Almost sure global well-posedness for the cubic wave equation

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Wave equation

We consider the cubic wave equation on $\mathbb{R}^3$:

$$\begin{cases} \partial_t^2 f - \triangle f + f^3 = 0 \\ f_{|t=0} = f_0 \ , \ \partial_t f_{|t=0} = f_1 \end{cases} .$$

The critical exponent of this equation is $s = \frac{1}{2}$.

Our aim is to use probabilities to prove that this equation is almost surely (with regard to a certain measure) globally well-posed in subcritical spaces $H^\sigma \times H^{\sigma-1}$ with $\sigma \in [0, 1/2)$.

Result

Theorem There exist probability measures μ on spaces of low regularity such that $\mu(H^{1/2} \times H^{-1/2}) = 0$ and for μ-almost every (f_0, f_1), the cubic wave equation with initial datum (f_0, f_1) has a unique global solution in $L(t)(f_0, f_1) + C(\mathbb{R}, H^1(\mathbb{R}^3))$ where $L(t)$ is the flow of the linear wave equation $\partial_t^2 - \triangle = 0$.

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Compactification

The first step is to use the Penrose transform (conformal) to turn the problem on $\mathbb{R}^3$ into a problem on the sphere S^3 :

$$\begin{cases} \partial_T^2 u + (1 - \triangle_{S^3})u + u^3 = 0 \\ u_{|T=0} = u_0 \ , \ \partial_T u_{|T=0} = u_1 \end{cases} .$$

Remark : This step is probably unnecessary. Though, skipping it implies using objects that seem less natural or at least not canonical.
The transform that maps (u_0, u_1) to (f_0, f_1) is an isometry between $H^s \times H^{s-1}$ of the sphere and $\mathcal{H}_0^s \times \mathcal{H}_1^{s-1}$ of $\mathbb{R}^3$ where $\mathcal{H}_i^s$ is very similar to H^s. In particular, if (u_0, u_1) is not in $H^{1/2} \times H^{-1/2}$ then (f_0, f_1) can not be in critical or super critical spaces.
Remark 2 : The existence of a solution of this compact equation gives the existence of a solution on $\mathbb{R}^3$. Uniqueness has to be treated separately.

Reduction

The second step is to reduce the equation on u on an equation on $v = u - U(T)(u_0, u_1)$ where $U(T)$ is the flow of the linear equation $\partial_T^2 + 1 - \triangle_{S^3} = 0$. We get

$$\partial_T^2 v + (1 - \triangle_{S^3})v + (U(T)(u_0, u_1) + v)^3 = 0$$

with initial datum $v_{|T=0} = v_0 = 0$ and $\partial_T v_{|T=0} = v_1 = 0$.

Local well-posedness

The Duhamel form of this equation is given by :

$$v(T) = U(T)(v_0, v_1) - \int_0^T \frac{\sin((T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}\Big(U(\tau)(u_0,u_1) + v(\tau)\Big)^3 d\tau \ .$$

The local theory yields that the Cauchy problem associated with this equation is well-posed in H^1 as soon as $v_0 \in H^1$, $v_1 \in L^2$ and $\frac{1}{(1+T^2)^{1/3}} U(T)(u_0,u_1) \in L^3_T, L^6(S^3)$.

Global theory on v

We use energy estimates with

$$\mathcal{E}(T) = \int_{S^3} (\partial_T v)^2 + \int v(1-\triangle)v + \frac{1}{2}\int v^4 \, .$$

Gronwall lemma yields

$$\mathcal{E}(T) \lesssim \Big(\int_0^T \|U(\tau)(u_0,u_1)\|_{L^6}^3 d\tau \Big) e^{c\int_0^T (\|U(\tau)(u_0,u_1)\|_{L^6}^2 + \|U(\tau)(u_0,u_1)\|_{L^\infty}) d\tau} \, .$$

We have global well posedness in $U(T)(u_0,u_1) + C(\mathbb{R}, H^1)$ as soon as $U(T)(u_0,u_1)$ belongs to $L^1_{\mathrm{loc},T}, L^\infty(S^3)$.

Conditions on the measure

We want to find a non trivial measure ρ on the topological σ-algebra of $H^\sigma \times H^{\sigma-1}$ such that :

- $\rho(H^{1/2} \times H^{-1/2}) = 0$,
- $\frac{1}{(1+T^2)^{1/3}} U(T)(u_0, u_1) \in L^3_T, L^6(S^3)$,
- $U(T)(u_0, u_1) \in L^1_{\mathrm{loc},T}, L^\infty(S^3)$.

For this, we will randomize the initial data.

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Notations

- $(e_{n,k})_{n,k}$ is a L^2 orthogonal basis composed of spherical harmonics : $-\triangle_{S^3} e_{n,k} = n^2 e_{n,k}$, $1 \le k \le (n+1)^2$,
- $(a_{n,k})_{n,k}$, $(b_{n,k})_{n,k}$ are two sequences of independent real Gaussian variables of law $\mathcal{N}(0,1)$ in a probability space $\Omega, \mathcal{A}, \mathbb{P}$,
- $\overline{u}_0 = \sum \lambda_{n,k} e_{n,k}$ belongs to H^σ for some $\sigma \in (0, 1/2)$ but does not belong to $H^{1/2}$,
- $\overline{u}_1 = \sum \mu_{n,k} e_{n,k}$ belongs to $H^{\sigma-1}$ (for the same σ) but not to $H^{-1/2}$.

Remark : The Gaussian condition can be released. We can take $(a_{n,k})_{n,k}$, $(b_{n,k})_{n,k}$ two sequences of i.i.d random variables that satisfy : "there exists c such that for all γ and all (n, k)

$$E(e^{\gamma a_{n,k}}), E(e^{\gamma b_{n,k}}) \le e^{c\gamma^2} \text{"} .$$

We can also take $\sigma = 0$.

Randomization

We then build two random variables :

$$u_0(\omega) = \sum_{n,k} \lambda_{n,k} a_{n,k}(\omega) e_{n,k}$$
$$u_1(\omega) = \sum_{n,k} \mu_{n,k} b_{n,k}(\omega) e_{n,k} .$$

We then have a measure ρ on $H^\sigma \times H^{\sigma-1}$, the image measure of $\mathbb{P}$ by (u_0, u_1), that is

$$\rho(A_0 \times A_1) = \mathbb{P}(u_0^{-1}(A_0))\mathbb{P}(u_1^{-1}(A_1)) .$$

Using the continuity of the space-time compactification on the initial datum, we can define the image measure μ of ρ by this transform and get back on the Euclidean space this way.
For almost all $\omega \in \Omega$, $u_0(\omega)$ does not belong to $H^{1/2}$ and $u_1(\omega)$ does not belong to $H^{-1/2}$. For μ almost all (f_0, f_1), f_0 does not belong to $H^{1/2}$ and f_1 does not belong to $H^{-1/2}$.

Properties of the randomization

With a good choice of $e_{n,k}$ and $\sigma > 0$, we have for all $(p,q) \in [1,\infty)$

$$\rho\Big(\Big\{(\tilde{u}_0,\tilde{u}_1) \in H^\sigma \times H^{\sigma-1} \;\Big|\; \|\frac{1}{(1+T^2)^{1/p}} U(T)(\tilde{u}_0,\tilde{u}_1)\|_{L_T^p, W^{\sigma,q}(S^3)} < \infty\Big\}\Big) = 1 \; .$$

In other words, for ρ almost every $(\tilde{u}_0,\tilde{u}_1)$ taken in $H^\sigma \times H^{\sigma-1}$, the function $\frac{1}{(1+T^2)^{1/p}} U(T)(\tilde{u}_0,\tilde{u}_1)$ belongs to $L_T^p, W^{\sigma,q}(S^3)$ and since $\sigma > 0$ to $L_T^p, L^\infty(S^3)$.

Hence, ρ almost every $(\tilde{u}_0,\tilde{u}_1)$ is a good candidate to be an initial datum of the equation on the sphere.

We prove that

$$I := \| \frac{1}{(1+T^2)^{1/p}} D^{\sigma} U(T)(u_0, u_1) \|_{L^r_\omega, L^p_T, L^q(S^3)}$$

is finite with $r = \max(p, q)$ and $D = (1 - \triangle_{S^3})^{1/2}$.
The Minkowski inequality yields

$$I \leq \| \frac{1}{(1+T^2)^{1/p}} D^{\sigma} U(T)(u_0, u_1) \|_{L^p_T, L^q(S^3), L^r_\omega} .$$

Then, at $x \in S^3$ and T fixed, with $\langle n \rangle = (1 + n^2)^{1/2}$

$$\frac{1}{(1+T^2)^{1/p}} D^\sigma U(T)(u_0, u_1) = \sum_{n,k} \frac{1}{(1+T^2)^{1/p}} \langle n \rangle^\sigma \times \left(\cos(\langle n \rangle T) a_{n,k} \lambda_{n,k} e_{n,k}(x) + \frac{\sin(\langle n \rangle T)}{\langle n \rangle} b_{n,k} \mu_{n,k} e_{n,k}(x) \right)$$

is a Gaussian variable as a linear combination of independent Gaussian variables. Hence, its L^r_ω norm is bounded by $C\sqrt{r}$ times its L^2_ω norm.

It gives

$$\|\frac{1}{(1+T^2)^{1/p}} D^\sigma U(T)(u_0,u_1)\|_{L^r_\omega} \lesssim \sqrt{r}\frac{1}{(1+T^2)^{1/p}}\Big(\sum_{n,k}(\langle n\rangle^{2\sigma}|\lambda_{n,k}|^2 + \langle n\rangle^{2\sigma-2}|\mu_{n,k}|^2)|e_{n,k}(x)|^2\Big)^{1/2}.$$

It remains to take the $L^p_T, L^q(S^3)$ of the right hand side of the inequality.

$$I \leq C\sqrt{r}\Big(\sum_{n,k}(\langle n\rangle^{2\sigma}|\lambda_{n,k}|^2 + \langle n\rangle^{2\sigma-2}|\mu_{n,k}|^2)\|e_{n,k}\|^2_{L^q}\Big)^{1/2}.$$

We need a L^2 basis $(e_{n,k})_{n,k}$ uniformly bounded in L^q.

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A measure on basis

(Technique and result by Burq and Lebeau)
The idea is to build a measure on the basis of L^2 composed of spherical harmonics such that the probability that the basis is uniformly bounded in L^p is non 0.

We consider the L^2 orthonormal basis of spherical harmonics of degree n as the orthogonal group $O_{N_n}(\mathbb{R})$ where N_n is the dimension of spherical harmonics of degree n : $N_n = (n+1)^2$.

Take ν_n the Haar measure on $O_{N_n}(\mathbb{R})$ and

$$\nu = \otimes_{n\in\mathbb{N}} \nu_n \ .$$

To evaluate probabilities on one element $b_{n,k}$ of the basis, we have to take the k-th column of the matrix associated to $(b_{n,j})_j$.

The image measure of ν_n by the map that takes the k-th column is (thanks to invariances) the uniform probability measure on the sphere S^{N_n} : p_{N_n}.

Measure concentration phenomenon

The Lipschitz-continuous functions F from S^N to $\mathbb{R}$ concentrate on their median M_F in the sense that

$$p_N(\{x \mid |F(x) - M_F| \geq R\}) \leq 2e^{-(N-1)R^2/(2\|F\|_{lip}^2)}$$

with

$$|F(x) - F(y)| \leq \|F\|_{lip}\|x - y\|_2 \ .$$

The L^p norm is Lipschitz continuous on spherical harmonics of degree n :

$$\|x - y\|_{L^p} \leq Cn^{1-1/p}\|x - y\|_{L^2}$$

and its median M_p is bounded by $C\sqrt{p}$ (independent from n). Both are consequences of the fact that S^3 has a finite volume and that for all x, y in S^3, there exists a transformation R on S^3 that preserves the metrics such that $x = Ry$.

We get then

$$\begin{aligned} \nu_n(\|b_{n,k}\|_{L^p} - M_p \geq R) &\leq p_n(|\,\|x\|_{L^p} - M_p| \geq R) \\ &\lesssim e^{-cR^2 n^{4/p}} \ . \end{aligned}$$

By summing over k

$$\nu_n(\exists k \mid \|b_{n,k}\|_{L^p} - M_p \geq R) \lesssim n^2 e^{-cR^2 n^{4/p}}$$

and over n

$$\nu(\exists n, k \mid \|b_{n,k}\|_{L^p} - M_p \geq R) \leq C_p R^{-2} \ .$$

Conclusion

We have used probabilities in two ways :

- randomizing the initial datum makes it almost surely in L^p spaces,
- randomizing the basis enables us to take one uniformly bounded in L^p.

If σ (the regularity of the ID) is 0, then we do not have $U(T)(u_0, u_1)$ almost surely in L^∞ but being careful with the choice of the basis and using a bootstrap argument in the energy estimates, we still have the same result as before.