}_{r,r}$, *where* $\Theta_r(t)$ *is a polynomial of degree* r *with positive coefficients that depend only on* η, β *and* r.
(b.) If $r \geq 2$ *and* $\phi \in \mathcal{F}_{r,r}$, *the function* $S(t)\phi$ *belongs to* $C([0, \infty); \mathcal{F}_{r,r})$ *if, and only if,*

$$(\partial_\xi^j \widehat{\phi})(0) = 0, \qquad\qquad j = 0, 1, 2, \cdots, r-2. \tag{22}$$

In this case we have the next estimate

$$\|S(t)\phi\|_{\mathcal{F}_{r,r}} \leq \left(e^{\eta t}\Theta_r(t) + \sum_{l=0}^{3r-2} C_{l,\eta,\beta}\, t^{(l-r+2)/2} \right) \|\phi\|_{\mathcal{F}_{r,r}}, \tag{23}$$

Theorem

Let $\eta > 0$ *and* $\beta > 0$ *fixed and* $\phi \in \mathcal{F}_{s,r}$ *with* $s, r \in \mathbb{N}$ *and* $s \geq r$.

Theorem

Let $\eta > 0$ and $\beta > 0$ fixed and $\phi \in \mathcal{F}_{s,r}$ with $s, r \in \mathbb{N}$ and $s \geq r$.
If $r = 0, 1$ the unique solution of the linear problem associated to PBO in $\mathcal{F}_{s,r}$ is given by $u(t) = S(t)\phi$.

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If $r = 0, 1$ the unique solution of the linear problem associated to PBO in $\mathcal{F}_{s,r}$ is given by $u(t) = S(t)\phi$.
If $r \geq 2$, the linear problem associated to PBO has a solution in $\mathcal{F}_{s,r}$ if, and only if,

$$(\partial_\xi^j \widehat{\phi})(0) = 0, \qquad j = 0, 1, 2, \cdots, r-2.$$

is satisfied. In this case the solution is unique and is again given by $u(t) = S(t)\phi$.

Now let us enunciate a global result for the initial value problem PBO in $\mathcal{F}_{2,1}(\mathbb{R})$.

Theorem

Let $\phi \in \mathcal{F}_{2,1}(R)$. Then there exists an unique solution of the problem PBO, $u \in C([0,\infty);\mathcal{F}_{2,1}(\mathbb{R}))$ such that $\partial_t u \in C(0,\infty;\mathcal{F}_{0,1}(\mathbb{R}))$.

Theorem

Let β, $\eta > 0$ be fixed and let $T > 0$. Assume that $u \in C([0, T]; \mathcal{F}_{2,2}(\mathbb{R}))$ is the solution of PBO. Then, $\widehat{u}(t, 0) = 0$, for all $t \in [0, T]$.

Theorem

Let β, $\eta > 0$ be fixed and let $T > 0$. Assume that $u \in C([0, T]; \mathcal{F}_{2,2}(\mathbb{R}))$ is the solution of PBO. Then, $\widehat{u}(t, 0) = 0$, for all $t \in [0, T]$.

Theorem

Let β, $\eta > 0$ be fixed and let $T > 0$. Assume that $u \in C([0, T]; \mathcal{F}_{3,3}(\mathbb{R}))$ is the solution of PBO. Then, $u(t) = 0$, for all $t \in [0, T]$.

Theorem

Let β, $\eta > 0$ be fixed and let $T > 0$. Assume that $u \in C([0,T];\mathcal{F}_{2,2}(\mathbb{R}))$ is the solution of PBO. Then, $\widehat{u}(t,0) = 0$, for all $t \in [0,T]$.

Theorem

Let β, $\eta > 0$ be fixed and let $T > 0$. Assume that $u \in C([0,T];\mathcal{F}_{3,3}(\mathbb{R}))$ is the solution of PBO. Then, $u(t) = 0$, for all $t \in [0,T]$.

We prove that if the solution $u(t)$ is sufficiently smooth $(u(t) \in H^3(\mathbb{R}))$ and falls off sufficiently fast as $|x| \to \infty$ $(u(t) \in L^2_3(\mathbb{R}))$ for all $t \in [0,T]$, then $u(t) = 0$, for all $t \in [0,T]$.

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OBRIGADO A TODOS!