The quasi-parabolic nature of the KdV equation in the asymmetrically weighted Sobolev space

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In 1983, Tosio Kato in the paper *On the Cauchy Problem for the (Generalized) Korteweg-de Vries Equation*, considers the initial value problem for (KdV):

$$\frac{\partial u}{\partial t} + D^3 u + u\,Du = 0,\ t > 0,\ x \in \mathbb{R},\ \ u(0) = \phi, \qquad (1)$$

where $D = \dfrac{\partial}{\partial x}$, for initial data in asymmetric spaces with the resulting irreversibility in time. Specifically $\phi \in Y = H^s(\mathbb{R}) \cap L_b^2(\mathbb{R})$ where $L_b^2(\mathbb{R}) = L^2\left(e^{2b\,x}\,dx\right)$ for $s \geq 0,\ b > 0$.

He notes that the semigroup $\exp\left(-t\,D^3\right)$ in $L_b^2(\mathbb{R})$, is *formally* equivalent to the semigroup

$$U_b(t) = \exp\left[-t\,(D-b)^3\right],\ t \geq 0, \qquad (2)$$

considered in $L^2(\mathbb{R}) = H^0$. Really of

$$(D-b)\, e^{bx}\, u = e^{bx}\, Du,$$

there follows

$$(D-b)^3\, e^{bx}\, u = e^{bx}\, D^3 u.$$

where $U_b(t)$ satisfies
Lemma 1 (T. Kato)**.**

$\{U_b(t)\colon t>0\}$ *is an infinitely differentiable semigroup on* $H^s(\mathbb{R})$ *for each real* s*, with*

$$\begin{aligned} ||D^n U_b(t)||_{B(L^2(\mathbb{R}),\, L^2(\mathbb{R}))} \;&\leq\; c_n\, t^{-n/2}\, \exp(b^3\, t),\;\; n = 1, 2, 3, \ldots \\ ||(d/dt) U_b(t)||_{B(L^2(\mathbb{R}),\, L^2(\mathbb{R}))} \;&\leq\; c\, t^{-3/2}\, \exp(b^3 t). \end{aligned}$$

$U_b(t)$ *is bounded on* H^s *to* $H^{s'}$*, with*

$$||U_b(t)||_{B(H^s,\, H^{s'})} \leq c\, t^{-(s'-s)/2}\, \exp(b^3\, t),\;\; s \leq s'. \quad (3)$$

These results are easy consequences of the factorization

$U_b(t) = \exp(b^2 t)\, \exp(-3b^2 t\, D)\, \exp(3\, b\, t\, D^2)\, \exp(-t\, D^3)$,

where $\exp(-3b^2 t\, D)$ and $\exp(-t\, D^3)$ are unitary on H^s and $\exp(3\, b\, t\, D^2)$ is heat semigroup, which is holomorphic in $t > 0$.

Moreover T. Kato (see Lemma 9.2.) shows which, if $e^{bx}\, u \in L^\infty\Big([0,\, T] \colon L^2(\mathbb{R})\Big)$, $e^{bx}\, f \in L^\infty\Big([0,\, T] \colon H^{-1}(\mathbb{R})\Big)$ and u satisfies

$$\frac{\partial u}{\partial t} + D^3\, u = f,\ 0 < t < T,$$

then

$$e^{bx} \in C\Big([0,\, T] \colon L^2(\mathbb{R})\Big) \cap C\left(]0,\, T] \colon H^s(\mathbb{R})\right) \quad \forall s < 1$$

and

$$e^{bx}\, u = U_b(t)\, e^{bx} u(0) + \int_0^t U_b(t-r) e^{bx} f(r) dr.$$

Now for the case not autonomous

$$\frac{\partial u}{\partial t} + D^3\, u + a(t)\, Du = 0, \;\; t \in I, \tag{4}$$

where $I \subset \mathbb{R}$ be an open interval, T. Kato shows which (see Lemma 9.3. in [Kato]), if $(a - c) \in C\,(I; H^{\infty}(\mathbb{R}))$, where c is a constant and u satisfies (4), then

$$e^{bx}\, u \in C\,(I; H^{\infty}(\mathbb{R}))\,,$$

this lemma shows the quasi-parabolic nature of the equation (4).

Results for the globally well posed is obtained by Kato (see Theorem 10.1 in [Kato]), if $\phi \in H^s \cap L^2_b$, $s \geq 2$ and $b > 0$, then exists a unique solution u to (1) such that $u \in C\left([0, +\infty); H^s \cap L^2_b\right)$, with the map $\phi \to u$ continuous. Moreover, $e^{bx}\, u \in C\left([0, +\infty); H^{s'}\right)$ for any $s' < s+2$. In the case $\phi \in H^0 \cap L^2_b$, $b > 0$, exists a unique solution

$$u \in C_w\left([0, +\infty); H^0\right),$$

(see Theorem 12.1 in [Kato]).

In 2002, Kenig - Ponce - Vega in the work *On the support of solutions to the generalized KdV equation* [KPV] showed that if $u(x, t)$ is solution of the k-gKdV equation

$$\partial_t u + \partial_x^3 u + u^k \, \partial_x u = 0,$$

such that

$$\sup_{t \in [0,1]} ||u(\cdot, t)||_{H^1(\mathbb{R})} < +\infty,$$

and such that for a given $\beta > 0$ $e^{\beta x} u_0 \in L^2(\mathbb{R})$ then $e^{\beta x} u \in C\left([0, 1] \colon L^2(\mathbb{R})\right)$ (Lemma 2.1 in [KPV]) and an extension to higher derivatives.

We use the ideas of the proof of the following Carleman estimates (see Lemma 2.3 in [KPV])

$$||e^{\lambda x} f||_{L^8(\mathbb{R}^2)} \leq c \, ||e^{\lambda x} \left\{\partial_t + \partial_x^3\right\} f||_{L^{8/7}(\mathbb{R}^2)}$$

for all $\lambda \in \mathbb{R}$, where $f \in C_0^{3,1}(\mathbb{R}^2)$ this is, $\partial_x f$, $\partial_x^2 f$, $\partial_x^3 f$, $\partial_t f \in C_b(\mathbb{R}^2)$ with compact support.

We also follow the ideas contained in the work *On uniqueness properties of solutions of the k-generalized KdV equations* by Escauriaza - Kenig - Ponce - Vega '2007 [EKPV] and *Lower bounds for non-trivial travelling wave solutions of equations of KdV type* by Kenig - Ponce - Vega '2012 [KPV2].

Also use arguments analogous to those found in the work of Carvajal - Panthee *Well-posedness for some perturbations of the KdV equation with low regularity data* '2008, they considering the initial value problem

$$\begin{aligned} u_t + u_{xxx} + \eta\, Lu + u\, u_x &= 0, \;\; x \in \mathbb{R},\; t \geq 0, \\ u(x, 0) &= 0 \end{aligned} \qquad (5)$$

where $\eta > 0$ is a constant and the linear operator L is defined via the Fourier transform by $\widehat{Lu}(\xi) = -\Phi(\xi)\,\hat{u}(\xi)$. The Fourier symbol

$$\Phi(\xi) = \sum_{j=0}^{n} \sum_{i=0}^{2m} c_{i,j}\xi^i\,|\xi|^j, \quad c_{i,j} \in \mathbb{R},\ c_{2m,n} = -1, \qquad (6)$$

is a real valued function which is bounded above, they proved in [CP] the IVP (5) with $\eta > 0$ and $\Phi(\xi)$ given by (6) is locally well-posed for any data $u_0 \in H^s(\mathbb{R})$, $s > -3/4$ (see Theorem 1.1 in [CP]).

We consider the Cauchy problem for the forced Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + D^3 u + u\, Du = f, \;\; t > 0, \;\; x \in \mathbb{R}. \tag{7}$$

with initial data in $Y = X^s \cap L^2_b$ where X^s is the Sobolev space $H^s(\mathbb{R})$ or the Zhidkov spaces

$$\left\{\phi \in \mathcal{D}(\mathbb{R})\colon \phi \in L^\infty(\mathbb{R}), \;\; \phi' \in H^{(s-1)}(\mathbb{R})\right\}.$$

Without loss of generality we consider $b = 1$, since

$$u_b(y,\, t') = b^{-2}\, u(b^{-1}y,\, b^{-3}t'),$$

satisfies

$$\frac{\partial u_b}{\partial t'} + \frac{\partial^3 u_b}{\partial y^3} + u_b \frac{\partial u_b}{\partial y} = f_b, \tag{8}$$

where $f_b(y,\, t') = b^{-5} f(b^{-1} y,\, b^{-3} t')$.

Multiplying by e^x (7) obtain

$$\frac{\partial}{\partial t}(e^x\, u) + (D-1)^3\,(e^x\, u) + u\,(D-1)\,(e^x\, u) = e^x\, f, \tag{9}$$

We denote by $v = e^x\, u$ and $g = e^x\, f$ obtaining,

$$\frac{\partial v}{\partial t} + (D-1)^3\, v + u\,(D-1)\, v = g. \tag{10}$$

Since the linear symbol of (10) is $i\,\tau + (i\,\xi - 1)^3$, by analogy with the spaces introduced by Molinet and Ribaud '2002 (see [MR]) for Korteweg- de Vries - Burgers equation, we define the function space $X^{a,\,s}$ endowed with the norm

$$||v||_{X^{a,\,s}} = ||\langle i\,\tau + (i\,\xi - 1)^3\rangle^a\,\langle \xi\rangle^s\,\widehat{v}||_{L^2(\mathbb{R}^2)},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$, so that

$$||v||_{X^{a,\,s}} = ||\langle|\tau - \xi^3 + \xi| + |\xi^2 - 1|\rangle^a \langle\xi\rangle^s \,\widehat{v}||_{L^2(\mathbb{R}^2)}.$$

We can re-express the norm of $X^{a,\,s}$ as

$$||v||_{X^{a,\,s}} \sim ||U(-t)v||_{H^{a,\,s}} + ||v||_{L^2_t\, H^{s+2a}},$$

where $U(t) = \exp\left(-t\, D^3\right)$ and

$$||v||^2_{H^{a,\,s}} = \int_{\mathbb{R}^2} \langle\tau\rangle^{2a} \langle\xi\rangle^{2s} |\widehat{v}(\xi,\,\tau)|^2 \,d\xi\, d\tau.$$

We denote by W the semigroup $U_b(t)$ in (2) for $b = 1$, associated with the free evolution of (10), $\forall t > 0$

$$\mathcal{F}_x\left(W(t)\phi\right)(\xi) = \exp\left[-3\xi^2 t + t + i\left(\xi^3 - 3\xi\right) t\right],\ \phi \in S',$$

and we extend W to a linear operator defined on the whole real axis by setting $\forall t \in \mathbb{R}$,

$$\mathcal{F}_x \left(W(t)\phi\right)(\xi) = \exp\left[-3\xi^2|t| + t + i\left(\xi^3 - 3\xi\right)t\right],\ \phi \in S',$$

Using Duhamel´s principle, we will mainly work on the integral formulation of the equation (10)

$$\begin{aligned} v(t) \;&=\; W(t)\phi - \frac{1}{2}\int_0^t W(t-t')\left[D(u\,v) - u\,v\right](t')\,dt' + \\ &\int_0^t W(t-t')f(t')\,dt',\ t \geq 0. \end{aligned} \tag{11}$$

For $T > 0$ consider the localized spaces $X_T^{a,\,s}$ endowed with the norm

$$||v||_{X_T^{a,\,s}} = \inf_{w \in X^{a,\,s}} \left\{\, ||w||_{X^{a,\,s}} \colon w(t) = v(t) \text{ on } [0,\,T] \,\right\}.$$

If $f \in H^{\infty}(\mathbb{R})$ time independent and $\phi \in H^s(\mathbb{R})$ for $s \geq -3/4$, using the linear estimates and bilinear estimates in Zihua Guo '2009 (see [GUO]) and Colliander - Keel - Staffilani - Takaoka- Tao '2003 (see [CKSTT]), we can adapt the proofs to show the existence the $T > 0$ and $u \in C\left([-T, T] \colon H^{-s}(\mathbb{R})\right)$ unique solution of (7) for $s \geq -3/4$.

If $g = e^{bx} f \in H^{\infty}(\mathbb{R})$,

$$\phi \in H_b^s(\mathbb{R}) = \left\{\psi \in S' \colon e^{bx}\psi \in H^s(\mathbb{R})\right\},$$

and $u \in C([0, T]) \colon H^s(\mathbb{R}))$ be a solution to (7) for $s \geq -3/4$ and using the argument used in Molinet and Ribaud '2002 ([MR]) find estimates analogous to (2.1) for example, exists $C > 0$ such that

$$||\psi(t)\, W(t)\phi||_{X^{1/2,\, s}} \leq C\, ||\phi||_{H^s(\mathbb{R})}, \quad \forall \phi \in H^s(\mathbb{R}),$$

$s \in \mathbb{R}$, where ψ is a time cutoff function satisfying

$$\psi \in C_0^\infty(\mathbb{R}),\ \ \sup \psi \subset [-2,\, 2],\ \ \psi \equiv 1 \text{ on } [-1,\, 1].$$

Also show estimates analogous to (2.2), (2.9), (2.33), (2.34) and bilinear estimates as in Proposition 3.1, show the existence the $T' > 0$ such that, there exists a unique solution

$$v \in C\left([0,\, T']\colon H^s(\mathbb{R})\right) \cap X_{T'}^{1/2,\, s},$$

of (10).

Combining these results we prove the local existence result

Theorem 2. *Let* $f \in H^\infty(\mathbb{R})$ *time independent,* $e^{bx} f \in H^\infty(\mathbb{R})$, $\phi \in H^s(\mathbb{R}) \cap H_b^s(\mathbb{R})$ *for* $s \geq -3/4$ *and* $b > 0$*. Then there exist* $T > 0$ *and unique solution* $u(t)$ *of the IVP (7) in the time interval* $[0, T]$ *in*

$$C\left([0, T] \colon H^s(\mathbb{R}) \cap H_b^s(\mathbb{R})\right).$$

Moreover, the map $\phi \mapsto u$ *is smooth from* $H^s(\mathbb{R}) \cap H_b^s(\mathbb{R})$ *to* $C\Big([0, T] \colon H^s(\mathbb{R}) \cap H_b^s(\mathbb{R})\Big)$*;*

u *and* $e^{bx} u$ *belongs to* $C\left(]0, T] \colon H^\infty(\mathbb{R})\right)$*.*

Using the results for the local existence of KdV for $s \geq 0$ and ideas of Kato [Kato] (see Theorem 11.1), follows easily

Theorem 3. *Let* $f \in H^\infty(\mathbb{R})$ *time independent,* $e^{bx} f \in H^\infty(\mathbb{R})$, $\phi \in H^s(\mathbb{R}) \cap L^2_b(\mathbb{R})$ *for* $s \geq 0$ *and* $b > 0$. *Then there exist* $T > 0$ *and unique solution* $u(t)$ *of the IVP (7) in the time interval* $[0, T]$ *in*

$$C\left([0, T] \colon H^s(\mathbb{R}) \cap L^2_b(\mathbb{R})\right).$$

Moreover, the map $\phi \mapsto u$ *is smooth from* $H^s(\mathbb{R}) \cap L^2_b(\mathbb{R})$ *to* $C\left([0, T] \colon H^s(\mathbb{R}) \cap L^2_b(\mathbb{R})\right)$;

u *and* $e^{bx}\, u$ *belongs to* $C\left(]0, T] \colon H^\infty(\mathbb{R})\right)$.

To prove the global well-posedness in $H^s(\mathbb{R}) \cap L^2_b(\mathbb{R})$ for $s \geq 0$, first we establish a series of a *priori* estimates.

We can adapt Fourier proof that $||u(t)||_{L^2(\mathbb{R})} = ||\phi||_{L^2(\mathbb{R})}$, $\forall t \in \mathbb{R}$ for u solution the KdV equation (see [CKSTT]).

By Plancherel,

$$||u(t)||^2_{L^2(\mathbb{R})} = \int_{\xi_1+\xi_2=0} \widehat{u}(\xi_1)\, \widehat{u}(\xi_2)\, d\xi_1\, d\xi_2.$$

Hence, for u local solution of (7), we apply ∂_t, use symmetry, and the equation to find

$$\partial_t \left(||u(t)||^2_{L^2(\mathbb{R})} \right) = 2 \int_{\xi_1+\xi_2=0} \widehat{f}(\xi_1)\, \widehat{u}(\xi_2)\, d\xi_1\, d\xi_2,$$

we have

$$||u(t)||_{L^2(\mathbb{R})} \leq ||\phi||_{L^2(\mathbb{R})} + ||f||_{L^2(\mathbb{R})}\, t, \ \forall t \in [0,\, T]. \quad (12)$$

For $t > 0$, multiplying by v and integrating by parts in $\mathbb{R}$ with respect to x the equation

$$\frac{\partial v}{\partial t} + (D-b)^3\, v + u\ (D-b)\, v = g,$$

we have

$$\frac{1}{2}\frac{d}{dt}\int v^2\, dx \ = \ -3b\int (Dv)^2\ dx - \int u\, v\, Dv\, dx + b\int u\, v^2\, dx + \int g\, v\, dx.$$

Using the Cauchy-Schwartz inequality and Gagliardo-Niremberg interpolation, we obtain the estimate

$$\frac{1}{2}\frac{d}{dt}\int v^2\,dx \;=\; C\left(\frac{1}{b^3}\,||u||^4_{L^2(\mathbb{R})} + b\,||u||^{4/3}_{L^2(\mathbb{R})}\right)||v||^2_{L^2(\mathbb{R})} + ||g||_{L^2(\mathbb{R})}||v||_{L^2(\mathbb{R})}.$$

An application of Gronwall´s inequality, using (12) and theorem (3) gives

Theorem 4. *Let $f \in H^\infty(\mathbb{R})$ time independent, $e^{bx} f \in H^\infty(\mathbb{R})$, $\phi \in L^2_b(\mathbb{R}) \cap H^2(\mathbb{R})$ for $s \geq 0$ and $b > 0$. Then exist a unique solution $u(t)$ of the IVP (7) in*

$$C\left([0, +\infty[: L^2_b(\mathbb{R}) \cap H^s(\mathbb{R})\right).$$

Moreover, the map $\phi \mapsto u$ is smooth from $H^s(\mathbb{R})$ to $L^2_b(\mathbb{R}) \cap H^s(\mathbb{R})$;

u and $e^{bx} u$ belongs to $C\left(]0, +\infty[: H^\infty(\mathbb{R})\right)$.

For initial value problem in spaces Zhidkov, we can adapt the estimates in $H^s(\mathbb{R}) \cap L^2_b(\mathbb{R})$ and apply the methods used in Iorio-Linares-Scialom '1998 (see [ILS]) and Gallo '2005 (see [G]) for establish existence the global solutions of (7) for $s \geq 1$.

We are interested in the case forced for the existence of global attractors in Y, this is, a compact invariant set $\mathcal{A}$ attracts an open set of initial conditions and Hausdorff dimension finite, is a consequence of the quasi-parabolic nature of the KdV equation in the asymmetrically weighted Sobolev space.

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