Stability of standing waves of a nonlinear Schrödinger equation with a Dirac Delta potential

César A. Hernández Melo

First Workshop on Nonlinear Dispersive Equations
October 30 2013

3.5-Schrödinger equation in $H^1(\mathbb{R})$ (3.5-SE)

$$iu_t + u_{xx} + Z\delta(x)u + u(|u|^2 + |u|^4) = 0$$

Here, $u = u(x,t) \in \mathbb{C}$, $x, t \in \mathbb{R}$, $Z \in \mathbb{R}$ and δ is given by

$$\delta : H^1(\mathbb{R}) \to \mathbb{C}, \qquad \langle \delta, g \rangle = g(0).$$

The non-linearity in the equation: $u|u|^2 + u|u|^4$

Physical applications for the 3.5-Schrödinger equation

1-Study of nonlinear resonance of light propagation with localized defects

Problems to be addressed

1- Existence of solutions to the 3.5-Schrödinger equation
2- Existence of standing waves solutions
3- Stability/instability

Standing waves

By a standing wave, we mean global solutions to the 3.5-Schrödinger equation in the form:

$$u(x,t) = e^{-iwt}\phi(x)$$

where $w \in \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$.

The function ϕ (the profile of the standing wave) must satisfy the ordinary equation:

$$\phi'' + Z\delta(x)\phi + w\phi + \phi^3 + \phi^5 = 0$$

Jump condition on the derivative of the function ϕ

$$\phi'' + Z\delta(x)\phi + w\phi + \phi^3 + \phi^5 = 0$$

Assuming that $\phi \in C^2(\mathbb{R} - \{0\}) \bigcap C(\mathbb{R})$, integrating the last equation on the interval $(-\epsilon, \epsilon)$, we have

$$\phi'(\epsilon) - \phi'(-\epsilon) + Z\phi(0) + \int_{-\epsilon}^{\epsilon} w\phi + \phi^3 + \phi^5 dx = 0,$$

taking $\epsilon \to 0$, the profile of the standing wave ϕ must satisfy:

$$\phi'(0+) - \phi'(0-) = -Z\phi(0)$$

The standing wave solutions are not smooth functions. The 3.5-Schrodinger equation loses translation symmetry. The second derivative at the ODE has to be considered in the weak sense.

Properties of the solutions of the ODE (Jeanjean):

$$\phi'' + Z\delta(x)\phi + w\phi + \phi^3 + \phi^5 = 0$$

Lemma

Let $Z \in \mathbb{R}$ and $-w > \frac{Z^2}{4}$. Then every solution $g \in H^1(\mathbb{R})$ of the ODE, satisfies the following properties

$$g \in C^j(\mathbb{R} - \{0\}) \cap C(\mathbb{R}), \quad j = 1, 2. \tag{1a}$$

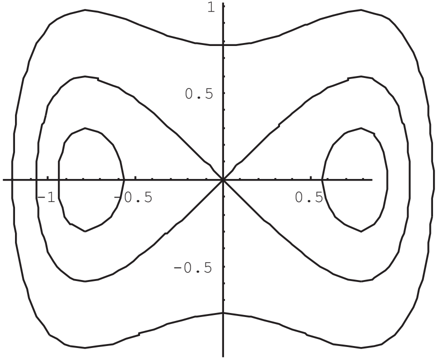
$$-g'' - wg - g^3 - g^5 = 0, \quad \text{for } \; x \neq 0. \tag{1b}$$

$$g'(0+) - g'(0-) = -Zg(0). \tag{1c}$$

$$g'(x), g(x) \to 0, \quad \text{if } \; |x| \to \infty. \tag{1d}$$

Dynamic of the ODE when $Z = 0$ and $w = -1$

$$H[\phi, \phi'] = \frac{[\phi']^2}{2} + w\frac{\phi^2}{2} + \frac{\phi^4}{4} + \frac{\phi^6}{6}$$



Standing wave profiles in $H^1(\mathbb{R})$, $Z = 0$:

The level curve $H[\cdot,\cdot] = 0$ represents the profile of the standing wave solution when $w = -1, Z = 0$:

$$\phi_{-1,0}(x) = \left[\frac{1}{4} + \frac{\sqrt{57}}{12}\cosh(2x)\right]^{-\frac{1}{2}}.$$

In general, for $-w > 0$ the function

$$\phi_w(x) = \left[-\frac{1}{4w} - \frac{\sqrt{9-48w}}{12w}\cosh(2\sqrt{-w}x)\right]^{-\frac{1}{2}},$$

represents the positive profile in $H^1(\mathbb{R})$ of the standing wave to the 3.5-Schrödinger equation when $Z = 0$.

Standing wave profiles in $H^1(\mathbb{R})$, $Z \in \mathbb{R}$:

For w, Z satisfying the relation $-w > \frac{Z^2}{4}$, the function: $\phi_{w,Z} =$

$$\left[-\frac{1}{4w} - \frac{\sqrt{9-48w}}{12w}\cosh\left(2\sqrt{-w}\left(|x| + R^{-1}\left(\frac{Z}{2\sqrt{-w}}\right)\right)\right)\right]^{-\frac{1}{2}},$$

where $\alpha = \frac{-1}{4w}$, $\beta = \frac{\sqrt{9-48w}}{-12w}$ and $b \to R(b)$ being the strictly increasing function given by:

$$R(b) = \frac{\beta\sinh(2\sqrt{-w}b)}{\alpha + \beta\cosh(2\sqrt{-w}b)},$$

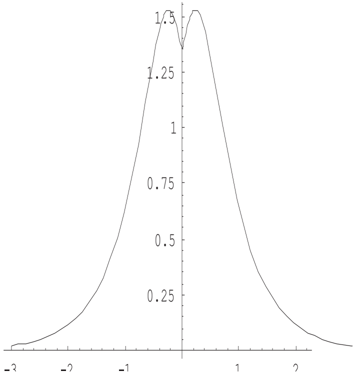
is a standing wave profile.

Profile of the standing waves $\phi_{w,Z}$

- $w = -3, Z = -2$

Profile of the standing waves $\phi_{w,Z}$

- $w = -3, Z = -2$

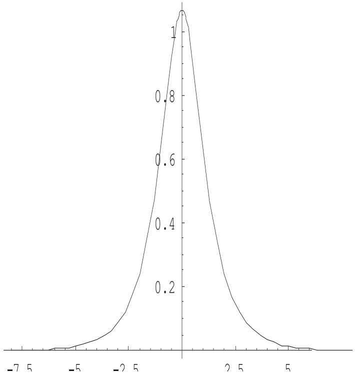


Profile of the standing waves $\phi_{w,Z}$

- $w = -3, Z = -2$
- $w = -3, Z = 0$

Profile of the standing waves $\phi_{w,Z}$

- $w = -3, Z = -2$
- $w = -3, Z = 0$

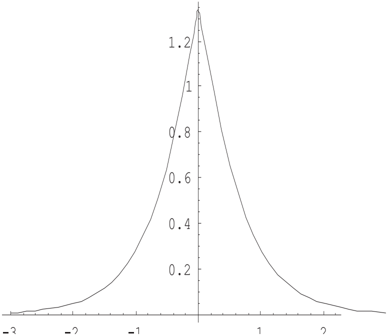


Profile of the standing waves $\phi_{w,Z}$

- $w = -3, Z = -2$
- $w = -3, Z = 0$
- $w = -3, Z = 2$

Profile of the standing waves $\phi_{w,Z}$

- $w = -3, Z = -2$
- $w = -3, Z = 0$
- $w = -3, Z = 2$



Symmetries of the equation

Formally, if $u(t)$ is a solution of the 3.5-Schrödinger equation, then the function

$$T(\theta)u(t), \qquad \theta \in \mathbb{R},$$

is also a solution of the 3.5-Schrödinger equation.

We observe that for $\theta \in \mathbb{R}$, the family of unitary linear operators $T(\theta)$ defined by:

$$T(\theta) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \qquad T(\theta)g = e^{-\theta i}g,$$

form a one-parameter group.

Stability definition

We consider the orbit

$$\Omega_{\phi_{w,Z}} = \{T(\theta)\phi_{w,Z} | \theta \in [0, 2\pi]\},$$

We say that $\Omega_{\phi_{w,Z}}$ is stable in $H^1(\mathbb{R})$, if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $u(0) \in H^1(\mathbb{R})$ with

$$\inf_{\theta\in[0,2\pi]} ||T(\theta)\phi_{w,Z} - u(0)||_1 < \delta,$$

then

$$\inf_{\theta\in[0,2\pi]} ||T(\theta)\phi_{w,Z} - u(t)||_1 < \epsilon$$

for all $t \in \mathbb{R}$. Other case, the orbit $\Omega_{\phi_{w,Z}}$ is called instable.

Main results

Let $-w > \frac{Z^2}{4}$ and $Z^* \approx -0.8660254$, then we get

I- For $Z \geq 0$, the orbit $\Omega_{\phi_{w,Z}}$ is stable in $H^1(\mathbb{R})$.

II- For $Z \in (Z^*, 0)$, the orbit $\Omega_{\phi_{w,Z}}$ is instable in $H^1(\mathbb{R})$.

III- For $Z \in (Z^*, \infty)$, the orbit $\Omega_{\phi_{w,Z}}$ is stable in $H^1_{even}(\mathbb{R})$.

IV- For $Z < Z^*$, the orbit $\Omega_{\phi_{w,Z}}$ is instable in $H^1_{even}(\mathbb{R})$.

The self-adjoint operator $\Delta_\gamma := \partial_{xx} + \gamma\delta(x)$ and the second distributional derivative d^2

Let $g \in C^2(\mathbb{R} - \{0\}) \bigcap C(\mathbb{R})$ with $g'(0+) - g'(0-) = -\gamma g(0)$ and $h \in C_0^\infty(\mathbb{R})$, then:

$$\int_{-\infty}^{\infty} d^2 g(x)h(x)dx = -\gamma g(0)h(0) + \int_{-\infty}^{\infty} g''(x)h(x)dx$$

therefore, $\Delta_\gamma g := g'' = (d^2 + Z\delta(x))g$

The operator Δ_γ does not recognize the singularity of the function g

Theorem

Let $-\infty < \gamma \leq \infty$. Then, the essential spectrum of Δ_γ is given by

$$\sigma_{ess}(\Delta_\gamma) = [0, \infty).$$

If $-\infty < \gamma < 0$, Δ_γ, has exactly one negative simple eigenvalue. Then the point spectrum $\sigma_p(\Delta_\gamma)$ of Δ_γ , is given by

$$\sigma_p(\Delta_\gamma) = \left\{ -\frac{\gamma^2}{4} \right\}, \quad \textit{with} \quad \psi_\gamma(x) = e^{\frac{\gamma|x|}{2}}$$

as its corresponding eigenfunction.

For $\gamma < 0$, the linear equation $u_t = \Delta_\gamma u$ has a standing wave in $H^1(\mathbb{R})$.

General conditions to obtain a stability theory

Albert, Bona, Souganidis, Grillakis, Shatah, Straus, Weinstein

General conditions to obtain a stability theory

Albert, Bona, Souganidis, Grillakis, Shatah, Straus, Weinstein

1. Existence of a smooth curve $w \to \phi_w \in H^1$.
2. Existence of flow in $H^1(\mathbb{R})$ (near to the orbit).
3. Existence of conserved quantities $E, F : H^1 \to \mathbb{R}$ satisfying $\Psi'(\phi_w) = E'(\phi_w) - wF'(\phi_w) = 0$, where $\Psi := E - wF$.
4. Detailed spectral study of the self-adjoint operator $\Psi''([\phi_w, 0]) : D(\Psi''([\phi_w, 0])) \to L^2 \times L^2$.
5. Detailed study of the convexity of the function $d : I \subset \mathbb{R} \to \mathbb{R}$, defined by $d(w) = E(\phi_w) - wF(\phi_w)$.

Conserved quantities

Let $E, F : H^1(\mathbb{R}) \to \mathbb{R}$ defined by,

$$E(u) = \frac{1}{2}\int |u_x|^2 dx - \frac{Z}{2}\int \delta(x)|u(x)|^2 dx - \frac{1}{4}\int |u|^4 dx - \frac{1}{6}\int |u|^6 dx$$

and

$$F(u) = \frac{1}{2}\int |u|^2 dx,$$

then, we have that E, F are conserved quantities for the flow of the 3.5-Schrodinger equation and invariant under the group of rotations T.

Well-posedness results

1- The Cauchy problem associated to the 3.5-Schrödinger equation is locally well-posed in $H^1(\mathbb{R})$. In addition, if the initial data u_0 is even, then the solution $u(t)$ to the 3.5-Schrödinger equation with initial data $u_0 = u(0)$ is also even.

2- Let $g \in H^1(\mathbb{R})$. Then the solution to the 3.5-Schrödinger equation u is globally well defined whenever the initial data $u(0) = g$, be small in $L^2(\mathbb{R})$.

3-For $Z > 0$, any solution u of the 3.5-Schrodinger equation with initial data $u(0) = f$ near to the orbit Ω_{ϕ_w}, is globally well defined.

The second derivative of $\Psi = E - wF$, at $\phi_{w,z}$:

$$\Psi''([\phi_{w,z}, 0]^t) = \begin{bmatrix} \mathcal{L}_{1,z} & 0 \\ 0 & \mathcal{L}_{2,z} \end{bmatrix},$$

where the self-adjoint operators $\mathcal{L}_{i,z} : \mathcal{D} \to L^2(\mathbb{R})$ are defined by

$$\mathcal{L}_{1,z} g = -\frac{d^2}{dx^2} g - wg - 3\phi_{w,z}^2 g - 5\phi_{w,z}^4 g,$$

$$\mathcal{L}_{2,z} g = -\frac{d^2}{dx^2} g - wg - \phi_{w,z}^2 g - \phi_{w,z}^4 g,$$

with

$$\mathcal{D} = \left\{ g \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}) | g'(0+) - g'(0-) = -Zg(0) \right\}$$

Spectral properties of the operator $\mathcal{L}_{2,Z}$

$$\mathcal{L}_{2,Z}(\phi_{w,Z}) = 0$$

Theorem

Let $w < 0$, $Z \in \mathbb{R} - \{0\}$ and $-w > \frac{Z^2}{4}$. Then, $\mathcal{L}_{2,Z}$ is a nonnegative operator with spectrum given by

$$\sigma_p(\mathcal{L}_{2,Z}) = \{0\}, \quad \sigma_{ess}(\mathcal{L}_{2,Z}) = [-w, \infty).$$

Here, zero is a simple eigenvalue with corresponding eigenfunction $\phi_{w,Z}$.

Specral properties of the operator $\mathcal{L}_{1,0}$

Theorem

Let $w < 0$, the operator $\mathcal{L}_{1,0}$ has only one simple and negative eigenvalue τ_0, the second eigenvalue is zero, it is also simple with corresponding eigenfunction $\frac{d}{dx}\phi_w$. The rest of the spectrum is essential and it is away from zero. More accurately

$$\sigma_{ess}(\mathcal{L}_{1,0}) = [-w, \infty).$$

Spectral properties of the operator $\mathcal{L}_{1,Z}$ in $H^1(\mathbb{R})$

Theorem

Let $w < 0$, $Z \in \mathbb{R} - \{0\}$ and $-w > \frac{Z^2}{4}$. Then, $Ker(\mathcal{L}_{1,Z}) = \{0\}$.

$$\text{Set } n(\mathcal{L}_{1,Z}) = \text{number of negative eigenvalues of } \mathcal{L}_{1,Z}$$

Theorem

Let $w < 0$ such that $-w > \frac{Z^2}{4}$. Then,
1- For $Z \geq 0$, $n(\mathcal{L}_{1,Z}) = 1$.

2- For $Z < 0$, $n(\mathcal{L}_{1,Z}) = 2$.

Proof: Continuation argument based on the spectral structure of $\mathcal{L}_{1,0}$, the study of the negative spectrum of $\mathcal{L}_{1,Z}$ for Z small, analytic perturbation theory ($\mathcal{L}_{1,Z} \to \mathcal{L}_{1,0}$) and Riesz projections.

Number of negative eigenvalues of the operator $\mathcal{L}_{1,Z}$ in $H^1_{even}(\mathbb{R})$

Properties of the second eigenfunction $\Omega(Z)$

Theorem

The eigenfunction $\Omega(Z)$ corresponding to the second eigenvalue of the operator $\mathcal{L}_{1,Z}$ is a odd function for all $Z \in (-\infty, \infty)$.

We remark that the first eigenfunction of $\mathcal{L}_{1,Z}$ is even

We can conclude that $n(\mathcal{L}_{1,Z}) = 1$ for all $Z \in \mathbb{R}$

Formula to analyse d

$$d''(w) = -\frac{d}{dw}||\phi_{w,z}||^2$$

$$||\phi_{w,z}||^2 = -2\sqrt{3}\left[arctg\left(\theta(w)\right) - arctg\left(\theta(w)\, tagh(\sqrt{-w}b)\right)\right],$$

with $\theta(w) = \frac{\sqrt{3-\sqrt{3-16w}}}{4\sqrt{-w}}$ and $b = R_w^{-1}\left(\frac{z}{2\sqrt{-w}}\right)$ where

$$R_w(b) = \frac{\beta(w) senh(2\sqrt{-w}b)}{\alpha(w) + \beta(w) cosh(2\sqrt{-w}b)},$$

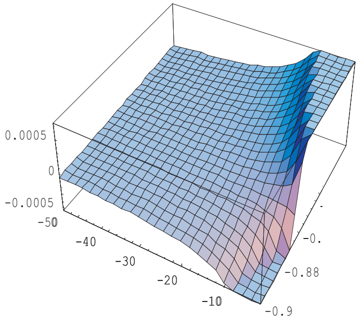
here $\alpha(w) = \frac{-1}{4w}$ and $\beta(w) = \frac{\sqrt{9-48w}}{-12w}$.

Sign of $d''(w) = d''(w, Z)$

- $w \in (-50, -2)$,
  $Z \in (-0.9, -0.8)$

Sign of $d''(w) = d''(w, Z)$

- $w \in (-50, -2)$,
  $Z \in (-0.9, -0.8)$

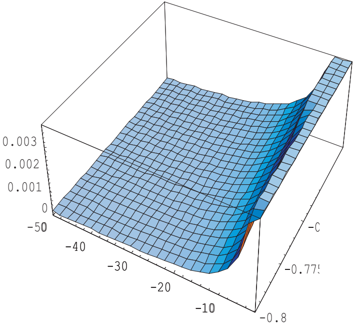


Sign of $d''(w) = d''(w, Z)$

- $w \in (-50, -2)$,
  $Z \in (-0.9, -0.8)$
- $w \in (-50, -2)$,
  $Z \in (-0.8, -0.7)$

Sign of $d''(w) = d''(w, Z)$

- $w \in (-50, -2)$,
  $Z \in (-0.9, -0.8)$
- $w \in (-50, -2)$,
  $Z \in (-0.8, -0.7)$

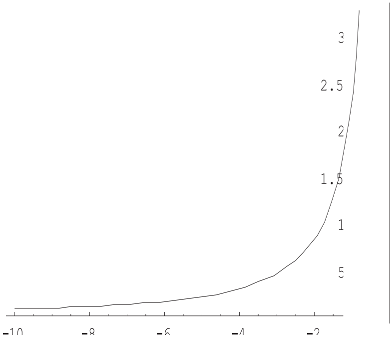


Sign of $d''(w) = d''(w, Z)$

- $w \in (-50, -2)$,
  $Z \in (-0.9, -0.8)$
- $w \in (-50, -2)$,
  $Z \in (-0.8, -0.7)$
- $w \in (-10, -0.8)$,
  $Z = -0.86602$

Sign of $d''(w) = d''(w, Z)$

- $w \in (-50, -2)$,
  $Z \in (-0.9, -0.8)$
- $w \in (-50, -2)$,
  $Z \in (-0.8, -0.7)$
- $w \in (-10, -0.8)$,
  $Z = -0.86602$

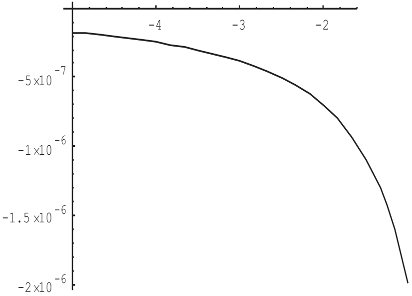


Sign of $d''(w) = d''(w, Z)$

- $w \in (-50, -2)$,
  $Z \in (-0.9, -0.8)$
- $w \in (-50, -2)$,
  $Z \in (-0.8, -0.7)$
- $w \in (-10, -0.8)$,
  $Z = -0.86602$
- $w \in (-5, -1)$,
  $Z = -0.86603$

Sign of $d''(w) = d''(w, Z)$

- $w \in (-50, -2)$,
  $Z \in (-0.9, -0.8)$
- $w \in (-50, -2)$,
  $Z \in (-0.8, -0.7)$
- $w \in (-10, -0.8)$,
  $Z = -0.86602$
- $w \in (-5, -1)$,
  $Z = -0.86603$



Theorem

Let $w < 0$, $Z \in \mathbb{R}$ satisfying $\frac{Z^2}{4} < -w$. Then for $Z^* \approx -0.866025403784$, the function $(w, Z) \to -||\phi_{w,Z}||^2$ satisfies the following properties

$$\begin{cases} -\partial_w ||\phi_{w,Z}||^2 > 0, & \text{if } Z > Z^*, \\ -\partial_w ||\phi_{w,Z}||^2 < 0, & \text{if } Z < Z^*. \end{cases}$$

Proof of the stability theorem

$$p_Z(w_0) := \begin{cases} 1, & \text{if } -\partial_w ||\phi_{w,Z}||^2 > 0 \text{ at } w = w_0, \\ 0, & \text{if } -\partial_w ||\phi_{w,Z}||^2 < 0 \text{ at } w = w_0. \end{cases}$$

$$H_{w_0,Z} := \Psi''([\phi_{w,Z}, 0]^t) = \begin{bmatrix} \mathcal{L}_{1,Z} & 0 \\ 0 & \mathcal{L}_{2,Z} \end{bmatrix},$$

Set $n(H_{w_0,Z})$ = number of negative eigenvalues of $H_{w_0,Z}$

Theorem

Let $-w_0 > \frac{Z^2}{4}$. Suppose that $Ker(\mathcal{L}_{1,Z}) = \{0\}$, $Ker(\mathcal{L}_{2,Z}) = [\phi_{w,Z}]$.

(1) *The standing wave $e^{-iw_0 t}\phi_{w_0,Z}$ is stable in $H^1(\mathbb{R})$ if $n(H_{w_0,Z}) = p_Z(w_0)$.*

(2) *The standing wave $e^{-iw_0 t}\phi_{w_0,Z}$ is instable in $H^1(\mathbb{R})$ if $n(H_{w_0,Z}) - p_Z(w_0)$ is odd.*

Proof of stability/instability theorem

Proof of stability/instability theorem

- For $Z \geq 0$, $\Omega_{\phi_{w,Z}}$ is stable in $H^1(\mathbb{R})$: $n(H_{w,Z}) = 1 = P_Z(w)$.

Proof of stability/instability theorem

I For $Z \geq 0$, $\Omega_{\phi_{w,Z}}$ is stable in $H^1(\mathbb{R})$: $n(H_{w,Z}) = 1 = P_Z(w)$.

II For $Z \in (Z^*, 0)$, $\Omega_{\phi_{w,Z}}$ is instable in $H^1(\mathbb{R})$: $n(H_{w,Z}) = 2$ and $P_Z(w) = 1$.

Proof of stability/instability theorem

I For $Z \geq 0$, $\Omega_{\phi_{w,Z}}$ is stable in $H^1(\mathbb{R})$: $n(H_{w,Z}) = 1 = P_Z(w)$.

II For $Z \in (Z^*, 0)$, $\Omega_{\phi_{w,Z}}$ is instable in $H^1(\mathbb{R})$: $n(H_{w,Z}) = 2$ and $P_Z(w) = 1$.

III For $Z \in (Z^*, \infty)$, $\Omega_{\phi_{w,Z}}$ is stable in $H^1_{even}(\mathbb{R})$: $n(H_{w,Z}|_{H^1_{even}}) = 1$ and $P_Z(w) = 1$.

Proof of stability/instability theorem

I For $Z \geq 0$, $\Omega_{\phi_{w,Z}}$ is stable in $H^1(\mathbb{R})$: $n(H_{w,Z}) = 1 = P_Z(w)$.

II For $Z \in (Z^*, 0)$, $\Omega_{\phi_{w,Z}}$ is instable in $H^1(\mathbb{R})$: $n(H_{w,Z}) = 2$ and $P_Z(w) = 1$.

III For $Z \in (Z^*, \infty)$, $\Omega_{\phi_{w,Z}}$ is stable in $H^1_{even}(\mathbb{R})$: $n(H_{w,Z}|_{H^1_{even}}) = 1$ and $P_Z(w) = 1$.

IV For $Z \in (-\infty, Z^*)$, $\Omega_{\phi_{w,Z}}$ is instable in $H^1_{even}(\mathbb{R})$: $n(H_{w,Z}|_{H^1_{even}}) = 1$ and $P_Z(w) = 0$.

Referências Bibliográficas

- S. Albeverio, F. Gesztesy, R. Krohn, H. Holden, Solvable Models in quantum mechanics, AMS Chelsea publishing. 2004.
- Jaime Angulo Pava, Nonlinear Dispersive Equations: Existence and Stability of Solitary and Periodic Travelling Wave Solutions, AMS 2009.
- T. Cazenave, Semilinear Schrödinger Equations, American Mathematical Society, (AMS). Lecture Notes, v. 10, 2003.
- R. Fukuizumi, and L. Jeanjean, Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta Potential, Discrete Contin. Dyn. Syst. 21, 2008.

Referências Bibliográficas

- H. Goodman, P.J. Holmes and M. I. Weinstein, Strong NLS soliton-defect interaction, Physica D, 192, 215-248, 2004.
- M. Grillakis, J. Shatah, and W. Strauss, Stability theory of solitary waves in the presence of symmetry, I, J. Funct. Anal., 160-197, 74 1987.
- T. Kato, Perturbation Theory for Linear Operators. 2nd edition, Springer, Berlin, 1984.

Thanks