ORBITAL STABILITY OF PERIODIC WAVES FOR THE KLEIN-GORDON TYPE EQUATIONS

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This is a joint work with E. Cardoso Jr. - UEM.

A BRIEF INTRODUCTION.

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$$E'(\Phi) - cF'(\Phi) = 0. \tag{2}$$

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where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a complex valued function. We assume that u is an $L-$periodic function, that is, $u(x+L,t) = u(x,t)$ for all $x, t \in \mathbb{R}$.

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$$F(U) = \mathrm{Im} \int_0^L \bar{u} u_t \; dx = \int_0^L (\mathrm{Re}\; u \;\mathrm{Im}\; u_t - \mathrm{Im}\; u \;\mathrm{Re}\; u_t) \; dx.$$

An important qualitative aspect regarding the Hamiltonian systems (1) is the orbital stability.

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DEFINITION OF STABILITY.

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$$\sup_{-\infty<t<\infty} \inf\{||U(t) - \mathbf{T}_1(s_1)\mathbf{T}_2(s_2)\Phi||_X,\ -\infty < s_1, s_2 < \infty\} < \varepsilon.$$

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Otherwise, we say that the periodic wave is orbitally unstable in X (or $X-$unstable).

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(3) $d''(c) > 0$, for all $c \in \mathcal{I}$.

In order to deduce the orbital stability of periodic waves, we need to determine (at least) existence and uniqueness of (weak) solutions.

We suppose that $f:\mathbb{R}\to\mathbb{R}$ is a real function and the Cauchy problem

$$\begin{cases} u_{tt}-u_{xx}+u-f(|u|^2)u=0 \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_0'(x), \end{cases} \tag{7}$$

has a unique (local) solution

$$u\in C([0,T];H^1_{per}([0,L]))\cap C^1([0,T];L^2_{per}([0,L])).$$

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$$u \in C([0,T]; H^1_{per}([0,L])) \cap C^1([0,T]; L^2_{per}([0,L])).$$

In addition, we assume that problem (7) has two (convenient) conserved quantities E and F.

- Let us consider $f(|u|^2) = |u|^{2p}$, $p \geq 1$, integer.

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Theorem 1.

Consider $(u_0, u_0') \in H^1_{per}([0,L]) \times L^2_{per}([0,L])$. The Cauchy Problem (7) has a unique (local) weak solution $u \in C([0,T]; H^1_{per}([0,L])) \cap C^1([0,T]; L^2_{per}([0,L]))$. In addition, we have the following conserved quantities

$$E(U) = \frac{1}{2}\int_0^L \left[|u_x|^2 + |u_t|^2 + |u|^2\left(2 - \log(|u|^2)\right)\right] \, dx,$$

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- In addition, we need to use the logarithmic Sobolev inequality

$$\begin{aligned} 4\pi^2 \int_0^L |u|^2 \log|u| \; dx \quad &\leq \quad \int_0^L |u_x|^2 \; dx \\ &+ \quad 2\pi^2 \int_0^L |u|^2 \log\left(\frac{2\pi}{L} \int_0^L |u|^2 \; dx\right) \; dx, \end{aligned}$$

$u \in H^1_{per}([0, L])$.

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$$\begin{cases} \varphi_{tt} - \varphi_{xx} + \varphi - u\log(|u|^2) + v\log(|v|^2) = 0 \\ \varphi(x,0) = 0, \quad \varphi_t(x,0) = 0. \end{cases} \tag{8}$$

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$$\varphi(x,t)=\frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau}\left[u\log(|u|^2)-v\log(|v|^2)\right]dyd\tau, \quad (9)$$

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In addition, the same approach determines sufficient conditions to obtain the spectral property associated with the operator $\mathcal{L}_1$.

In order to find a local minimum, we define the orbit generated by Φ

$$\mathcal{O}_{\phi_c} := \left\{ e^{i\theta}(\phi_c(\cdot + y), ic\phi_c(\cdot + y));\ (y, \theta) \in [0, L] \times [0, 2\pi) \right\}.$$

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Taking $v := u_t$, let us consider $(y,\theta) \in [0,L] \times [0,2\pi)$. Let $t \in [0,T]$ be arbitrary but fixed. We define the continuous function

$$\begin{aligned} \Omega_t(y,\theta) &:= \|u_x(\cdot + y, t)e^{i\theta} - \phi_c'\|^2_{L^2_{per}} \\ &+ (1-c^2)\|u(\cdot + y, t)e^{i\theta} - \phi_c\|^2_{L^2_{per}} \\ &+ \|v(\cdot + y, t)e^{i\theta} - ic\phi_c\|^2_{L^2_{per}}. \end{aligned} \qquad (11)$$

In order to find a local minimum, we define the orbit generated by Φ

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Since Ω_t, $t \in [0,T]$, is continuous and $[0,L] \times [0,2\pi)$ is bounded, we can write,

$$\Omega_t(y(t),\theta(t)) = \inf_{(y,\theta)\in[0,L]\times[0,2\pi)} \Omega_t(y,\theta) := \left[\rho_c(\vec{u}(\cdot,t), \mathcal{O}_{\phi_c})\right]^2. \tag{12}$$

Furthermore, the map

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$$u(x+y,t)e^{i\theta} := \phi_c(x) + w(x,t) \quad \text{where } w := A + iB \tag{13}$$

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Denoting

$$\vec{w} = (w,z) = (\mathrm{Re}\ w, \mathrm{Im}\ z, \mathrm{Im}\ w, \mathrm{Re}\ z) = (A,D,B,C).$$

By using the minimum property (above) one has

$$\left\langle \left(\begin{array}{c} A(\cdot,t) \\ D(\cdot,t) \end{array} \right), \left(\begin{array}{c} \log(\phi_c^2)\phi_c' + 2\phi_c' \\ c\phi_c' \end{array} \right) \right\rangle_{2,2} = 0 \qquad (15)$$

and

$$\left\langle \left(\begin{array}{c} B(\cdot,t) \\ C(\cdot,t) \end{array} \right), \left(\begin{array}{c} \phi_c \log(\phi_c^2) \\ -c\phi_c \end{array} \right) \right\rangle_{2,2} = 0, \qquad (16)$$

$\forall\ t \in [0, T]$.

Next, since $G = E - cF$ is a conserved quantity and $G'(\phi_c, ic\phi_c) = (E' - cF')(\phi_c, ic\phi_c) = 0$, we deduce from Taylor's Theorem

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\begin{aligned}
\Delta G &:= G(u_0, u_1) - G(\phi_c, ic\phi_c) \\
&= G(w(\cdot,t) + \phi_c, z(\cdot,t) + ic\phi_c) - G(\phi_c, ic\phi_c) \\
&\geq \frac{1}{2}\left\langle \mathcal{L}_R \begin{pmatrix} A(\cdot,t) \\ D(\cdot,t) \end{pmatrix}, \begin{pmatrix} A(\cdot,t) \\ D(\cdot,t) \end{pmatrix} \right\rangle_{2,2} \\
&+ \frac{1}{2}\left\langle \mathcal{L}_I \begin{pmatrix} B(\cdot,t) \\ C(\cdot,t) \end{pmatrix}, \begin{pmatrix} B(\cdot,t) \\ C(\cdot,t) \end{pmatrix} \right\rangle_{2,2} \\
&- \beta_3 \|\vec{w}(\cdot,t)\|^3 - \beta_4 \|\vec{w}(\cdot,t)\|^4 - \mathcal{O}(\|\vec{w}(\cdot,t)\|^5),
\end{aligned}
$$

Here, operators $\mathcal{L}_R$ and $\mathcal{L}_I$ are defined as

$$\mathcal{L}_R = \begin{pmatrix} -\partial_x^2 + 1 - \log(|\phi_c|^2) - 2 & -c \\ -c & 1 \end{pmatrix} \tag{17}$$

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- The fact that ϕ_c is positive enable us to conclude that zero is the first eigenvalue of $\mathcal{L}_I$ which is simple.

Thus, classical methods of orbital stability (in the sense of definition above) is established on the set

$$\mathcal{A} = \{(u,v) \in H^1_{per} \times L^2_{per};\ F(u,v) = F(\phi_c, ic\phi_c)\},$$

provided that

$$\left\langle \mathcal{L}^{-1}_{R,\phi_c} \begin{pmatrix} c\phi_c \\ \phi_c \end{pmatrix}, \begin{pmatrix} c\phi_c \\ \phi_c \end{pmatrix} \right\rangle_{2,2} = \underbrace{\left\langle \begin{pmatrix} M \\ N \end{pmatrix}, \begin{pmatrix} c\phi_c \\ \phi_c \end{pmatrix} \right\rangle_{2,2}}_{:=-d''(c)} < 0,$$

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where

$$\begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} \dfrac{d}{dc}(\phi_c) \\ \phi_c + c\dfrac{d}{dc}(\phi_c) \end{pmatrix}.$$

However, one has

$$-d''(c) = \int_0^L \phi_c^2 \, dx + c \underbrace{\frac{d}{dc}\left(\int_0^L \phi_c^2 \, dx\right)}_{I_c}. \tag{19}$$

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$$-\eta_c'' - 2c\phi_c + (1-c^2)\eta_c - \log(\phi_c^2)\eta_c - 2\eta_c = 0, \qquad (20)$$

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$$I_c = -2c \int_0^L \phi_c^2 \, dx. \tag{21}$$

Collecting the results in (19) and (21) we deduce

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that is, $-d''(c) < 0$ if, and only if, $|c| > \frac{\sqrt{2}}{2}$.

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A simple application of the triangle inequality and the fact that G is C^1 map in a neighborhood of the point $(\phi_c, ic\phi_c)$ give us the orbital stability if $(u, v) \notin \mathcal{A}$

THANK YOU VERY MUCH!

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