On the Dynamics of the Symmetric Solutions of the Schrödinger-Korteweg de Vries system in the Energy Space

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The Model

We study the interactions between long and short waves that arise in different physics contexts. Specifically, we study interactions modeled by the Initial Value Problem (IVP) for the Schrödinger-Korteweg de Vries system, that is,

$$\begin{cases} i\phi_\tau + \phi_{\xi\xi} = \alpha\phi\eta + \beta|\phi|^2\phi, \quad \tau,\ \xi \in \mathbb{R}, \\ \eta_\tau + \eta_{\xi\xi\xi} + \eta\eta_\xi = \gamma(|\phi|^2)_\xi, \\ \phi(\xi,0) = \phi_0(\xi), \quad \eta(\xi,0) = \eta_0(\xi), \end{cases} \tag{1}$$

where the short wave $\phi = \phi(\xi,\tau)$ is a complex valued function, the long wave $\eta = \eta(\xi,\tau)$ is a real valued function and α, β and γ are real constants with $\alpha\gamma \neq 0$.

The resonant interactions ($\beta = 0$) appear, for instance, in plasma physics, in a diatomic lattice system and in the study of interactions for capillary-gravity waves on water of uniform finite depth.

Focusing and Defocusing Regime

Under the transformations:

$$\phi(\xi,\tau)=\sqrt{\tfrac{8}{|\alpha\gamma|}}\,u(2\xi,4\tau)=\sqrt{\tfrac{8}{|\alpha\gamma|}}\,u(x,t)$$

and

$$\eta(\xi,\tau)=\tfrac{4}{\alpha}v(2\xi,4\tau)=\tfrac{4}{\alpha}v(x,t)$$

the system (1) becomes into the coupled equations:

$$\begin{cases} iu_t+u_{xx}=uv+p|u|^2u, & t,\ x\in\mathbb{R},\\ v_t+2v_{xxx}+3q(v^2)_x=\epsilon(|u|^2)_x, & \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & \end{cases} \tag{2}$$

with $p=\frac{2\beta}{\alpha\gamma}$, $q=\frac{1}{3\alpha}$ and $\epsilon=sgn(\alpha\gamma)$.

We will refer to the cases $\epsilon=-1$ and $\epsilon=1$ as *focusing* and *defocusing* case, respectively.

Conservation Laws

Mass:

$$E_1(t) = \int_{-\infty}^{+\infty} |u|^2 dx = E_1(0).$$

Moment:

$$E_2(t) = \int_{-\infty}^{+\infty} \Big\{ v^2 + 2\epsilon \ \mathrm{Im}(u\bar{u}_x) \Big\} dx = E_2(0).$$

Energy:

$$E_3(t) = \int_{-\infty}^{+\infty} \Big\{ |u_x|^2 + \epsilon |v_x|^2 + v|u|^2 + \epsilon p |u|^4 - \epsilon q v^3 \Big\} dx = E_3(0).$$

Overview about well-posedness

- M. Tsutsumi (**1993**) showed LWP in $H^{s+1/2} \times H^s$ for $s \in \mathbb{Z}^+$ and GWP in the same spaces if $\epsilon = 1$.
- D. Bekiranov -T. Ogawa - G. Ponce (**1997**) showed LWP in $H^{s+1/2} \times H^s$ for $s \geq -1/2$.
- B. Guo - Ch. Miao (**1999**) showed GWP in the energy space $H^1 \times H^1$ if $p = 0$ and $\epsilon = 1$.
- H. Pecher (**2005**) showed LWP in for $s > 0$ and GWP for $\epsilon = 1$ in $H^s \times H^s$ for
  - $s > 3/5$ in the case $p = 0$,
  - $s > 2/3$ in the case $p \neq 0$.
- A. J. Corcho - F. Linares (**2007**) showed LWP in $H^{s_1} \times H^{s_2}$ for $s_2 > -3/4$, $s_1 - 1 \leq s_2 \leq 2s_1 - 1/2$ if $0 \leq s_1 \leq 1/2$ and $s_1 - 1 \leq s_2 < s_1 + 1/2$ if $s_1 > 1/2$.
- Z. Guo - Y. Wang (**2010**) showed LWP in $L^2 \times H^{-3/4}$.

Theorem (*Y. Wu, 2010*)

Let $(u_0, v_0) \in H^s(\mathbb{R}) \times H^\kappa(\mathbb{R})$ *provided:*

- $\kappa > -3/4$ *and* $s - 2 \leq \kappa < \min\{4s,\ s+1\}$ *if* $p = 0$*;*
- $\kappa > -3/4$, $s - 2 \leq \kappa < \min\{4s,\ s+1\}$ *and* $s \geq 0$ *if* $p \neq 0$.

Then, there exist a positive time $T_0(\|u_0\|_{H^s}, \|v_0\|_{H^\kappa})$ *and a unique solution for the Cauchy problem (2) in the space* $C([0, T_0];\ H^s(\mathbb{R}) \times H^\kappa(\mathbb{R}))$*. Moreover, the map* $(u_0, v_0) \longmapsto (u(\cdot, t), v(\cdot, t))$ *is locally Lipchitz. Also, if* $(\epsilon = 1)$ *the solutions can be extended to any time interval* $[0, T]$ *for data* $(u_0, v_0) \in H^s \times H^s$ *when* $s > 1/2$ *for all* $p \in \mathbb{R}$*.*

Global existence for $\epsilon = -1$?

Here we focus our attention on the *focusing case* ($\epsilon = -1$), where it is not clear the existence of an a-priori estimate in the energy space $H^1 \times H^1$ since the terms $\int |u_x|^2 dx$ and $\int |v_x|^2 dx$ appear with oppositive signals in the energy conservation law E_3:

$$E_3(t) = \int_{-\infty}^{+\infty} \Big\{ |u_x|^2 - |v_x|^2 + v|u|^2 - p|u|^4 + qv^3 \Big\} dx = E_3(0).$$

Global Existence under symmetry assumptions

Theorem (*Global Extension for Local Symmetric Solutions*)

Assume that a local solution (u, v) of the IVP (2) in the energy space $H^1 \times H^1$ is symmetric (even or odd function in the spatial variable), then this local solution can be extended to all time interval $[0, T]$.

Auxiliary Lemma

Lemma

Let $(u(\cdot,t), v(\cdot,t))$ *a solution in* $C([0,T_{max}); H^1 \times H^1)$ *of the IVP (2) provided by Theorem (Y. Wu, 2010). Then, for any* $\lambda_1, \lambda_2, \lambda_3 > 0$ *there exist positive constants* $A_{\lambda_1,\lambda_2}(\|u_0\|_{L^2}, E_2(0))$ *and* $B_{\lambda_1,\lambda_2}(\|u_0\|_{L^2}, E_2(0))$ *(only depend on* $\|u_0\|_{L^2}$ *and* $E_2(0)$*) and* $C_{\lambda_3}(\|u_0\|_{L^2})$ *only depending on* $\|u_0\|_{L^2}$ *such that*

$$\Big|\int |u|^2 v dx\Big| \le \lambda_1 \|v_x\|_{L^2}^2 + \lambda_2 \|u_x\|_{L^2}^2 + A_{\lambda_1,\lambda_2}(\|u_0\|_{L^2}, E_2(0)), \tag{3}$$

$$\Big|\int v^3 dx\Big| \le \lambda_1 \|v_x\|_{L^2}^2 + \lambda_2 \|u_x\|_{L^2}^2 + B_{\lambda_1,\lambda_2}(\|u_0\|_{L^2}, E_2(0)), \tag{4}$$

$$\Big|\int |u|^4 dx\Big| \le \lambda_3 \|u_x\|_{L^2}^2 + C_{\lambda_3}(\|u_0\|_{L^2}). \tag{5}$$

Proof: By the conserved quantities $E_1(t)$ and $E_2(t)$, we have

$$\|v\|_{L^2} \leq |E_2(0)|^{1/2} + \sqrt{2}\|u_0\|_{L^2}^{1/2}\|u_x\|_{L^2}^{1/2}. \tag{6}$$

Then, using the conserved quantity $E_1(t)$, Sobolev's immersion and Young's inequality we get

$$\begin{aligned} \left|\int |u|^2 v dx\right| &\leq \sqrt{2}\|v_x\|_{L^2}^{1/2}\|v\|_{L^2}^{1/2}\|u_0\|_{L^2}^2 \\ &= (4\lambda_1)^{1/4}\|v_x\|_{L^2}^{1/2}\sqrt{2}(4\lambda_1)^{-1/4}\|v\|_{L^2}^{1/2}\|u_0\|_{L^2}^2 \\ &\leq \lambda_1\|v_x\|_{L^2}^2 + C\lambda_1{}^{-1/3}\|v\|_{L^2}^{2/3}\|u_0\|_{L^2}^{8/3}. \end{aligned}$$

Auxiliary Lemma

Combining the last inequality with (6) and using Young's inequality we obtain

$$\begin{aligned}\left|\int |u|^2 v dx\right| &\leq \lambda_1 \|v_x\|_{L^2}^2 + C{\lambda_1}^{-1/3} \|u_0\|_{L^2}^{8/3} \left(|E_2(0)|^{1/3} + \|u_0\|_{L^2}^{1/3} \|u_x\|_{L^2}^{1/3}\right) \\ &\leq \lambda_1 \|v_x\|_{L^2}^2 + C{\lambda_1}^{-1/3} |E_2(0)|^{1/3} \|u_0\|_{L^2}^{8/3} + C{\lambda_1}^{-1/3} \|u_0\|_{L^2}^{3} \|u_x\|_{L^2}^{1/3} \\ &\leq \lambda_1 \|v_x\|_{L^2}^2 + \lambda_2 \|u_x\|_{L^2}^2 + A_{\lambda_1,\lambda_2}(\|u_0\|_{L^2}, E_2(0)),\end{aligned}$$

where

$$A_{\lambda_1,\lambda_2}\left(\|u_0\|_{L^2}, E_2(0)\right) = C{\lambda_1}^{-2/5}{\lambda_2}^{-1/5} \|u_0\|_{L^2}^{18/5} + C{\lambda_1}^{-1/3} |E_2(0)|^{1/3} \|u_0\|_{L^2}^{8/3},$$

which yields (3).■

Proof of global existence: In this situation, $\int \mathrm{Im}(u\bar{u}_x)\,dx = 0$, if $t \in [0, T_{\max})$. Then, from the conservation law $E_2(t)$ we have

$$\int v^2(x,t)dx = \int v^2(x,0)dx. \tag{7}$$

Let $T \in (0, T_{\max})$ and consider the integral equation associated to the first equation in the IVP (2), then using the admissible pairs $(\infty, 2)$ and $(4, \infty)$ for the Strichartz's estimates we obtain

$$\begin{aligned} \|u_x(t)\|_{L^2_x} &\leq \|u_x(0)\|_{L^2_x} + \left\| \int_0^t U(t-t')(u_x v + u v_x)(t')dt' \right\|_{L^2_x} \\ &\leq \|u_x(0)\|_{L^2_x} + C\|u_x v\|_{L^{4/3}_T L^1_x} + C\|u v_x\|_{L^{4/3}_T L^1_x} \\ &\leq \|u_x(0)\|_{L^2_x} + CT^{3/4}\|u_x\|_{L^\infty_T L^2_x}\|v\|_{L^\infty_T L^2_x} + \\ &\quad + CT^{3/4}\|v_x\|_{L^\infty_T L^2_x}\|u\|_{L^\infty_T L^2_x} \end{aligned} \tag{8}$$

Global existence under symmetry assumptions

On the other hand by the conservation law $E_3(t)$ and Lemma 0.1 it is easy see that

$$\|v_x(t)\|_{L^2_x}^2 \leq -2E_3(0) + 3\|u_x(t)\|_{L^2_x}^2 + F\big(\|u_0\|_{L^2}, |E_2(0)|\big).$$

Hence

$$\|v_x(t)\|_{L^2_x} \leq 2\|u_x(t)\|_{L^2_x} + F_0\big(\|u_0\|_{L^2}, |E_2(0)|, |E_3(0)|\big). \tag{9}$$

Global existence under symmetry assumptions

Finally using the conservation laws $E_1(t)$ and (7), from (8) and (9) we obtain

$$\|u_x(t)\|_{L^2_x} \leq \|u_x(0)\|_{L^2_x} + CT^{3/4}F_0 + CT^{3/4}\|u_x\|_{L^\infty_T L^2_x}\left(\|u_0\|_{L^2_x} + \|v_0\|_{L^2_x}\right),$$

thus if

$$CT^{3/4}\left(\|u_0\|_{L^2_x} + \|v_0\|_{L^2_x}\right) \leq \frac{1}{2}, \tag{10}$$

then

$$\|u_x(t)\|_{L^2_x} \leq 2\|u_x(0)\|_{L^2_x} + CT^{3/4}F_0\left(\|u_0\|_{L^2}, |E_2(0)|, |E_3(0)|\right), \tag{11}$$

for all $t \in [0, T]$. Notice that the norms $\|u(t)\|_{L^2_x}$ and $\|v(t)\|_{L^2_x}$ are conserved; hence, iterating this process and using (10) we obtain the a-priori estimate (11) for all $t \in [0, T_{\max})$. ■

Global existence under symmetry assumptions

Remark

If the solution $(u, v) \in H^1 \times H^1$ *of the IVP* (2) *has the form* $u(x,t) = U(x - \gamma(t), t)$ *and* $v(x,t) = V(x - \gamma(t), t)$, *where* U *is a symmetric function in the spatial variable, then making a change of variables, we have*

$$\int Im(u\bar{u}_x)(x,t)\,dx = \int Im(U\bar{U}_x)(x,t)\,dx = 0$$

and also, in this case, the solution can be extended to all time.

Theorem (*Viriel Identity*)

Let $(u(\cdot,t), v(\cdot,t))$ *a solution in* $C\left([0,T_0]; H^{1+}\times H^{2+}\right)$ *of the IVP (2) provided by the local theory given in the Theorem (Y. Wu, 2010) and assume in addition that the initial data* $(u_0, v_0)\in L^2(x^2dx)\times L^2(x^2dx)$. *Then, for all* $t\in[0,T_0]$, *we have that*

(a) $\frac{d}{dt}\int x^2|u|^2dx = 4\mathcal{I}m\int x\bar{u}u_x dx.$

(b) $f''(t) = 8\int|u_x|^2dx - 12\int|v_x|^2dx + 4\int v|u|^2dx + 8q\int v^3dx + 2p\int|u|^4dx.$

where

$$f(t) := \int x^2|u|^2dx + 2\int_0^t\int xv^2dxdt'.$$

Remark

If v is **symmetric** *solution of the system (2). From inequality*

$$\left|\int xv^2(x,t)dx\right| \le \int |xv(x,t)||v(x,t)|dx$$
$$\le \|xv\|_{L^2_x}\|v\|_{L^2},$$

it's follows that $\int xv^2(x,t)dx = 0$*, for all* $t \in [0, T_0]$*.*
Thus, in this context, the viriel identity takes the form

$$\frac{d^2}{dt^2}\int x^2|u|^2dx = 8\int |u_x|^2dx - 12\int |v_x|^2dx + 4\int |u|^2vdx + 8q\int v^3dx + 2p\int |u|^4dx.$$

Persistence property

Proposition (*Persistence property*)

Let $(u(\cdot,t),v(\cdot,t))$ *a local solution in the space* $C\left([0,T_0];H^{1+}\times H^{2+}\right)$ *of the IVP (2) provided by Theorem (Y. Wu, 2010) and assume in addition that* $(u_0,v_0)\in L^2(x^2dx)\times L^2(x^2dx)$. *We then have*

$$(u(\cdot,t),v(\cdot,t))\in C\left([0,T_0];H^{1+}\cap L^2(x^2dx)\times H^{2+}\cap L^2(x^2dx)\right),$$

for all $t\in[0,T_0]$.

Viriel estimate

Theorem (Viriel estimate)

Let $(u(\cdot,t), v(\cdot,t))$ *a solution in* $C\left([0,T_{max}); H^{1+}\times H^{2+}\right)$ *of the IVP (2) provided by Theorem (Y. Wu, 2010) with* $(u_0, v_0)\in L^2(x^2dx)\times L^2(x^2dx)$ *and assume that that* v *is symmetric. Then for any* $8<\delta<12$ *there exists a positive function* F*, only depending on* δ*,* $\|u_0\|_{L^2}$ *and* $|E_2(0)|$ *such that*

$$\frac{d^2}{dt^2}\int x^2|u(x,t)|^2dx \le \delta E_3(0) + F\left(\delta, \|u_0\|_{L^2}, |E_2(0)|\right). \tag{12}$$

Proof of the Viriel estimate

Proof: Initially, we let $\delta \in \mathbb{R}$. From (0.2) we get

$$\frac{d^2}{dt^2}\int x^2|u(x,t)|^2dx = \delta E_3(0) + (8-\delta)\int |u_x|^2dx + (\delta-12)\int (v_x)^2dx + (4-\delta)\int v|u|^2dx + q(8-\delta)\int v^3dx + p(\delta+2)\int |u|^4dx. \tag{13}$$

Now we note that if $8 < \delta < 12$ then using the auxiliary Lemma with positive numbers $\lambda_1, \lambda_2, \lambda_1', \lambda_2'$ and λ_3' verifying

$$(\delta-4)\lambda_1 + |q|(\delta-8)\lambda_2 = 12-\delta \tag{14}$$

$$(\delta-4)\lambda_1' + |q|(\delta-8)\lambda_2' + |p|(\delta+2)\lambda_3' = \delta-8 \tag{15}$$

we get from (13) the following estimate:

Proof of the Viriel estimate

$$\frac{d^2}{dt^2}\int x^2|u(x,t)|^2dx \leq \ \delta E_3(0)+$$
$$+(\delta-4)A_0(\lambda_1,\lambda_1')+|q|(\delta-8)B_0(\lambda_2,\lambda_2')+|p|(\delta+2)C_0(\lambda_3').$$

Now we define $F\big(\delta,\|u_0\|_{L^2},|E_2(0)|\big)$ as the minimum in the variables λ_i and λ_i' of the continuous function

$$(\delta-4)A_0(\lambda_1,\lambda_1')+|q|(\delta-8)B_0(\lambda_2,\lambda_2')+|p|(\delta+2)C_0(\lambda_3')$$

in the bounded region defined by (14) and (15) to get

$$\frac{d^2}{dt^2}\int x^2|u(x,t)|^2dx \leq \delta E_3(0)+F\big(\delta,\|u_0\|_{L^2},|E_2(0)|\big),$$

as claimed. ■

Blow-up assumptions

Corollary (*Blow-up assumptions*)

Assume the same hypotheses of the Theorem 0.2 and set $K_\delta := \delta E_3(0) + F_\delta\left(\|u_0\|_{L^2}, |E_2(0)|\right)$ *such that verifies at least one of the following assumptions:*

$$(a)\ K_\delta < 0,$$

$$(b)\ K_\delta = 0 \quad \text{and} \quad \mathcal{I}m \int x\bar{u}_0 u_{0x} dx < 0,$$

$$(c)\ K_\delta > 0 \quad \text{and} \quad \mathcal{I}m \int x\bar{u}_0 u_{0x} dx \leq -\sqrt{K_\delta}\left\|xu_0\right\|_{L^2}.$$

Then, there is T^* *such that* $\lim_{t \to T^*} \|u_x(\cdot, t)\|_{L^2} = \infty$.

Examples

Remark

The set of the initial data that satisfy the blow-up assumptions in the Corollary 3 is not empty. For example, taking $u_0(x) \in H^1$ *and* $v_0(x) = \sqrt{N}e^{-(Nx)^2}$, *with* $N \in \mathbb{N}$, *the condition (a) in Corollary 3 is satisfied for enough large* N. *Indeed, we have*

$$\|v_0\|_{L^2}^2 = \int e^{-2(Nx)^2} N dx = \int e^{-2y^2} dy,$$

so $F_\delta\left(\|u_0\|_{L^2}, |E_2(0)|\right)$ *not depends on* N. *On the other hand, computing* $E_3(0)$ *we obtain*

$$E_3(0) = \|u_0'\|_{L^2}^2 - 4N^2 \int y^2 e^{-2y^2} dy + \frac{1}{\sqrt{N}} \int e^{-2y^2} |u_0(\tfrac{y}{N})|^2 dy - p\|u_0\|_{L^4}^4$$
$$+ q\sqrt{N} \int e^{-3y^2} dy,$$

which verifies $\delta E_3(0) < -F_\delta\left(\|u_0\|_{L^2}, |E_2(0)|\right)$ *for* $N \gg 1$.

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THANKS !