On the Schrödinger equation with singular potentials

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Schrödinger equation with singular potentials

We consider the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u - \mu(x)u = F(u), \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $t \in \mathbb{R}$, $\lambda = \pm 1$, μ is a given potential, and:

- In the continuous case $x \in \mathbb{R}^n$, $F(u) = \lambda |u|^{\rho-1} u$ with $\rho > 1$, or $F(u) = \lambda u^\rho$, with $\rho \in \mathbb{N}$;
- In the periodic case $x \in \mathbb{T}^n$, $F(u) = \lambda |u|^{\rho-1} u$ with $\rho \in \mathbb{N}$ odd, or $F(u) = \lambda u^\rho$ with $\rho \in \mathbb{N}$.

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- The second type is bounded potentials that do not decay to zero or go to zero very slowly at infinity.

Delta-type potentials

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- From an experimental viewpoint, nanoscale devises have caused an interest in point-like impurities (defects) that are associated to Delta-type potentials.
- We have the case repulsive ($\sigma > 0$) and attractive ($\sigma < 0$).

Delta-type potentials cont...

Results on fundamental solutions, global existence in H^s ($s \geq 0$), standing waves, and stability have been obtained in dimension $n = 1$ by several authors.

See e.g. Albeverio-Gestezy-Krohn-Holden (Texts Monog. Phys. '88), Albeverio-Brzezniak-Dabrowski(JFA 1995), Caudrelier-Mintchev-Ragoucy (J. Math. Phys '05), Hölmer-Marzuola-Zworski (CMP '07), Fukuizumi-Ohta-Ozawa (AIHP '08), Adami-Noja (CMP '09), Datchev-Hölmer (CPDE'09), Kovarik-Sacchetti (J.Phys.A '10), Adami-Noja-Visciglia (DCDS-B '13), among others.

Delta-type potentials cont...

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For $n = 2, 3$, the fundamental solution is well known (see Albeverio-Gestezy-Krohn-Holden '88), however there is no good formula depending explicitly on $e^{i\Delta t}\phi$.

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and so they could be realized in the physical space in any region with finite volume, though some of them may have infinite L^2-norm.

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Precisely, the Banach space $\mathcal{L}_{\vartheta}^{\infty}$ of all Bochner measurable functions $u : \mathbb{R} \to L^{(\rho+1,\infty)}$ endowed with the norm

$$\|u\|_{\mathcal{L}_{\vartheta}^{\infty}} = \sup_{-\infty<t<\infty} |t|^{\vartheta} \|u(t)\|_{(\rho+1,\infty)},$$

where $\vartheta = \frac{1}{\rho-1} - \frac{1}{2(\rho+1)}$.

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Define also the initial data class $\mathcal{E}_0$ as the set of all $u \in \mathcal{S}'(\mathbb{R})$ such that the norm

$$\|u_0\|_{\mathcal{E}_0} = \sup_{-\infty<t<\infty} |t|^\vartheta \|G_\sigma(t)u_0\|_{(\rho+1,\infty)} < \infty,$$

where $G_\sigma(t)$ is the linear group associated to (1).

Existence and asymptotic stability...

Based on L^p-spaces and time-decay estimates for the associated linear group, spaces like $\mathcal{L}^{\infty}_{\vartheta}$ were first used by Kato-Fujita ('62 and '84) and F. Weissler ('80) in the context of Navier-Stokes and semilinear parabolic equations.

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See also Cazenave-Vega-Vilela (CCM '01) for another approach in weak-L^p spaces via Strichartz type estimates.

Existence and asymptotic stability...

Our results read as follows.

Theorem (A)

(Global-in-time existence) Let $n = 1$, $\sigma \geqq 0$, $\rho_0 = \frac{3+\sqrt{17}}{2}$, *and* $\rho_0 < \rho < \infty$. *There is* $\varepsilon > 0$ *such that if* $\|u_0\|_{\mathcal{E}_0} \leq \varepsilon$ *then* (1) *has a unique global-in-time mild solution* $u \in \mathcal{L}^\infty_\vartheta$ *satisfying* $\|u\|_{\mathcal{L}^\infty_\vartheta} \leq 2\varepsilon$.

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Theorem (B)

(Asymptotic Stability) Let u *and* v *be two solutions obtained from Theorem (A) with initial data* u_0 *and* v_0, *respectively. We have that*

$$\lim_{|t|\to\infty} |t|^{\vartheta} \|u(\cdot,t) - v(\cdot,t)\|_{(\rho+1,\infty)} = 0$$

if only if $\lim_{|t|\to\infty} |t|^{\vartheta} \|G_{\sigma}(t)(u_0 - v_0)\|_{(\rho+1,\infty)} = 0$. *This last condition holds, in particular, for* $u_0 - v_0 \in L^{(\frac{\rho+1}{\rho},\infty)}$.

Some steps in the proof of Thm (A)

The IVP is formally converted to (mild solutions)

$$u(t) = G_\sigma(t)u_0 - i\lambda \int_0^t G_\sigma(t-s)[|u(s)|^{\rho-1}u(s)]ds. \qquad (2)$$

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For $\mu = \sigma\delta$, $\sigma \geqq 0$, and $n = 1$, Holmer-Marzuola-Zworski (CMP 2007) proved the formula (there are similar ones for the other potentials)

$$G_\sigma(t)\phi(x) = e^{it\Delta}(\phi * \tau_\sigma)(x)\chi_+^0 + \left[e^{it\Delta}\phi(x) + e^{it\Delta}(\phi * \rho_\sigma)(-x)\right]\chi_-^0 \qquad (3)$$

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where

$$\rho_\sigma(x) = -\frac{\sigma}{2}e^{\frac{\sigma}{2}x}\chi_-^0, \quad \tau_\sigma(x) = \delta(x) + \rho_\sigma(x),$$

with χ_+^0 and χ_-^0 the characteristic function of $[0, +\infty)$ and $(-\infty, 0]$, respectively.

Some steps in the proof...

From (3) and following Ferreira-VillamizarRoa-Silva (PAMS '09), one can obtain the dispersive estimate in Lorentz spaces

$$\|G_\sigma(t)f\|_{(p',d)} \leq C|t|^{-\frac{1}{2}(\frac{2}{p}-1)}\,\|f\|_{(p,d)}\,. \tag{4}$$

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From (4) and Hölder inequality in weak-L^p spaces, one can prove that the nonlinear part $\mathcal{N}(u)$ of (2) verifies

$$\|\mathcal{N}(u)-\mathcal{N}(v)\|_{\mathcal{L}^\infty_\vartheta} \leq K\|u-v\|_{\mathcal{L}^\infty_\vartheta}(\|u\|^{\rho-1}_{\mathcal{L}^\infty_\vartheta}+\|v\|^{\rho-1}_{\mathcal{L}^\infty_\vartheta}). \tag{5}$$

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Using (4) and (5), one proves that the map

$$\Psi(u) = G_\sigma(t)u_0 - i\lambda \int_0^t G_\sigma(t-s)[|u(s)|^{\rho-1}u(s)]ds$$

is a contraction on a small ball of $\mathcal{L}^\infty_\vartheta$.

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- In the case $\mu = \delta'$, if a homogeneous function of degree $-\frac{2}{\rho-1}$ belonged to $\mathcal{E}_0$ then one could prove existence of self-similar solutions and asymptotic self-similar ones by means of Theorem (A) and (B).

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- The case $\mu = \delta$ for $n = 2$ is more delicate. Here, besides needing homogeneous data in $\mathcal{E}_0$, one would need the dispersive estimate (4) with $n = 2$ which is not known to be true.

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(Local-in-time solutions) Let $n = 1$, $1 < \rho < \rho_0$, $d_0 = \frac{1}{2}(\frac{\rho-1}{\rho+1})$, and $d_0 < \zeta < \frac{1}{\rho}$.

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For $0 < T < \infty$, consider the Banach space $\mathcal{L}_\zeta^T$ of all Bochner measurable functions $u : (-T, T) \to L^{(\rho+1,\infty)}$ endowed with the norm

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A local-in-time existence result in $\mathcal{L}_\zeta^T$ can be proved for (1) by considering $u_0 \in L^{(\frac{\rho+1}{\rho},\infty)}(\mathbb{R})$ and small $T > 0$.

Nondecaying potentials and periodic solutions

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We consider the Banach space

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with norm $\|f\|_{\mathcal{I}} = \|\widehat{f}\,\|_{\mathcal{M}}$, and its periodic version

$$\mathcal{I}_{per} = \{f \in \mathcal{D}'(\mathbb{T}^n) : \widehat{f} \in l^1(\mathbb{Z}^n)\}$$

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with the norm $\|f\|_{\mathcal{I}_{per}} = \|\widehat{f}\,\|_{l^1(\mathbb{Z}^n)}$.

In general $u_0 \in \mathcal{I}$ may not belong to $L^p(\mathbb{R}^n)$, nor to $L^{p,\infty}(\mathbb{R}^n)$, with $p \neq \infty$. In particular, $u_0 \in \mathcal{I}$ may have infinite L^2-mass. Also, $\mu \equiv 1$ then $\widehat{\mu} = \delta \in \mathcal{M}(\mathbb{R}^n)$.

Nondecaying potentials and periodic solutions

Our local-in-time well-posedness result in $\mathcal{I}$ reads as follows.

Theorem (C)

(Periodic case) Let $1 \leq \rho < \infty$*,* $u_0 \in \mathcal{I}_{per}$*, and* $\mu \in \mathcal{I}_{per}$*. There is* $T > 0$ *such that the IVP* (1) *has a unique mild solution* $u \in L^\infty((-T, T); \mathcal{I}_{per})$ *satisfying*

$$\sup_{t \in (-T,T)} \|u(\cdot, t)\|_{\mathcal{I}_{per}} \leq 2 \|u_0\|_{\mathcal{I}_{per}} .$$

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(Nonperiodic case) Let $u_0 \in \mathcal{I}$ *and* $\mu \in \mathcal{I}$. *The same conclusion of item (1) holds true by replacing* $\mathcal{I}_{per}$ *by* $\mathcal{I}$.

Some steps of the proof of Thm (C)

The IVP is formally converted to (mild solution)

$$u(t) = S_{per}(t)u_0 + B_{per}(u) + L_{\mu,per}(u), \tag{6}$$

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where the operators are defined via Fourier transform in $\mathcal{D}'(\mathbb{T}^n)$:

$$S_{per}(t)u_0 = \sum_{m\in\mathbb{Z}^n} \widehat{u}_0(m)e^{-4\pi^2 i|m|^2 t}e^{2\pi i x\cdot m}, \tag{7}$$

$$\widehat{L_{\mu,per}(u)}(m,t) = -i\int_0^t e^{-4\pi^2 i|m|^2(t-s)}(\widehat{\mu} * \widehat{u})(m,s)ds \tag{8}$$

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$$\widehat{B_{per}(u)}(m,t) = -i\lambda\int_0^t e^{-4\pi^2 i|m|^2(t-s)}(\underbrace{\widehat{u}*\widehat{u}*\ldots*\widehat{u}}_{\rho-times})(m,s)ds, \tag{9}$$

where the symbol $*$ denotes the discrete convolution

$$\widehat{f}*\widehat{g}(m) = \sum_{\xi\in\mathbb{Z}^n} \widehat{f}(m-\xi)\widehat{g}(\xi).$$

Some steps in the proof...

A basic tool is the Young inequality for measures and discrete convolutions:

$$\|\mu * \nu\|_{\mathcal{M}} \leq \|\mu\|_{\mathcal{M}} \|\nu\|_{\mathcal{M}} \tag{10}$$

$$\|f * g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1} . \tag{11}$$

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The operator $L_{\mu,per}$ can be estimated as

$$\begin{aligned} \|L_{\mu,per}(u)\|_{\mathcal{I}_{per}} &= \left\| \widehat{L_{\mu,per}(u)} \right\|_{l^1(\mathbb{Z}^n)} \\ &\leq \int_0^t \sum_{m\in\mathbb{Z}^n} |(\widehat{\mu} * \widehat{u})(m,s)| \, ds \\ &\leq \int_0^t \|\widehat{\mu}\|_{l^1(\mathbb{Z}^n)} \|\widehat{u}(\cdot,s)\|_{l^1(\mathbb{Z}^n)} \, ds \\ &\leq T \|\mu\|_{\mathcal{I}_{per}} \|u\|_{L^\infty(0,T;\mathcal{I}_{per})} . \end{aligned}$$

Some steps in the proof...

By elementary convolution properties and Young inequality,

$$\left\| \underbrace{(\widehat{u} * \widehat{u} * \ldots * \widehat{u})}_{\rho - times} - \underbrace{(\widehat{v} * \widehat{v} * \ldots * \widehat{v})}_{\rho - times} \right\|_{l^1(\mathbb{Z}^n)}$$

$$\leq \left\| [(\hat{u} - \hat{v}) * \widehat{u} * \ldots * \widehat{u} + \ldots + \hat{v} * \widehat{v} * \ldots * (\widehat{u} - \widehat{v}) \right\|_{l^1(\mathbb{Z}^n)}$$

$$\leq \|(\hat{u} - \hat{v})\|_{l^1} \|\widehat{u}\|_{l^1}^{\rho-1} + \|(\hat{u} - \hat{v})\|_{l^1} \|\widehat{u}\|_{l^1}^{\rho-2} \|\widehat{v}\|_{l^1} + \ldots$$

$$+ \|(\hat{u} - \hat{v})\|_{l^1} \|\widehat{u}\|_{l^1} \|\widehat{v}\|_{l^1}^{\rho-2} + \|(\hat{u} - \hat{v})\|_{l^1} \|\widehat{v}\|_{l^1}^{\rho-1}$$

$$\leq K \|(\hat{u} - \hat{v})\|_{l^1} \left(\|\widehat{u}\|_{l^1}^{\rho-1} + \|\widehat{v}\|_{l^1}^{\rho-1} \right)$$

Some steps in the proof...

It follows that

$$
\begin{aligned}
&\|B_{per}(u)(t) - B_{per}(v)(t)\|_{\mathcal{I}_{per}} \\
&\leq \left\| \int_0^t e^{-4\pi^2 i|\xi|^2(t-s)} \left[\underbrace{(\widehat{u} * \widehat{u} * \ldots * \widehat{u})}_{\rho-times} - \underbrace{(\widehat{v} * \widehat{v} * \ldots * \widehat{v})}_{\rho-times} \right] ds \right\|_{l^1} \\
&\leq K \int_0^t \|\hat{u} - \hat{v}\|_{l^1} \left(\|\widehat{u}\|_{l^1}^{\rho-1} + \|\widehat{v}\|_{l^1}^{\rho-1} \right) ds \\
&\leq KT \|u - v\|_{L^\infty(0,T;\mathcal{I}_{per})} \left(\|u\|_{L^\infty(0,T;\mathcal{I}_{per})}^{\rho-1} + \|v\|_{L^\infty(0,T;\mathcal{I}_{per})}^{\rho-1} \right).
\end{aligned}
$$

Some steps in the proof...

It follows that

$$
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\end{aligned}
$$

Now one can show that

$$
\Psi(u) = S_{per}(t)u_0 + B_{per}(u) + L_{\mu,per}(u) \tag{12}
$$

has a fixed point in $L^\infty((-T,T);\mathcal{I}_{per})$ for $T > 0$ small enough.

Some comments

- Let us denote by $\mathcal{I}_0$ the subspace of $\mathcal{I}$ whose elements have Fourier transform with no point mass at the origin $x = 0$.

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- Y. Giga *at all* (IMUJ '08 and Meth. Appl. Anal '05) showed local solvability for Coriolis-Navier-Stokes equations in $\mathcal{I}_0$.

- As far as we know, the analysis on spaces $\mathcal{I}$ and $\mathcal{I}_{per}$ seems to be new in the context of dispersive equations, and in particular for the nonlinear Schrödinger equation $\mu = 0$.

Some comments...

- In $\mathbb{R}^n$, Theorem (C) provides a framework for NLS type equations that contains functions with high oscillation and infinite L^2-mass (but with finite local one).

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- For instance, $f(x) = \sum_{j=1}^{\infty} a_j e^{2\pi i x \cdot b_j}$ where $x \in \mathbb{R}^n$, $\sum_{j=1}^{\infty} |a_j| < \infty$ and $(b_j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$ can grow arbitrarily fast as $j \to \infty$. These functions are called almost periodic.

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- The approach used by us could be employed to treat (1) with $|u|^{\rho-1} u$ and ρ odd, instead of u^ρ. For that, it would be enough to write $|u|^{\rho-1} u$ as

$$[\left(|u|^2\right)^{\frac{\rho-1}{2}} u]^\wedge = (\underbrace{\widehat{u} * \widehat{u} * \ldots * \widehat{u}}_{\frac{\rho-1}{2}-times}) * (\underbrace{\widehat{\overline{u}} * \widehat{\overline{u}} * \ldots * \widehat{\overline{u}}}_{\frac{\rho-1}{2}-times}) * u$$

and to note that $\widehat{\overline{u}}(\xi) = \overline{\widehat{u}}(-\xi)$ and $\|\overline{u}(\xi)\|_{\mathcal{I}_{per}} = \|u(\xi)\|_{\mathcal{I}_{per}}$.

Some comments...

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- For DNLS, they used the norms $\|\langle\xi\rangle^s\widehat{u}(\xi)\|_{L^p(\mathbb{R})}$ with $2 \leq p < \infty$ and $\|\langle\xi\rangle^s\widehat{u}(\xi)\|_{l^p}$ with $2 < p < 4$, where $s \geq \frac{1}{2}$.

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- For NLS (continuous case), Grünrock (IRMN '05) used the norm $\|\langle\xi\rangle^s \widehat{u}(\xi)\|_{L^p(\mathbb{R})}$ with $1 < p < \infty$ and $s \geq 0$.

- Comparing with the continuous case for NLS in $n = 1$, the space $\mathcal{I}$ is not contained in the above ones, and in fact

$$\|\widehat{u}\|_{\mathcal{I}} \leq C \, \|\langle\xi\rangle^s \widehat{u}(\xi)\|_{L^p(\mathbb{R})} \, ,$$

for $1 < p < \infty$ and $s > 1 - \frac{1}{p}$.

Thank you for your attention