Existence of minimal blowup solutions for the nonlinear $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ wave equation

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Consider the nonlinear wave equation in $\mathbb{R}^{d+1}$,

$$\begin{cases} u_{tt} - \Delta u + \gamma u|u|^p = 0 \\ u(0) = u_0 \in \dot{H}^s(\mathbb{R}^d), \ u_t(0) = u_1 \in \dot{H}^{s-1}(\mathbb{R}^d), \end{cases}$$

with $\gamma \in \{1, -1\}$.

The equation is invariant under the scaling

$$u_r(x,t) = r^{\frac{2}{p}} u(rx, rt).$$

This invariance determines the critical Sobolev space for the initial data (u_0, u_1). We want

$$\|u_r(0)\|_{\dot{H}^s} = \|u(0)\|_{\dot{H}^s}, \quad \|\partial_t u_r(0)\|_{\dot{H}^{s-1}} = \|\partial_t u(0)\|_{\dot{H}^{s-1}}.$$

A calculation shows that the critical regularity corresponds to the case where

$$s_c = \frac{d}{2} - \frac{2}{p}.$$

Therefore our problem is critical if the initial data is in $(\dot{H}^{s_c} \times \dot{H}^{s_c-1})$.

The energy $E(u)$ is conserved, where

$$E(u) = \int_{\mathbb{R}^d} \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \gamma\frac{1}{2}|u|^{p+2} dx.$$

Since this energy scales like $s = 1$, we say that the equation is energy critical if $s = 1$, energy subcritical if $s < 1$ or energy supercritical if $s > 1$.

Here, we consider the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ critical, energy subcritical nonlinear wave equation

$$NLWE \quad \begin{cases} u_{tt} - \Delta u + \gamma u|u|^{\frac{4}{d-1}} = 0 \\ u(0) = u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d), \; u_t(0) = u_1 \in \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d), \end{cases}$$

for $\gamma \in \{1, -1\}$ and $d \geq 2$.

We notice that the we can not use the energy since our solution is not regular enough (energy subcritical).

Conjecture

Assume $u : \mathbb{R}^d \times I \to \mathbb{R}$ *is a solution to NLWE with maximal interval of existence* $I \subset \mathbb{R}$ *which satisfies*

$$(u, u_t) \in L_t^\infty(I; \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)). \tag{1}$$

Then u *is global, and*

$$\|u\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d \times \mathbb{R})} \leq C$$

for some constant $C = C(\|(u, u_t)\|_{L_t^\infty(\mathbb{R}; \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d))})$. *In particular,* u *scatters as* $t \to \pm\infty$.

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Previous results

- For the energy critical:
  - Defocusing: Struwe, Grillakis, Shatah-Struwe, Bahouri-Shatah, Kapitankski, Bahouri-Gérard, Ginibre-Velo, Rauch and others.
  - Focusing: global well-posedness and scattering may not hold. Levine and Krieger-Schlag-Tataru. Kenig–Merle developp the concentration-compactness argument.
- For the energy subcritical and supercritical:
  - Energy supercritical: Kenig–Merle, Duyckaerts-Kenig-Merle, Visan–Killip and Bulut.
  - Energy subcritical: Shen proved global well-posedness and scattering in dimension $d = 3$ for radial data for the $\dot{H}^s \times \dot{H}^{s-1}$ with $s > \frac{1}{2}$.

Idea of the proof.

By contradiction, assume the Conjecture fails.

- Proving the existence of a critical solution with especial properties.
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We define

$$L(E) := \sup\{\|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times I)} : u \ \text{ is a solution of NLWE such that} \sup_{t\in I}\|(u(t),\partial_t u(t))\|_{\dot H^{\frac12}\times \dot H^{-\frac12}} \le E\}.$$

By stability results, L is continuous non-decreasing function. Moreover by the small data theory $L(E) \le E^{\frac{d+3}{d+1}}$ for enough small E.

Therefore, if Conjecture fails (i.e there are blow-up solutions), there exists a critical E_c such that $L(E) < \infty$ if $E < E_c$ and $L(E) = \infty$ for $E \ge E_c$.

We can find a sequence $u_n : \mathbb{R}^d \times I_n \to \mathbb{C}$ of solutions to NLWE with I_n compact such that

$$\lim_{n\to\infty}\sup_{t\in I_n}\|(u_n(t),\partial_t u_n(t))\|_{\dot H^{\frac12}\times \dot H^{-\frac12}} = E_c,$$

$$\text{and} \quad \lim_{n\to\infty}\|u_n\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times I_n)} = \infty.$$

Is that critical value attained for any blow-up solution? That is, can we find a solution $u : \mathbb{R}^d \times I \to \mathbb{C}$ to the NLWE such that

$$\sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} = E_c,$$
$$\text{and } \|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d \times I)} = \infty.$$

The wave equation $\partial_{tt} u = \Delta u$, in $\mathbb{R}^{d+1}$, with initial data $u(\cdot, 0) = u_0$, $\partial_t u(\cdot, 0) = u_1$, has solution which can be written as

$$
\begin{aligned}
u(\cdot, t) = & \; S(u_0, u_1)(\cdot, t) \\
= & \; \frac{1}{2}\Big(e^{it\sqrt{-\triangle}} u_0 + \frac{1}{i}\frac{e^{it\sqrt{-\triangle}} u_1}{\sqrt{-\triangle}}\Big) + \frac{1}{2}\Big(e^{-it\sqrt{-\triangle}} u_0 - \frac{1}{i}\frac{e^{-it\sqrt{-\triangle}} u_1}{\sqrt{-\triangle}}\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
e^{\pm it\sqrt{-\triangle}} u_0(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi \pm t|\xi|)} \widehat{u_0}(\xi) d\xi, \\
\frac{e^{\pm it\sqrt{-\triangle}} u_1}{\sqrt{-\triangle}}(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi \pm t|\xi|)} \frac{\widehat{u_1}(\xi)}{|\xi|} d\xi.
\end{aligned}
$$

Let $r \in (0, \infty)$, $\alpha \in (-1,1)$, $x_0 \in \mathbb{R}^d$ and $\theta \in SO(d)$, we define the transformations $G_{r,\alpha,x_0,\theta} : \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d) \to \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ by

$$G_{r,\alpha,x_0,\theta}(f(x), g(x))$$
$$= r^{\frac{d-1}{2}}(S(f,g)(R_\theta^{-1}L^\alpha R_\theta(r(x-x_0),0)), \partial_t S(f,g)(R_\theta^{-1}L^\alpha R_\theta(r(x-x_0),0)),$$

where L^α is the Lorentz transform

$$L^\alpha(x_1, \underline{x}, t) = (\frac{x_1 + \alpha t}{\sqrt{1-\alpha^2}}, \underline{x}, \frac{t + \alpha x_1}{\sqrt{1-\alpha^2}}),$$

and R_θ is the rotation by angle θ around the t-axis.

Definition

A solution u of NLWE with lifespan I is almost periodic modulo symmetries if and only if there exists $r : I \to \mathbb{R}^+$, $\alpha : I \to (-1, 1)$, $x_0 : I \to \mathbb{R}^d$ and $\theta : I \to SO(d)$ such that the set

$$K = \{(G_{r(t),\alpha(t),x_0(t),\theta(t)}(u(x,t), \partial_t u(x,t)), t \in I\}$$

has compact closure in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$.

Another way to say the same: Let G be the collection of transformations $G_{r,\alpha,x_0,\theta}$, then the quotiented orbit $\{G(u(t), \partial_t u(t)) : t \in I\}$ is a precompact subset of $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$.

These transformations are the only responsible of the defect of compactness of $\{(u(t), \partial_t u(t)) : t \in I\}$.

Theorem

Suppose that Conjecture fails, then there exists a maximal-lifespan blowup solution $u : \mathbb{R}^d \times I \to \mathbb{C}$, *such that*

$$\sup_{t\in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} \leq \sup_{t\in J} \|(v(t), \partial_t v(t))\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}$$

for every maximal blowup solution $v : \mathbb{R}^d \times J \to \mathbb{C}$. *Moreover,* u *is almost periodic modulo symmetries.*

Main ingredients of the proof:

- Profile decomposition for the linear wave equation: captures the defect of compactness due to the symmetries of the equation.
    - The proof relies on a refinement of the Strichartz inequality for the wave equation.
- Profile decomposition for the nonlinear wave equation.
    - Stability result.
    - Lorentz nonlinear profiles.

In 1977, Strichartz proved his fundamental inequality

$$\|S(u_0, u_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq C(\|u_0\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)} + \|u_1\|^2_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)})^{\frac{1}{2}},$$

where

$$\|f\|_{\dot{H}^s} = (\sum_k 2^{2ks}\|P_k f\|_2^2)^{\frac{1}{2}},$$

with $\widehat{P_k f} = \chi_{\mathcal{A}_k}\widehat{f}$ and $\mathcal{A}_k = \{\xi \in \mathbb{R}^d;\ \ 2^k \leq |\xi| \leq 2^{k+1}\}$.

We improve this inequality to

$$\|S(u_0, u_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq C(\|u_0\|^2_{\dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^d)} + \|u_1\|^2_{\dot{B}^{-\frac{1}{2}}_{2,q}(\mathbb{R}^d)})^{\frac{1}{2}},$$

where $q = 2\frac{d+1}{d-1}$ for $d \geq 3$, and $q = 3$ for $d = 2$.

Here $\dot{B}^s_{2,q}$ is defined by

$$\|f\|_{\dot{B}^s_{2,q}} = (\sum_k 2^{qks}\|P_k f\|_2^q)^{\frac{1}{q}} \quad \left(\|f\|_{\dot{B}^s_{2,q}} \leq \left(\sup_k 2^{ks(q-2)}\|P_k f\|_2^{q-2}\right)^{\frac{1}{q}}\|f\|_{\dot{H}^s}^{\frac{2}{q}}\right)$$

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Better refinement

Let $S = \{w_m\}_m \subset \mathbb{S}^{d-1}$ be maximally 2^{-j}-separated, and define $\tau_m^{j,k}$ by

$$\tau_m^{j,k} := \left\{ \xi \in \mathcal{A}_k : \left|\frac{\xi}{|\xi|} - w_m\right| \leq \left|\frac{\xi}{|\xi|} - w_{m'}\right| \text{ for every } w_{m'} \in S,\ m' \neq m \right\}.$$

We also set $\widehat{P_k g_m^j} = \chi_{\tau_m^{j,k}} \widehat{g}$.

For our applications the following refinement will be of more use.

There exist $p < 2$ and $q(1-\theta) > 2$ such that

$$\begin{aligned}\|S(u_0,u_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq & C\Big(\sup_{j,k,m} 2^{k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_0)_m^j}\|_p^{\theta} \|u_0\|_{B^{\frac{1}{2}}_{2,q(1-\theta)}}^{1-\theta} \\ & + \sup_{j,k,m} 2^{-k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_1)_m^j}\|_p^{\theta} \|u_1\|_{B^{-\frac{1}{2}}_{2,q(1-\theta)}}^{1-\theta} \Big).\end{aligned}$$

Previous Strichartz's refinements in the literature

- For the Schrödinger equation:
  - Bourgain in 1989 in dimension $d = 2$.
  - Moyua–Vargas–Vega first in 1996 and then in 1999 improved that refinement.
  - Begout–Vargas in 2007 extended the result to dimensions $d > 2$ and Carles–Keraani in 2007 to dimension $d = 1$.
- For other equations:
  - Kenig–Ponce–Vega in 2000 for the Airy equation.
  - Rogers–Vargas in 2006 for the nonelliptic Schrödinger equation.
  - Chae–Hong-Lee in 2009 for higher order Schrödinger equations.
  - Killip-Stovall-Visan in 2011 for the Klein–Gordon equation.

- Bilinear approach

Theorem (Tao 2001)

Let $\frac{d+3}{d+1} \leq r_1 \leq 2$, *and suppose that* $\angle(w_m, w_{m'}) \sim 1$. *Then for all* $\epsilon > 0$,

$$\|e^{it\sqrt{-\triangle}} P_0 g_m^1 e^{it\sqrt{-\triangle}} P_\ell g_{m'}^1\|_{L^{r_1}(\mathbb{R}^{d+1})} \lesssim 2^{\ell(\frac{1}{r_1}-\frac{1}{2}+\epsilon)} \|\widehat{P_0 g_m^1}\|_{L^2(\mathbb{R}^d)} \|\widehat{P_\ell g_{m'}^1}\|_{L^2(\mathbb{R}^d)}$$

- Refined orthogonality.

$$\|\sum_k f_k\|_p \lesssim C^{1-\frac{2}{p^*}} \left(\sum_k \|f_k\|_p^{p_*}\right)^{\frac{1}{p_*}}$$

- Atomic decomposition.

Lemma

Let $q > 2$, *and* $1 < p < 2$. *Then*

$$\sum_j \Big(\sum_m |\tau_m^{j,k}|^{q\frac{p-2}{2p}} \|\widehat{P_k g_m^j}\|_p^q\Big)^{\frac{2}{q}} \lesssim \|P_k g\|_2^2.$$

THANK YOU !!