Orbital stability of solitary waves

(Benjamin 1972, Bona 1975)

Suppose $c>0$ and $\theta \in \mathbb{R}$.

For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^1$ and

$$\| u_0 - \phi^{(1)}(x;\theta,c) \|_{H^1} < \delta,$$

then

$$\inf_{y \in \mathbb{R}} \| u(x,t) - \phi^{(1)}(x;y,c) \|_{H^1} < \varepsilon$$

for all $t>0$.

(This says that

$$d(u(x,t), \mathcal{O}) < \varepsilon \quad \text{for all } t>0,$$

where $\mathcal{O}$ is the orbit $\{ u^{(1)}_{c,\theta}(x,t) : t \in \mathbb{R} \}$ of the solitary-wave solution $u^{(1)}_{c,\theta}$.)

"Stability" of 2-solitons

<u>Theorem</u> (Maddocks + Sachs, 1993)

Suppose $\phi^{(2)}(x;\theta,c)$ is a given 2-solitary wave profile. For every $\varepsilon>0$, There exists $\delta>0$ such that if

$$u_0 \in H^2 \text{ and } \|u_0 - \phi^{(2)}(x;\theta,c)\|_{H^2} < \delta,$$

Then

$$\inf_{y\in\mathbb{R}^2} \|u(x,t) - \phi^{(2)}(x;y,c)\|_{H^2} < \varepsilon$$

for all $t>0$.

(This is not orbital stability.)

Cf. Martel, Merle, Tsai (2001)

Neves + Lopes (2006).

The KdV equation

$$u_t + uu_x + u_{xxx} = 0 \qquad (x \in \mathbf{R},\ t \geq 0)$$

has the infinite sequence of conserved functionals:

$$I_2(u) := \int_{-\infty}^{\infty} \frac{1}{2} u^2 \, dx,$$

$$I_3(u) := \int_{-\infty}^{\infty} \left(\frac{1}{2} u_x^2 - \frac{1}{6} u^3 \right) \, dx,$$

$$I_4(u) := \int_{-\infty}^{\infty} \left(\frac{1}{2} u_{xx}^2 - \frac{5}{6} u u_x^2 + \frac{5}{72} u^4 \right) dx,$$

$$\ldots$$

For each $n \in \mathbf{N}$, $I_n : H^{n-2} \to \mathbf{R}$ is continuous.

For N-soliton solutions, $\phi^{(N)}(x;\theta,c)$,

$\theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$ and

$c = (c_1, \dots, c_N) \in \mathbb{R}^N$ with $c_i > 0 \ \forall i$
and $c_i \neq c_j \ (i \neq j)$,

we have

$$I_2(\phi^{(N)}) = 12\left[c_1^3 + c_2^3 + \dots + c_N^3\right],$$

$$I_3(\phi^{(N)}) = \frac{-36}{5}\left[c_1^5 + c_2^5 + \dots + c_N^5\right],$$

$$I_4(\phi^{(N)}) = \frac{37}{7}\left[c_1^7 + c_2^7 + \dots + c_N^7\right],$$

etc.

Theorem: (Cazenave + Lions)

~~[illegible]~~ Suppose $c > 0$.

Let $s = I_2(c)$, and

$$m_s = \inf \{ I_3(u) : u \in H_1 \text{ and } I_2(u) = s \}.$$

Let

$$G_s = \{ u \in H_1 : I_3(u) = m_s \text{ and } I_2(u) = s \}.$$

Then

$$G_s = \{ \phi^{(1)}(x; \theta, c) : \theta \in \mathbb{R} \}.$$

Moreover, if $\{u_n\}$ is any sequence in H_1 such that $I_2(u_n) = s$ and

$$\lim_{n \to \infty} I_3(u_n) = m_s \text{ ; then}$$

$$\lim_{n \to \infty} \operatorname{dist}_{H_1}(u_n, G_s) = 0.$$

<u>Theorem</u>: Suppose $c_1, c_2 > 0$.

Let $s = I_2(c)$ and $t = I_3(c)$,

and

$$m_{s,t} = \inf \{ I_4(u) : u \in H_2, \ I_2(u) = s, \text{ and } I_3(u) = t \}.$$

Let

$$G_{s,t} = \{ u \in H_2 : I_4(u) = m_{s,t}, \ I_2(u) = s, \text{ and } I_3(u) = t \}.$$

Then

$$G_{s,t} = \{ \phi^{(2)}(x; \theta, c) : \theta \in \mathbb{R}^2 \}.$$

Moreover, if $\{u_n\}$ is any sequence in H^2 such that $I_2(u_n) = s$, $I_3(u_n) = t$, and $\lim_{n \to \infty} I_4(u_n) = m_{s,t}$; Then

$$\lim_{n \to \infty} \operatorname{dist}_{H^2} (u_n, G_{s,t}) = 0$$

<u>Lemma</u>: Suppose p, q are any real numbers.

If $G_{p,q}$ is not empty, then either

(i) $G_{p,q} = \{ \phi^{(2)}(x; \theta, c) : \theta \in \mathbb{R}^2 \}$

for some $c \in \mathbb{R}^2$; $c_1, c_2 > 0$; $c_1 \neq c_2$;

or

(ii) $G_{p,q} = \{ \phi^{(1)}(x; \theta, c) : \theta \in \mathbb{R}^1 \}$

for some $c > 0$.

Hence, in either case,

$$G_{p,q} = \frac{36}{7}(c_1^7 + c_2^7)$$

for some $c \in \mathbb{R}^2$; $c_1, c_2 \geq 0$.

Now consider the problem of minimizing $I_4(u)$ over the set of all $u \in H^2$ such that $I_3(u) = B$ and $I_2(u) = A$.

If minimizers exist, they are homoclinic solutions of the ODE

$$I_4'(u) + \mu I_3'(u) + \lambda I_2'(u) = 0,$$

or

$$\begin{aligned} u'''' &+ \frac{5}{6}(u')^2 + \frac{5}{3}uu'' + \frac{5}{18}u^3 \\ &+ \mu\left(-u'' - \frac{1}{2}u^2\right) + \lambda u = 0. \end{aligned}$$

If we define $v = u'$, $w = u''$, $z = u'''$, then the ODE can be rewritten as

$$\begin{pmatrix} u' \\ v' \\ w' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda & 0 & \mu & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} + \text{higher order terms.}$$

The eigenvalues of the linearized equation are the roots of the equation $\beta^4 - \mu\beta^2 + \lambda = 0$.

Eigenvalues of the linearized equation: