A sharp condition for global well-posedness of the Inhomogeneous Nonlinear Schrödinger Equation

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The Inhomogeneous Nonlinear Schrödinger Equation (INLS

We consider the following Inhomogeneous Nonlinear Schrödinger Equation (INLS)

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^{2\sigma} u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $(x,t) \in \mathbb{R}^N \times [0,\infty)$.

Remark

When $b = 0$ this is the well-known Nonlinear Schrödinger Equation (NLS).

Previous results

A more general form of this equation was considered by Merle (AIHP 1996) and Raphaël and Szeftel (JAMS 2011)

$$i\partial_t u + \Delta u + k(x)|u|^{2\sigma} u = 0,$$

where they study the problem of existence/nonexistence of minimal mass solutions.

However, in this both papers, they assume that

$$k(x) \text{ is bounded.}$$

Previous results

The (INLS) $i\partial_t u + \Delta u + |x|^{-b}|u|^{2\sigma}u = 0$, was already studied by Genoud (JAA 2012) in the case $\sigma = \frac{2-b}{N}$ (critical case). He proved global well-posedness in $H^1(\mathbb{R}^N)$ assuming

$$\|u_0\|_{L^2(\mathbb{R}^N)} < \|Q\|_{L^2(\mathbb{R}^N)},$$

where Q is the unique non-negative, radially-symmetric, decreasing solution of the equation

$$\Delta Q - Q + |x|^{-b}|Q|^{\frac{2(2-b)}{N}}Q = 0. \tag{2}$$

Remark

The existence and uniqueness of the ground state solution to (2) was proved by Genoud (PhD Thesis 2008), Toland (PRSE 1984) and Yanagida (ARMA 1991).

The Nonlinear Schrödinger Equation (NLS)

$$\begin{cases} i\partial_t u + \Delta u + |u|^{2\sigma} u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{3}$$

where $(x,t) \in \mathbb{R}^N \times [0,\infty)$.

Global well-posedness results in $H^1(\mathbb{R}^N)$

- Weinstein (CMP 83): If $\sigma = 2/N$ we have global solution if

$$\|u_0\|_{L^2(\mathbb{R}^N)} < \|Q\|_{L^2(\mathbb{R}^N)},$$

where Q is the unique non-negative, radially-symmetric, decreasing solution of (10) with $b = 0$.

Remark

Genoud's result is a generalization of the above result.

The Nonlinear Schrödinger Equation (NLS)

- Holmer and Roudenko (CMP 2008): If $\frac{2}{N} < \sigma < \frac{2}{N-2}$ we have global solution if

$$E[u_0]^{s_\sigma} M[u_0]^{1-s_\sigma} < E[Q]^{s_\sigma} M[Q]^{1-s_\sigma}, \quad E[u_0] \geq 0. \tag{4}$$

and

$$\|\nabla u_0\|^{s_\sigma}_{L^2(\mathbb{R}^N)} \|u_0\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)} < \|\nabla Q\|^{s_\sigma}_{L^2(\mathbb{R}^N)} \|Q\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)}, \tag{5}$$

where $s_\sigma = \frac{N}{2} - \frac{1}{\sigma}$ is the critical Sobolev index.

Remark

Note that if $\sigma = 2/N$ conditions (11) and (12) are the same and then we recover Weinstein's result.

Problem

Problem: Is it possible to prove a global well-posedness theorem for the (INLS) similar to Holmer and Roudenko (CMP 2008) result for the (NLS)?

Critical Sobolev index

- The $H^s(\mathbb{R}^N)$ space: $\|f\|^2_{H^s(\mathbb{R}^N)} \equiv \int_{\mathbb{R}^N}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2 dx$.
- Scaling: If u is a solution of (1) then, for all $\lambda > 0$,

$$u_\lambda(x,t) = \lambda^{\frac{2-b}{2\sigma}} u(\lambda x, \lambda^2 t)$$

is also a solution. Moreover,

$$\|u_\lambda(\cdot,0)\|_{\dot{H}^s} = \lambda^{s+\frac{2-b}{2\sigma}-\frac{N}{2}}\|u_0\|_{\dot{H}^s}.$$

- Critical Sobolev index: $s_\sigma = \frac{N}{2} - \frac{2-b}{2\sigma}$
- Assumption: If $\frac{2-b}{N} < \sigma < \frac{2-b}{N-2}$ then $0 < s_\sigma < 1$

(L^2-supercritical and H^1-subcritical).
We also assume $0 < b < \min\{2, N\}$.

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Conservation laws

We have following conserved quantities for the (INLS) equation

$$M[u(t)] = \int_{\mathbb{R}^N} |u(x,t)|^2 \, dx \tag{6}$$

and

$$E[u(t)] = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(x,t)|^2 \, dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |x|^{-b}|u(x,t)|^{2\sigma+2} \, dx. \tag{7}$$

- We need to show that the quantity (7) is well-defined for solutions in $H^1(\mathbb{R})$

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Sharp Gagliardo-Nirenberg inequality

Theorem

Let $k > 0$, then the Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{2\sigma+2}\,dx \leq K_{\mathrm{opt}} \,\|\nabla u\|_{L^2(\mathbb{R}^N)}^{N\sigma+b}\|u\|_{L^2(\mathbb{R}^N)}^{2\sigma+2-(N\sigma+b)}, \quad (8)$$

holds, and the sharp constant $K_{\mathrm{opt}} > 0$ is explicitly given by

$$K_{\mathrm{opt}} = \left(\frac{N\sigma+b}{2\sigma+2-(N\sigma+b)}\right)^{\frac{2-(N\sigma+b)}{2}} \frac{2\sigma+2}{(N\sigma+b)\|Q\|_{L^2(\mathbb{R}^N)}^{2\sigma}}, \quad (9)$$

where Q is the unique non-negative, radially-symmetric, decreasing solution of the equation

$$\Delta Q - Q + |x|^{-b}|Q|^{2\sigma}Q = 0. \quad (10)$$

Comments

(i) If $b = 0$ (NLS) and $\sigma = \frac{2}{N}$ we recover the sharp Gagliardo-Nirenberg inequality proved by Weinstein (CMP 83).

(ii) If $0 < b < \min\{2, N\}$ (INLS) and $\sigma = \frac{2-b}{N}$ we recover the sharp Gagliardo-Nirenberg inequality proved by Genoud (JAA 2012).

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Global well-posedness in $H^1(\mathbb{R}^N)$

Theorem

Let $\frac{2-b}{N} < \sigma < \frac{2-b}{N-2}$, $0 < b < \min\{2, N\}$ *and set* $s_\sigma = \frac{N}{2} - \frac{2-b}{2\sigma}$. *Suppose that* $u(t)$ *be the solution of* (1) *with initial data* $u_0 \in H^1(\mathbb{R}^N)$ *satisfying*

$$E[u_0]^{s_\sigma} M[u_0]^{1-s_\sigma} < E[Q]^{s_\sigma} M[Q]^{1-s_\sigma}, \quad E[u_0] \geq 0. \tag{11}$$

and

$$\|\nabla u_0\|^{s_\sigma}_{L^2(\mathbb{R}^N)} \|u_0\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)} < \|\nabla Q\|^{s_\sigma}_{L^2(\mathbb{R}^N)} \|Q\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)}, \tag{12}$$

where Q *is unique positive even solution of the elliptic equation* (10), *then* $u(t)$ *is a global solution in* $H^1(\mathbb{R}^N)$.

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Blow-up solutions in $H^1(\mathbb{R}^N)$

Theorem

Let $\frac{2-b}{N} < \sigma < \frac{2-b}{N-2}$, $0 < b < \min\{2, N\}$ *and set* $s_\sigma = \frac{N}{2} - \frac{2-b}{2\sigma}$. *Suppose that* $u(t)$ *be the solution of* (1) *with initial data*

$$u_0 \in H^1(\mathbb{R}^N) \cap \{u : |x|u \in L^2(\mathbb{R}^N)\}$$

satisfying

$$E[u_0]^{s_\sigma} M[u_0]^{1-s_\sigma} < E[Q]^{s_\sigma} M[Q]^{1-s_\sigma}, \quad E[u_0] \geq 0 \quad or \quad E[u_0] < 0. \tag{13}$$

and

$$\|\nabla u_0\|_{L^2(\mathbb{R}^N)}^{s_\sigma} \|u_0\|_{L^2(\mathbb{R}^N)}^{1-s_\sigma} > \|\nabla Q\|_{L^2(\mathbb{R}^N)}^{s_\sigma} \|Q\|_{L^2(\mathbb{R}^N)}^{1-s_\sigma}, \tag{14}$$

then the maximum existence time is finite and blow-up in $H^1(\mathbb{R}^N)$ *must occour.*

Virial type estimates for (INLS)

Proposition

Let $u(x,t)$ be a solution of (INLS) equation then

$$\frac{d}{dt}\int_{\mathbb{R}^N} |x|^2|u(x,t)|^2 dx = 4Im\int_{\mathbb{R}^N} \bar{u}(x,t)(\nabla u(x,t)\cdot x)dx \tag{15}$$

and

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2|u(x,t)|^2 dx = 8(N\sigma+b)E[u_0]-4(N\sigma+b-2)\|\nabla u(t)\|^2_{L^2(\mathbb{R}^N)}. \tag{16}$$

Remark

If $b=0$ (NLS) and $\sigma=\frac{2}{N}$ this is the Virial estimates obtained by Merle (AIHP 1996).

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If $b=0$ (NLS) and $\sigma=\frac{2}{N}$ this is the Virial estimates obtained by Merle (AIHP 1996).

Energy trapping

(see also Kenig and Merle (IM 2006) and Cazenave, Fang and Xie (Sci. China Math 2011))

Proposition

Let $u_0 \in H^1(\mathbb{R}^N)$ *such that*

$$\|\nabla u_0\|^{s_\sigma}_{L^2(\mathbb{R}^N)}\|u_0\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)} > \|\nabla Q\|^{s_\sigma}_{L^2(\mathbb{R}^N)}\|Q\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)}.$$

(a) *If* $E[u_0] \leq 0$ *then*

$$\|\nabla u(t)\|^{s_\sigma}_{L^2(\mathbb{R}^N)}\|u(t)\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)} > c_{\sigma,b,N}\|\nabla Q\|^{s_\sigma}_{L^2(\mathbb{R}^N)}\|Q\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)},$$

where $c_{\sigma,b,N} = \left(\frac{N\sigma+b}{2}\right)^{1/N\sigma+b-2}.$

Energy trapping

Proposition

(b) *If* $E[u_0] > 0$ *and* $E[u_0]^{s_\sigma} M[u_0]^{1-s_\sigma} < E[Q]^{s_\sigma} M[Q]^{1-s_\sigma}$ *then*

$$\|\nabla u(t)\|^{s_\sigma}_{L^2(\mathbb{R}^N)} \|u(t)\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)} > c_{\sigma,b,N} \|\nabla Q\|^{s_\sigma}_{L^2(\mathbb{R}^N)} \|Q\|^{1-s_\sigma}_{L^2(\mathbb{R}^N)},$$

where $c_{\sigma,b,N,Q,u} =$

$$\left(1 + \left(\left(\frac{N\sigma + b}{2}\right)^{1/N\sigma+b-2} - 1\right)\left(1 - \frac{E[u]M[u]^{\frac{s_\sigma}{1-s_\sigma}}}{E[Q]M[Q]^{\frac{s_\sigma}{1-s_\sigma}}}\right)^{1/2}\right)^{s_\sigma}.$$

Remark

Note that $c_{\sigma,b,N}, c_{\sigma,b,N,Q,u} > 1$ *since* $\sigma > \dfrac{2-b}{N}$.

Energy trapping

Proposition

(b) *If* $E[u_0] > 0$ *and* $E[u_0]^{s_\sigma} M[u_0]^{1-s_\sigma} < E[Q]^{s_\sigma} M[Q]^{1-s_\sigma}$ *then*

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^{s_\sigma} \|u(t)\|_{L^2(\mathbb{R}^N)}^{1-s_\sigma} > c_{\sigma,b,N} \|\nabla Q\|_{L^2(\mathbb{R}^N)}^{s_\sigma} \|Q\|_{L^2(\mathbb{R}^N)}^{1-s_\sigma},$$

where $c_{\sigma,b,N,Q,u} =$

$$\left(1 + \left(\left(\frac{N\sigma+b}{2}\right)^{1/N\sigma+b-2} - 1\right)\left(1 - \frac{E[u]M[u]^{\frac{s_\sigma}{1-s_\sigma}}}{E[Q]M[Q]^{\frac{s_\sigma}{1-s_\sigma}}}\right)^{1/2}\right)^{s_\sigma}.$$

Remark

Note that $c_{\sigma,b,N}, c_{\sigma,b,N,Q,u} > 1$ *since* $\sigma > \dfrac{2-b}{N}$.

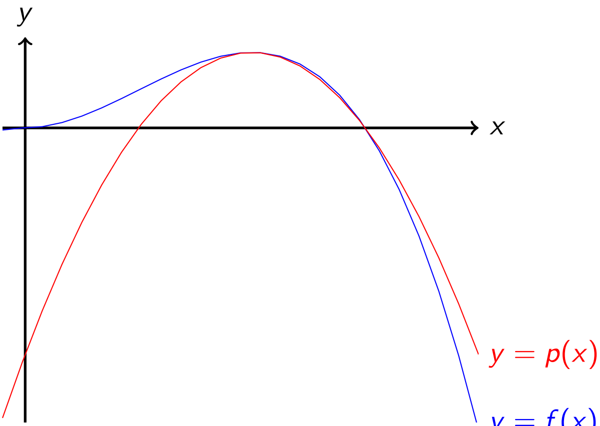
Calculus fact

Lemma

Let $f(x) = \frac{1}{2}x^2 - ax^\alpha$, where $a > 0$ and $\alpha > 2$. Define $p(x)$ the tangent parabola at the positive local maximum of f, namely $(x_{max}, f(x_{max}))$, that pass through the positive root of f, namely $(0, c)$ with $c > 0$, then

$$f(x) \geq p(x), \quad \textit{for all} \ \ x \in (x_{max}, c).$$

Calculus fact



Proof of Theorem 3

Suppose by contradiction that the solution $u(t)$ of equation (1) with initial data satisfying hypotheses (13)-(14) exists globaly.

Multiplying the Virial identity (16) by $M[u]^{\frac{s_\sigma}{1-s_\sigma}}$ and using Proposition 3.2 we have for all $t > 0$

$$\left(\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2 |u(x,t)|^2 dx\right) M[u_0]^{\frac{s_\sigma}{1-s_\sigma}}$$

$$< 8(N\sigma + b)E[Q]M[Q]^{\frac{s_\sigma}{1-s_\sigma}}$$

$$- 4(N\sigma + b - 2)A\left(\|\nabla Q\|_{L^2(\mathbb{R}^N)}\|Q\|_{L^2(\mathbb{R}^N)}^{\frac{s_\sigma}{1-s_\sigma}}\right)^2,$$

for some number $A = A(\sigma, b, N, Q, u_0) > 1$, given by Proposition 3.2.

Proof of Theorem 3

Since Q is a solution of (10)

$$8(N\sigma+b)E[Q]M[Q]^{\frac{s_\sigma}{1-s_\sigma}} = 4(N\sigma+b-2)\left(\|\nabla Q\|_{L^2(\mathbb{R}^N)}\|Q\|_{L^2(\mathbb{R}^N)}^{\frac{s_\sigma}{1-s_\sigma}}\right)^2$$

Therefore

$$\left(\frac{d^2}{dt^2}\int_{\mathbb{R}^N}|x|^2|u(x,t)|^2dx\right)M[u_0]^{\frac{s_\sigma}{1-s_\sigma}}$$

$$< -4(N\sigma+b-2)(A-1)\left(\|\nabla Q\|_{L^2(\mathbb{R}^N)}\|Q\|_{L^2(\mathbb{R}^N)}^{\frac{s_\sigma}{1-s_\sigma}}\right)^2 = -B, \tag{17}$$

for some number $B = B(\sigma, b, N, Q, u_0) > 0$.

Finally, integrating (17) twice and taking t large we reach a contradiction.

Obrigado!

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