Stabilizability and critical set restrictions for ZK

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Events in October-November 2013

Introduction

The Zakharov-Kuznetsov equation (ZK)

$$u_t + u_x + uu_x + \Delta u_x = 0 \tag{1}$$

is a nonlinear dispersive PDE describing magneto-acoustic waves in a cold plasma. It is a multi-dimensional version of KdV

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{2}$$

Energy does not increase: $\dfrac{1}{2}\dfrac{d}{dt}\|u\|^2(t) \leq 0.$

Decay issues: whether, how, why, where?

There are many analytical and numerical methods to study (1) and (2). They are mainly concerned with problems posed on a whole space $\mathbb{R}^2$ or $\mathbb{R}$ respectively. For numerical simulations, however, there appears the issue of cutting-off the spatial domain (Bona '09). This motivates the detail qualitative analysis of problems for (1) and/or (2) in bounded regions.

Note that for the Cauchy problem (whole space) the linear term u_x can be easily scaled out, while for the initial-boundary value problems posed on bounded domains this leads to changes in a domain geometry.

Previous: Faminskii '95, Saut-Temam '10, Linares-Pastor '11, Larkin-Tronco '13, among others.

Introduction

One of the difficulties in this field is a "critical size", i.e. the size of a spatial domain for which solutions to a simplest homogeneous problem may not decay. The well-known example is KdV: if, for instance, $L = 2\pi n, \; n \in \mathbb{N}$, then $v(x) = 1 - \cos x$ solves

$$\begin{cases} u_t + u_x + u_{xxx} = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \\ u(0,t) = u(L,t) = u_x(L,t) = 0, \\ u(x,0) = v(x), \quad x \in (0, L), \end{cases}$$

and clearly $v(x) \not\to 0$ as $t \to \infty$. Despite the valuable resent advances (Cerpa et al '09) the question whether solutions of undamped problems associated to nonlinear KdV decay as $t \to \infty$ for all finite $L > 0$ is open.

Introduction

In the case of (linear) KdV, the controllability and exponential decay are equivalent to the unique continuation, which means that the corresponding eigenvalue problem has trivial solutions only.

This occurs (Rosier '97) iff

$$L \notin \mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2};\ k, l \in \mathbb{N} \right\}.$$

Note: once (2) is scaled to be $u_t + uu_x + u_{xxx} = 0,\ \mathcal{N} := \emptyset$.

Critical sizes

For $L > 0$, $B > 0$ consider linearized 2D ZK

$$u_t + u_x + u_{xxx} + u_{xyy} = 0, \tag{3}$$

posed on a rectangle $\mathcal{D} = (0, L) \times (0, B) \subset \mathbb{R}^2$ with the simplest homogeneous boundary data.

Critical set: decay of solutions fails if $L > 0$ and $B > 0$ solve

$$\left(\frac{2\pi}{L\sqrt{3}}\sqrt{k^2 + kl + l^2}\right)^2 + \left(\frac{\pi n}{B}\right)^2 = 1; \quad k, l, n \in \mathbb{N}, \tag{4}$$

i.e., if $\mathcal{D}$ is of a critical size, like in the case of KdV posed on an interval. In other words, (4) is a 2D generalization of Rosier's set.

Problem 1: Let $\mathcal{D} = (0, L) \times (-B, B)$, $\alpha = 1$ or $\alpha = 0$.

$$u_t + (\alpha + u)u_x + u_{xxx} + u_{xyy} = 0, \text{ in } \mathcal{D} \times (0, T); \qquad (5)$$

$$u\,|_{y=-B} = u\,|_{y=B} = 0, \quad x \in (0, L), \ t > 0; \qquad (6)$$

$$u\,|_{x=0,L} = u_x\,|_{x=L} = 0, \quad y \in (-B, B), \ t > 0; \qquad (7)$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathcal{D}. \qquad (8)$$

Theorem

Let $\alpha = 1$ and $\pi^2\left[\frac{3}{L^2} + \frac{1}{4B^2}\right] > 1$. If $\|u_0\|$ is sufficiently small, then regular solutions to (5)-(8) satisfy the inequality

$$\|u\|^2(t) \leq \left((1 + x), u^2\right)(t) \leq e^{-\frac{A^2}{1+L}t}\left((1 + x), u_0^2\right).$$

Main result

Sketch of the proof:

- Existence: parabolic regularization, *a priori* estimates independent of $\varepsilon > 0$ (and $B > 0$).
- Passage to the limit to obtain solutions:

$$u \in L^\infty(0,T;H^2(\mathcal{D})) \cap L^2(0,T;H^3(\mathcal{D}));$$
$$\Delta u_x \in L^\infty(0,T;L^2(\mathcal{D})) \cap L^2(0,T;H^1(\mathcal{D}));$$
$$u_t \in L^\infty(0,T;L^2(\mathcal{D})) \cap L^2(0,T;H^1(\mathcal{D}))$$

- Uniqueness and decay: straightly. Use of regularity and u_{xyy}.
- Rate: $2A^2 = \pi^2 \left[\frac{3}{L^2} + \frac{1}{4B^2}\right] - 1$.
- Smallness of initial datum: $\|u_0\|^2 < \frac{9A^4L^2B^2}{4\pi^2(4B^2+L^2)}$.

Theorem

Let $\alpha = 0$ and L, B be arbitrary positive. If $\|u_0\|$ is sufficiently small, then regular solutions to (5)-(8) satisfy the inequality

$$\|u\|^2(t) \le \left((1+x), u^2\right)(t) \le e^{-\sigma t}\left((1+x), u_0^2\right)$$

Rate:

$$\sigma = \frac{\pi^2(12B^2 + L^2)}{8B^2L^2(1+L)}.$$

Smallness of initial function:

$$\|u_0\|^2 < \frac{81\pi^2(4B^2 + L^2)}{L^2B^2}$$

Problem 2: Let $\mathcal{S} = (0, L) \times \mathbb{R}$, $\alpha = 1$ or $\alpha = 0$.

$$u_t + (\alpha + u)u_x + u_{xxx} + u_{xyy} = 0, \text{ in } \mathcal{S} \times (0, T); \tag{9}$$

$$u\mid_{x=0,L} = u_x \mid_{x=L} = 0, \quad y \in \mathbb{R}, \; t > 0; \tag{10}$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathcal{S}. \tag{11}$$

Theorem

Similar claims as for Problem 1: restrictions for $L > 0$ if $\alpha = 1$ and no restrictions if $\alpha = 0$. Smallness of initial data required.

Comparison between size restrictions for linear and nonlinear models: taking $k = l = n = 1$, (4) becomes

$$\frac{4\pi^2}{L^2} + \frac{\pi^2}{4B^2} = 1, \tag{12}$$

and restrictions for nonlinear case read

$$\frac{3\pi^2}{L^2} + \frac{\pi^2}{4B^2} > 1. \tag{13}$$

Suppose $L^* > 0$ and $B^* > 0$ solve (12) and denote

$$\mathcal{D}^* = (0, L^*) \times (-B^*, B^*) \subset \mathbb{R}^2.$$

Call this set to be a *minimal critical rectangle*. If $\mathcal{D}$ is located inside the minimal critical rectangle, then a sufficiently small solution to nonlinear problem (5)-(8) necessarily stabilizes.

In this sense, "nonlinear" restrictions are stipulated by (12) and, therefore, smallness conditions for L, B can be interpreted as not only technical ones, but as close to be sharp. In particular, stabilizability holds for all rectangles $\mathcal{D}$ either with the width $L < \pi\sqrt{3}$, or with the height $2B < \pi$.

Furthermore, a small solution for problems posed on a sufficiently narrow strip $\mathcal{S}$ stabilizes as well. This partially responds an open question by Saut and Temam '10.

Observe also that (12) fits well with the stabilization result by Larkin and Tronco '13.

References

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