# Blow-up on manifolds for the nonlinear Schrödinger equation

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October, 31 2013

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Workshop on Nonlinear Dispersive Equations, Campinas

# Euclidean $L^2$-critical theory

Consider the one dimensional equation

$$i\partial_t u + \Delta u = -|u|^4 u, \quad t > 0, \quad x \in \mathbb{R}.$$

**Local well-posedness:** This equation is well-posed in $H^1$: if $u_0 \in H^1(\mathbb{R})$, there exists a maximal time $T > 0$ and a solution $u \in \mathcal{C}([0, T), H^1(\mathbb{R})) \cap \mathcal{C}^1([0, T), H^{-1}(\mathbb{R}))$ with $u(0) = u_0$ and with the criteria

$$T < \infty \text{ implies } \|\nabla u(t)\|_{L^2} \to \infty.$$

# Goal

**Blow up problematic:**

- Estimate blow-up rate $\|\nabla u(t)\|_{L^2}$,
- Find singularities (points of concentration of mass),
- How much mass is concentrated?

## $H^1$ known blow-up regimes

- **Pseudo-conformal regime**:

$$\|\nabla u(t)\|_{L^2(\mathbb{R})} \sim \frac{1}{T-t} \qquad \text{as } t \to T,$$

- **Log-log regime (Perelman, Merle-Raphaël)**:

$$\|\nabla u(t)\|_{L^2(\mathbb{R})} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{1/2} \qquad \text{as } t \to T.$$

# Questions

Do these regimes persist in other geometries?

- lack of strong dispersion (losses in Strichartz estimates)

- no smoothing effect

- loss of symmetries (translation in space, scaling, ...) and remarkable identity (Virial relation)

For the **pseudo-conformal regime**: Banica-Carles-Duyckaerts

For the **log-log regime**: Planchon-Raphaël

# Quintic Schrödinger equation on surfaces

$$i\partial_t u + \Delta u = -|u|^4 u, \qquad t > 0, \qquad x \in M,$$

where $(M, g)$ is a two dimensional complete Riemannian manifold.
**Hamiltonian structure:** $L^2$ and $H^1$ conservation laws:

$$M(u(t)) := \int_M |u(t)|^2 = M(u(0)),$$

$$E(u(t)) = \frac{1}{2}\int_M |\nabla u(t)|^2 - \frac{1}{6}\int_M |u(t)|^6 = E(u(0)).$$

**Local well-posedness in $H^1$ (Burq-Gérard-Tzvetkov):** if $u_0 \in H^1(M)$, there exists a maximal time $T > 0$ and a solution $u \in \mathcal{C}([0, T), H^1(M)) \cap \mathcal{C}^1([0, T), H^{-1}(M))$ with $u(0) = u_0$ and with the criteria

$$T < \infty \text{ implies } \|\nabla u(t)\|_{L^2} \to \infty.$$

# Log-log regime with radial symmetry

**Theorem**

*Let $(M, g)$ be a rotationally symmetric surface with $g = dr^2 + h^2(r)d\theta^2$, $0 < r < \rho \leq \infty$. Assume in the non-compact case ($\rho = \infty$) $h'(r) \leq Ch(r)$ for $r$ large. Consider the equation on $M$ ($\dim(M) = 2$):*

$$i\partial_t u + \Delta u = -|u|^4 u, \quad t > 0.$$

*Then there exists an open set $\mathcal{P}$ of $H^2_{\mathrm{rad}}(M)$ such that if $u(0) \in \mathcal{P}$ then $u$ blows up with the log log speed on a set $\{x \in M, r(x) = r_0\}$ for some $r_0 = r(pole) > 0$.*

**Prototypes.** Compact: sphere
Non-compact: hyperbolic space
Euclidean space (treated in $H^1_{\mathrm{rad}}(\mathbb{R}^2)$ by P. Raphaël).

# Heuristic

In radial coordonates and for a radial solution, the equation becomes

$$i\partial_t u + \partial_r^2 u + \frac{h'(r)}{h(r)}\partial_r u = -|u|^4 u, \quad u(t,x) = u(t,r),$$

• Outside poles

$$\frac{h'(r)}{h(r)}\partial_r \ll \partial_r^2.$$

The equation is almost $L^2$-critical outside poles: we prove a radial Sobolev embedding on $M$: for a radial function $u$ on $M$:

$$\|u\|_{L^p(\varepsilon<r(x)<\rho-\varepsilon)} \leq C(\varepsilon)\|u\|_{H^s(\varepsilon<r(x)<\rho-\varepsilon)}, \quad p \leq p^* = \frac{2}{1-2s}.$$

• Near poles: we are outside the blow-up set $\Rightarrow$ better estimates on $u$: "good" $H^{1/2}$ estimate.

# Modulation

**Splitting the dynamic:** we choose $u_0$ such that $\|u_0\|_{L^2(M)} \sim \|Q\|_{L^2(\mathbb{R})}$ so that until a time $t_1 > 0$:

$$u(t,r) = \frac{1}{\sqrt{\lambda(t)}}\left(Q_{b(t)}\left(\frac{r-r(t)}{\lambda(t)}\right) + \varepsilon\left(t, \frac{r-r(t)}{\lambda(t)}\right)\right)e^{i\gamma(t)},$$

with if $y = (r - r(t))/\lambda(t)$ and for $k = 0, 1, 2$:

$$\int |\partial_y^k \varepsilon(t)|^2 \mu(y) dy < \infty, \quad \mu(y) = h(\lambda(t)y + r(t)) 1_{\left\{-\frac{r(t)}{\lambda(t)} \leq y \leq \frac{\rho - r(t)}{\lambda(t)}\right\}}(y),$$

with orthogonality conditions:

$$\mathrm{Re}(\varepsilon(t), y^2 Q_{b(t)}) = \mathrm{Re}(\varepsilon(t), yQ_{b(t)}) = 0,$$
$$\mathrm{Im}(\varepsilon(t), \Lambda Q_{b(t)}) = \mathrm{Im}(\varepsilon(t), \Lambda^2 Q_{b(t)}) = 0.$$

where $\Lambda = \frac{1}{2} + y\partial_y$.

$Q_b$ is a troncated version of refined profiles:

$$\partial_y^2 Q_b - Q_b + ib\Lambda Q_b + |Q_b|^4 Q_b = \mathcal{O}(e^{-\frac{\pi}{|b|}}) \ll 1.$$

# Control of the finite dimensional part

**Modulation "equations":** Time rescaling: $s = \int_0^t \frac{d\tau}{\lambda^2(\tau)}$. Then

$$\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s| + \left|\frac{r_s}{\lambda}\right| \leq C\mathcal{E}(s) + \Gamma_b^{1-},$$

$$\left|\gamma_s - 1 - \frac{(\mathrm{Re}\ \varepsilon, L_+(\Lambda^2 Q))}{\|\Lambda Q\|_{L^2}^2}\right| \leq \delta\mathcal{E}^{1/2}(s) + \Gamma_b^{1-},$$

where

$$\delta \ll 1, \qquad \Gamma_b \sim e^{-\frac{\pi}{|b|}}, \qquad \text{as } b \to 0,$$

$$\mathcal{E}(s) = \int |\partial_y \varepsilon(s,y)|^2 h(\lambda(s)y + r(s))dy + \int_{|y| \leq \frac{10}{b(s)}} |\varepsilon(s,y)|^2 e^{-|y|} dy,$$

and $(L_+, L_-)$ is the linearized operator near $Q$:

$$L_- = -\partial_y^2 + 1 - 5Q^4, \qquad L_+ = -\partial_y^2 + 1 - Q^4.$$

Extra terms due to $\frac{h'}{h}\partial_r$ are controled by the smallness of $\lambda(s)$: $\lambda \ll \Gamma_b$.

# Control of the infinite dimensional part, I

**Goal:** use a virial type argument to control the rest $\varepsilon$

**Heuristic:** $u$ satisfies the approximation (at least outside poles):

$$i\partial_t u + \partial_{rr} u \sim -|u|^4 u.$$

For the 1D Euclidean equation

$$i\partial_t u + \Delta u = -|u|^4 u, \qquad x \in \mathbb{R},$$

we have the **Virial relation**: if $u \in \Sigma := \{u \in H^1, xu \in L^2\}$ then

$$\frac{d^2}{dt^2} \int |x|^2 |u|^2 dx = 4 \frac{d}{dt} \mathrm{Im} \left( \int x \cdot \nabla u \bar{u} \right) = 16 E(u_0).$$

# Control of the infinite dimensional part, II

Thus, we expect:

$$\frac{d}{dt}\mathrm{Im}\left(\int \phi(r) r\partial_r u\overline{u}\right) \sim 4E_0,$$

where $\phi$ is a cut-off function avoiding poles. We expand in term of $\varepsilon$ and use three arguments:

• almost coercivity of the second order part (coercivity modulo negative direction)

• conservation laws (needs an $H^{1/2}$ control)

• orthogonality conditions

**Conclusion.** Virial estimate:

$$Db_s \geq \mathcal{E}(s) - \Gamma_b^{1-}.$$

# Smallness of the critical norm near poles, I

To obtain a control from the conservation of energy, we need to prove:

$$\int |\tilde{u}|^6 \ll \int |\nabla \tilde{u}(t)|^2. \quad \tilde{u}(t,r) = \frac{1}{\sqrt{\lambda(t)}} e^{i\gamma(t)} \varepsilon\left(t, \frac{r - r(t)}{\lambda(t)}\right).$$

- **Outside poles:** this is the conservation of mass
- **Near poles:** $H^{1/2}$ control. Derivation of a **pseudo-energy** $E_2$ at level $H^2$ with

$$\|u(t)\|^2_{H^2(M)} \leq CE_2(u(t)),$$

and

$$\frac{d}{dt} E_2(u(t)) \leq C(\|u(t)\|_{H^{1/2}}, \|u(t)\|_{H^1}, \|u(t)\|_{H^{3/2}}) E_2(u(t))^{1-\theta}, \; \theta \in (0,1)$$

# Smallness of the critical norm near poles, II

This implies an $H^2$ estimate

$$\|u(t)\|_{H^2(M)} \leq \frac{1}{\lambda(t)^{2+\eta}}, \qquad \text{for some } \eta > 0.$$

Moreover, for all $0 < a_2 < a_1 < b_1 < b_2$, and $t > 0$,

$$\begin{aligned} \|D^s u\|_{L^\infty_{[0,t)} L^2(a_1,b_1)} &\leq C\Big(\|D^s u(0)\|_{L^2(a_2,b_2)} + \|u\|_{L^2_{[0,t)} H^{\max(1,s+\frac{1}{2})}(a_2,b_2)} \\ &\quad + \|D^s(u|u|^4)\|_{L^1_{[0,t)} L^2(a_2,b_2)}\Big), \end{aligned}$$

we get

$$\|u(t)\|_{H^{1/2}(|r-1|>1/2)} \ll 1.$$

# Integration of modulation equations

The integration of

$$b_s \sim e^{-\frac{\pi}{b}}, \qquad \frac{\lambda_s}{\lambda} \sim -b, \quad \left|\frac{r_s}{\lambda}\right| \ll 1,$$

gives with $s = \int_0^t \frac{d\tau}{\lambda^2(\tau)}$:

$$\lambda(t) \sim \frac{1}{\|\nabla u(t)\|_{L^2}} \sim C\left(\frac{T-t}{\log|\log(T-t)|}\right)^{1/2},$$

and $r_t$ integrable thus $r(t) \not\to r(\mathrm{poles})$.

**Thank you !!**