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Complex ADHM equations and sheaves on $\mathbb{P}^3$

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Abstract

We establish a 1-1 correspondence between certain classes of sheaves on $\mathbb{P}^3$ and solutions of a complex version of the Atiyah–Hitchin–Drinfeld–Manin equations. We also show how relaxing of the locally free condition on sheaves is reflected on partial completions of the space of regular solutions. We also introduce a new class of sheaves on $\mathbb{P}^3$ closely related to ideal sheaves for curves within $\mathbb{P}^3$.

Keywords: ADHM equations; Monads; Instanton sheaves

Contents

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Introduction

Atiyah, Hitchin, Drinfeld and Manin obtained their famous classification of $U(n)$-instantons on S^4 using the Penrose transform to holomorphic vector bundles on $\mathbb{P}^3$ subject to some reality and cohomological conditions [2]. Their result for framed instantons is given by the so-called regular ADHM data satisfying quadratic equations.

Furthermore, Donaldson has shown that the framed $U(n)$-instantons on S^4 can be described in terms of holomorphic vector bundles on $\mathbb{P}^2$ which are trivial on the line at infinity, and with a fixed trivialization there [6]. Nakajima later extended the classification of framed holomorphic vector bundles on $\mathbb{P}^2$ to framed torsion-free sheaves, which yields a partial completion of regular ADHM data to stable ADHM data [16]. Nakajima effectively used the moduli spaces of stable ADHM data for geometric constructions of various infinite dimensional algebras [15].

One can also classify rank r framed holomorphic vector bundles on $\mathbb{P}^3$ satisfying the cohomological restrictions arising from instantons but without imposing the reality conditions. These are called in the literature (mathematical) instanton bundles [10,17] and, in fact, correspond to (possibly singular) $GL(r,\mathbb{C})$-instantons on S^4 [21]. One of the first facts we verify on this paper is that framed instanton bundles on $\mathbb{P}^3$ can be constructed from the complex ADHM data first introduced by Donaldson in [6] but never again studied in the literature.

Complex ADHM data can also be presented as a family of real ADHM data parameterized by $\mathbb{P}^1$. Then the regular complex data is the same as a family of regular real data. We also define semiregular complex ADHM data by relaxing the regularity condition for real data at finitely many points of $\mathbb{P}^1$. Similarly, we define stable and semistable complex ADHM data, thus presenting different partial completions of the moduli space of instanton bundles on $\mathbb{P}^3$. Our main results can be collected in a theorem which gives a natural characterization of classes of sheaves constructed from the four types of complex ADHM data described above.

Main Theorem. *There are* 1-1 *correspondences between the following objects*:

- *regular complex ADHM data and framed locally-free instanton sheaves*;
- *semiregular complex ADHM data and framed reflexive instanton sheaves*;
- *stable complex ADHM data and framed torsion-free instanton sheaves.*

Moreover, to every semistable complex ADHM data one can associate a framed torsion-free weakly instanton sheaf.

As in the $\mathbb{P}^2$ case studied by Nakajima, the rank one instanton sheaves on $\mathbb{P}^3$ should be of special interest. We show, however, that this set is empty and from the Main Theorem we deduce that there are no stable solutions of the complex ADHM equations of rank one. For this reason, we also introduce a new class of sheaves in $\mathbb{P}^3$, so-called weakly instanton sheaves, which are defined by omitting the vanishing of the second cohomology group in the definition of instanton sheaves, while the remaining conditions are verified. We show how framed weakly instanton sheaves can be constructed from semistable complex ADHM data, and conjecture that every framed weakly instanton sheaf arises in this way (otherwise, one should strengthen our definition of framed weakly instanton sheaf). The existence of rank one semistable solutions implies the existence of rank one weakly instanton sheaves. Moreover, they admit a simple description, which complements the Main Theorem:

Theorem. *There is a* 1-1 *correspondence between rank one weakly instanton sheaves and ideal sheaves of curves in* $\mathbb{P}^3$.

This result reinforces the analogy with the rank one framed torsion free sheaves on $\mathbb{P}^2$ studied by Nakajima [16], which are in 1-1 correspondence with zero-dimensional closed subschemes of $\mathbb{C}^2$. Thus we conclude that the moduli of semistable solutions of the complex ADHM equations is a natural complex counterpart to the moduli of stable ADHM equations studied in [16]. This analogy suggests that the moduli of semistable solutions, in particular the rank one case, might be used for constructions of interesting, new infinite dimensional algebras and their representations. Such construction should certainly reflect the fact that the structure of curves in $\mathbb{P}^3$ is much richer then the one of points in $\mathbb{P}^2$.

In another direction, one can further study the deformation of the Penrose transform introduced in [8] that relates the commutative geometry encoded into instanton sheaves on $\mathbb{P}^3$ and the noncommutative geometry of the quantum Minkowski space-time. We expect that regular, semiregular, stable and semistable complex ADHM data also admit a natural realization in different types of quantum instantons in parallel with the Main Theorem of the present paper.

The construction and study of the moduli space of framed instanton sheaves (or, equivalently, of stable complex ADHM data) is beyond the scope of the present paper. Many authors have studied *unframed* rank $2n$ instanton bundles over $\mathbb{P}^{2n+1}$ over the past 25 years, see for instance [1,3,14,18,19]. Establishing irreducibility and smoothness of their moduli space and computing its dimension has proved to be a very hard problem; the introduction of [4] has a nice survey on this topic, as well as all relevant references. The only general result is due to Costa and Ottaviani, who proved that the moduli space of unframed rank $2n$ instanton bundles over $\mathbb{P}^{2n+1}$ is an affine variety [5]. We expect that the new parameterization of the moduli space of framed instanton sheaves provided by our Main Theorem will allow a deeper understanding of its structure; in particular, we will prove below that the moduli space of rank $r \geqslant 2$ instanton sheaves on $\mathbb{P}^3$ of charge 1 is a smooth, irreducible quasi-projective variety of dimension $4r$. Furthermore, based on the methods presented here and using geometric invariant theory, the second named author has established that the moduli space of framed instanton sheaves of arbitrary rank and charge is a quasi-projective variety (see the preprint [11]).

1. Complex ADHM equations

1.1. Real ADHM data

Let V and W be complex vector spaces, with dimensions c and r, respectively, and consider maps $B_1, B_2 \in \mathrm{End}(V)$, $i \in \mathrm{Hom}(W, V)$ and $j \in \mathrm{Hom}(V, W)$. We define

$$\mathbf{B} = \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W)$$

to be the space of all possible *ADHM datum* (B_1, B_2, i, j). A point $X = (B_1, B_2, i, j) \in \mathbf{B}$ is said to be:

- *stable*, if there is no proper subspace $S \subset V$ such that $B_k(S) \subset S$ $(k = 1, 2)$ and $i(W) \subset S$;
- *costable*, if there is no nonzero subspace $S \subset V$ such that $B_k(S) \subset S$ $(k = 1, 2)$ and $S \subset \ker j$;
- *regular*, if it is both stable and costable.

Notice that (B_1, B_2, i, j) is stable if and only if the dual data $(B_1^\dagger, B_2^\dagger, i^\dagger, j^\dagger)$ is costable.

Now fix Hermitian metrics on V and W. Recall the *real ADHM equations* are given by:

$$[B_1, B_2] + ij = 0, \tag{1}$$

$$\left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + ii^\dagger - j^\dagger j = 0. \tag{2}$$

We state the following proposition for further reference; proofs can be found at [8,16].

Proposition 1. *Every stable solution of* (1) *and* (2) *is regular. For $r \geqslant 2$ and for all c, the set of regular solutions is nonempty. For $r = 1$, every stable solution of* (1) *has $j = 0$; in particular, there are no regular solutions for $r = 1$.*

Finally, consider the map:

$$R : \mathbf{B} \longrightarrow \mathrm{Hom}\left(W^{\oplus c^2}, V\right) \tag{3}$$

which associates to $X = (B_1, B_2, i, j)$ the map $(0 \leqslant m, n \leqslant c - 1)$:

$$R(X) = i \oplus \cdots \oplus B_1^m B_2^n i \oplus \cdots \oplus B_1^{c-1} B_2^{c-1} i.$$

We will also need the following [20, Lemma 3.2]:

Lemma 2. *Suppose $X = (B_1, B_2, i, j) \in \mathbf{B}$ satisfies* (1)*. The map $R(X)$ is surjective if and only if X is stable.*

1.2. Complex ADHM data

As above, let V and W be complex vector spaces, with dimensions c and r, respectively; Hermitian structures are no longer required. Set $\tilde{W} = V \oplus V \oplus W$. Let $\mathbb{B} = \mathbf{B} \oplus \mathbf{B}$, with $\tilde{X} = (B_{kl}, i_k, j_k) \in \mathbb{B}$ $(k, l = 1, 2)$ being called a *complex ADHM datum*. As usual, the group $GL(V)$ acts naturally on $\mathbb{B}$, in the following way:

$$g(B_{kl}, i_k, j_k) = \left(gB_{kl}g^{-1}, gi_k, j_k g^{-1}\right), \quad g \in GL(V). \tag{4}$$

Equivalently, we can think of an element in $\mathbb{B}$ as a holomorphic section of the bundle $\mathbf{B} \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ by defining:

$$\tilde{B}_1 = zB_{11} + wB_{21} \quad \text{and} \quad \tilde{B}_2 = zB_{12} + wB_{22}, \tag{5}$$

$$\tilde{\imath} = zi_1 + wi_2 \quad \text{and} \quad \tilde{\jmath} = zj_1 + wj_2. \tag{6}$$

In other words, $\mathbb{B} = \mathbf{B} \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, with z, w denoting a basis of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ or, equivalently, a choice of homogeneous coordinates in $\mathbb{P}^1$. In particular, one can also view the maps (5) and (6) in the following way:

$$\tilde{B}_1, \tilde{B}_2 \in \mathrm{Hom}(V, V) \otimes H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big),$$
$$\tilde{\imath} \in \mathrm{Hom}(W, V) \otimes H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big) \quad \text{and} \quad \tilde{\jmath} \in \mathrm{Hom}(V, W) \otimes H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big).$$

Given any point $p \in \mathbb{P}^1$, the evaluation map

$$\mathrm{ev}_p : H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big) \to \mathcal{O}_{\mathbb{P}^1}(1)_p \simeq \mathbb{C}$$

can be tensored with the identity to yield maps $\mathrm{ev}_p : \mathbb{B} \to \mathbf{B} \otimes \mathcal{O}_{\mathbb{P}^1}(1)_p \simeq \mathbf{B}$ and $\mathrm{ev}_p : \mathrm{Hom}(V, V) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \to \mathrm{Hom}(V, V) \otimes \mathcal{O}_{\mathbb{P}^1}(2)_p \simeq \mathrm{Hom}(V, V)$. For simplicity, we use the notation $\tilde{X}_p = \mathrm{ev}_p(\tilde{X})$.

Definition. A complex ADHM datum $\tilde{X} = (B_{kl}, i_k, j_k)$ is said to be:

- $\mathbb{C}$-*semistable*, if there is $p \in \mathbb{P}^1$ such that $\tilde{X}_p$ is stable;
- $\mathbb{C}$-*stable*, if $\tilde{X}_p$ is stable for all $p \in \mathbb{P}^1$;
- $\mathbb{C}$-*costable*, if $\tilde{X}_p$ is costable for all $p \in \mathbb{P}^1$;
- $\mathbb{C}$-*semiregular*, if it is $\mathbb{C}$-stable and there is $p \in \mathbb{P}^1$ such that $\tilde{X}_p$ is regular;
- $\mathbb{C}$-*regular*, if $\tilde{X}_p$ is regular for all $p \in \mathbb{P}^1$.

In particular, notice that $\tilde{X} = (B_{kl}, i_k, j_k)$ is $\mathbb{C}$-regular if and only if it is both $\mathbb{C}$-stable and $\mathbb{C}$-costable. Moreover, if $\tilde{X} = (B_{kl}, i_k, j_k)$ is $\mathbb{C}$-stable then $(B_{11}, B_{12}, i_1, j_1)$ and $(B_{21}, B_{22}, i_2, j_2)$ are both stable.

Moreover, by the continuity of the evaluation map ev_p, it is easy to see that $\mathbb{B}^{\mathrm{ss}}$, the set of $\mathbb{C}$-semistable complex ADHM data, is open as a subset of $\mathbb{B}$ (in the Zariski topology). It then follows easily that $\mathbb{B}^{\mathrm{st}}$, the set of $\mathbb{C}$-stable complex ADHM data, is also open.

The first main goal of this paper is to study the *complex ADHM equations*:

$$[B_{11}, B_{12}] + i_1 j_1 = 0, \tag{7}$$

$$[B_{21}, B_{22}] + i_2 j_2 = 0, \tag{8}$$

$$[B_{11}, B_{22}] + [B_{21}, B_{12}] + i_1 j_2 + i_2 j_1 = 0 \tag{9}$$

which were first posed by Donaldson in [6]. It is important to note that Eqs. (7)–(9) are equivalent to:

$$[\tilde{B}_1, \tilde{B}_2] + \tilde{\imath}\tilde{\jmath} = 0, \quad \forall [z : w] \in \mathbb{P}^1. \tag{10}$$

Proposition 3. *Assume that the complex ADHM datum $\tilde{X}$ satisfies* (10).

- *If $\tilde{X}$ is $\mathbb{C}$-semistable, then there are at most finitely many points $p \in \mathbb{P}^1$ such that $\tilde{X}_p$ is not stable.*

- *If $\tilde{X}$ is $\mathbb{C}$-semiregular, then there are at most finitely many points $p \in \mathbb{P}^1$ such that $\tilde{X}_p$ is not regular.*

Proof. Choose any Hermitian metric on V and W. For each $p \in \mathbb{P}^1$, consider the map

$$R_p = R(\tilde{X}_p)R(\tilde{X}_p)^\dagger : V \to V.$$

The assignment $p \to \det(R_p)$ defines an algebraic map $D : \mathbb{P}^1 \to \mathbb{C}$. Since $\tilde{X}$ satisfies (10), we have that $\tilde{X}_p$ satisfies (1) for each p.

Therefore, by Lemma 2, $\tilde{X}_p$ is stable if and only if $\det(R_p) \neq 0$. Thus if $\tilde{X}_p$ is stable for some $p \in \mathbb{P}^1$, then $\det(R_p)$ may vanish only at finitely many points, which means that $\tilde{X}_p$ is stable away from finitely many points in $\mathbb{P}^1$.

The second statement follows by duality. $\square$

It is easy to see that solutions of (7)–(9) are preserved by the $GL(V)$ action (4). Therefore, we define the moduli space of solutions of the complex ADHM equations as the quotient:

$$\mathcal{M}_{\mathbb{C}}(r,c) := \left\{ \begin{array}{c} \mathbb{C}\text{-stable} \\ \text{solutions of (7)–(9)} \end{array} \right\} \Big/ GL(V).$$

Alternatively, we may consider the map

$$\tilde{\mu} : \mathbb{B}^{\mathrm{st}} \to \mathrm{Hom}(V,V) \otimes H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)\big),$$
$$\tilde{\mu}(B_{kl}, i_k, j_k) = [\tilde{B}_1, \tilde{B}_2] + \tilde{\imath}\tilde{\jmath}.$$

In this way we have $\mathcal{M}_{\mathbb{C}}(r,c) = \tilde{\mu}^{-1}(0)/GL(V)$.

1.3. Existence of solutions

The case $r = \dim W = 1$ is rather special due to the following result:

Proposition 4. *There are no $\mathbb{C}$-stable solutions of* (7)–(9) *for $r = 1$.*

The proof will be delayed until the end of Section 2.2 (see Proposition 15). However, let us consider here the simplest possible case: $r = c = 1$. In this case, a $\mathbb{C}$-stable solution of (7)–(9) reduces to six complex numbers (b_{kl}, i_k), since $j_1 = j_2 = 0$ by Proposition 1. Now $\tilde{\imath}$ is simply a section of $\mathcal{O}_{\mathbb{P}^1}(1)$, so it must vanish at one point $p \in \mathbb{P}^1$, which implies that $\mathrm{ev}_p(\tilde{X})$ is not stable. This example explicitly illustrates the fact that *there exist $\mathbb{C}$-semistable solutions of* (7)–(9) *which are not $\mathbb{C}$-stable*.

Fortunately, the existence of regular solutions for the real ADHM equations (1) and (2) for $r \geqslant 2$ can be used to guarantee the existence of $\mathbb{C}$-stable solutions of the complex equations. Indeed, note that if V are W are provided with a Hermitian inner product, then the space of complex ADHM data $\mathbb{B}$ acquires a natural involution $\sigma : \mathbb{B} \to \mathbb{B}$ given by:

$$\sigma(B_{11}, B_{12}, B_{21}, B_{22}, i_1, i_2, j_1, j_2) = \big(B_{22}^\dagger, -B_{21}^\dagger, -B_{12}^\dagger, B_{11}^\dagger, j_2^\dagger, -j_1^\dagger, -i_2^\dagger, i_1^\dagger\big).$$

The point $\tilde{X} \in \mathbb{B}$ is said to be *real* if it is fixed by σ.

Note that if $\tilde{X}$ is real, then complex ADHM equations (7)–(9) above reduce to the real ADHM (1)–(2) equations by setting $B_1 = B_{11} = B_{22}^\dagger$, $B_2 = B_{12} = -B_{21}^\dagger$, $i = i_1 = j_2^\dagger$ and $j = j_1 = -i_2^\dagger$.

Proposition 5. *If* (B_1, B_2, i, j) *is a stable* (*hence regular*) *solution of* (1), (2), *then* $\tilde{X} = (B_1, B_2, -B_2^\dagger, B_1^\dagger, i, -j^\dagger, j, i^\dagger)$ *is a* $\mathbb{C}$*-regular solution of* (7)–(9).

In particular, it follows from Proposition 1 that $\mathcal{M}_{\mathbb{C}}(r,c)$ is nonempty for all $r \geqslant 2$ and for all $c \geqslant 1$.

Proof. It is easy to see that (B_1, B_2, i, j) satisfies (1) and (2), if and only if $\tilde{X}$ as above satisfies (7)–(9). Now note that in this case:

$$\tilde{B}_1 = zB_1 - wB_2^\dagger, \quad \tilde{B}_2 = zB_2 + wB_1^\dagger, \quad \tilde{\imath} = zi - wj^\dagger.$$

If $\tilde{X}$ is not $\mathbb{C}$-stable, there is $[z:w] \in \mathbb{P}^1$ and a proper subspace $S \subset V$ such that $[\tilde{B}_1^\dagger, \tilde{B}_2^\dagger]|_S = 0$ and $S \subset \ker \tilde{\imath}^\dagger$. Thus $i^\dagger|_S = k \cdot j|_S$ for some $k \in \mathbb{C}$, hence $ii^\dagger|_S = k \cdot ij|_S = [B_1, B_2]|_S$. Hence $\mathrm{Tr}(ii^\dagger|_S) = 0$, so that $[B_1^\dagger, B_2^\dagger]|_S = 0$ and $S \subset \ker i^\dagger$ and (B_1, B_2, i, j) is not stable.

Thus we conclude that $\tilde{X}$ as above is $\mathbb{C}$-stable. However, it is not difficult to see that every real, $\mathbb{C}$-stable complex ADHM datum is $\mathbb{C}$-regular. Indeed, if $\tilde{X}$ is real, then:

$$\tilde{B}_1 = zB_{11} - wB_{12}^\dagger, \quad \tilde{B}_2 = zB_{12} + wB_{11}^\dagger,$$
$$\tilde{\imath} = zi_1 - wj_1^\dagger, \quad \tilde{\jmath} = zj_1 + wi_1^\dagger.$$

Thus if $(\tilde{B}_1, \tilde{B}_2, \tilde{\imath}, \tilde{\jmath})$ is not costable at $[z:w]$, then $(\tilde{B}_1, \tilde{B}_2, \tilde{\imath}, \tilde{\jmath})$ is not stable at $[-\overline{w}:\bar{z}]$. $\square$

Remark 6. It is interesting to note that, differently from the real ADHM equations, *there are* $\mathbb{C}$*-stable solutions of* (7)–(9) *which are not* $\mathbb{C}$*-semiregular*. Indeed, for $r = 2$ and $c = 1$, we can take:

$$B_{kl} = 0, \qquad i_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad i_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad j_1 = j_2 = 0.$$

Furthermore, *there are* $\mathbb{C}$*-semiregular solutions of* (7)–(9) *which are not* $\mathbb{C}$*-regular*; for $r = 3$ and $c = 1$, we can take:

$$B_{kl} = 0, \qquad i_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad i_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad j_1 = (001), \qquad j_2 = 0.$$

However, as we shall see in Section 3, every $\mathbb{C}$-semiregular solution of (7)–(9) for $r = 2$ is in fact $\mathbb{C}$-regular.

1.4. Solutions for $c = 1$

For $r \geqslant 2$ the varieties $\mathcal{M}_{\mathbb{C}}(r,1)$ can be described quite concretely. In this case, B_{kl} are just complex numbers, while i_k and j_k can be regarded as vectors in W; the complex ADHM equations reduce to:

$$i_1 j_1 = i_2 j_2 = i_1 j_2 + i_2 j_1 = 0. \tag{11}$$

$\mathbb{C}$-stability reduces to the condition of $\tilde{\imath} = z i_1 + w i_2$ not being the zero map for any $[z:w] \in \mathbb{P}^1$. The group acting is simply $G = \mathbb{C}^*$, and it acts trivially on the B_{kl}; it acts by multiplication by t on i_1, i_2; and by multiplication by t^{-1} on j_1, j_2. It is then easy to see that $\mathcal{M}_{\mathbb{C}}(r,1) = \mathbb{C}^4 \times \mathcal{B}(r)$, where $\mathcal{B}(r)$ is the set of solutions of Eqs. (11) modulo $\mathbb{C}^*$.

Proposition 7. *$\mathcal{B}(r)$ is a nonsingular quasi-projective variety of dimension* $4(r-1)$.

Proof. Setting

$$i_1 = (x_1 \cdots x_r), \qquad i_2 = (y_1 \cdots y_r),$$
$$j_1 = \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}, \qquad j_2 = \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix}$$

Eqs. (11) become:

$$\sum_{k=1}^{r} x_k z_k = \sum_{k=1}^{r} y_k w_k = \sum_{k=1}^{r} x_k w_k + y_k z_k = 0, \tag{12}$$

while $\mathbb{C}$-stability is equivalent to the vectors $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_r)$ being linearly independent.

Then $\mathcal{B}(r)$ is the complete intersection of the three quadrics (12) within the open subset of the $(4r-1)$-dimensional weighted projective space

$$X = \mathbb{P}(\underbrace{1, \ldots, 1}_{2r}, \underbrace{-1, \ldots, -1}_{2r})$$

consisting of $\mathbb{C}$-stable points. This shows that $\mathcal{B}(r)$ is quasi-projective.

Now consider the map:

$$\mu : X \to \mathbb{C}^3,$$
$$\mu(x_1, \ldots, x_r, y_1, \ldots, y_r, z_1, \ldots, z_r, w_1, \ldots, w_r)$$
$$= \left(\sum_{k=1}^{r} x_k z_k, \sum_{k=1}^{r} y_k w_k, \sum_{k=1}^{r} x_k w_k + y_k z_k \right).$$

The derivative of μ is given by

$$D\mu = \begin{pmatrix} z_1,\dots,z_r & 0\dots0 & x_1,\dots,x_r & 0\dots0 \\ 0\dots0 & w_1,\dots,w_r & 0\dots0 & y_1,\dots,y_r \\ w_1,\dots,w_r & z_1,\dots,z_r & y_1,\dots,y_r & x_1,\dots,x_r \end{pmatrix}.$$

It is then easy to see that $D\mu$ has maximal rank 3 if and only if $(x_1,\dots,x_r)$ and $(y_1,\dots,y_r)$ are linearly independent. This shows that $\mathcal{B}(r)$ is nonsingular, and that $\dim\mathcal{B}(r) = 4r-4$. $\square$

In general, one can show that $\mathcal{M}_{\mathbb{C}}(r,c)$ is a quasi-projective variety (see [11]). We conjecture that $\mathcal{M}_{\mathbb{C}}(r,c)$ is irreducible, nonsingular and of pure dimension $4rc$.

2. Instanton sheaves on $\mathbb{P}^3$

2.1. Basic definitions

In this section we will characterize $\mathcal{M}_{\mathbb{C}}(r,c)$ as a moduli space of certain sheaves on $\mathbb{P}^3$. First, we introduce the following definition, which is a refinement of a definition originally due to Manin [12].

Definition. A coherent sheaf $\mathcal{E}$ on $\mathbb{P}^3$ is said to be *instanton* if

- $c_1(\mathcal{E}) = 0$;
- $H^0(\mathcal{E}(-1)) = H^1(\mathcal{E}(-2)) = H^2(\mathcal{E}(-2)) = H^3(\mathcal{E}(-3)) = 0.$

The integer $c = c_2(\mathcal{E})$ is called the charge of $\mathcal{E}$.

This somewhat mysterious cohomological condition is made natural once we recall that, under Penrose transform, locally-free instanton sheaves on $\mathbb{P}^3$ correspond to (singular) $GL(r,\mathbb{C})$ instantons on the compactified complexified Minkowski space-time [12,21] (see also Section 2.6 below).

Proposition 8. *Let $\mathcal{E}$ be a torsion-free sheaf on $\mathbb{P}^3$ is instanton if and only if it satisfies the following conditions*:

- $c_1(\mathcal{E}) = c_3(\mathcal{E}) = 0$;
- *$\mathcal{E}$ has natural cohomology in the range $-3 \leqslant k \leqslant -1$, i.e. for each k in the range there is at most one p such that $H^p(\mathcal{E}(k))$ is nontrivial.*

This means that rank 2 locally free instanton sheaves are *mathematical instanton bundles* on $\mathbb{P}^3$, an object first introduced in [17,18] and later studied by many authors (see for instance [1,3–5,14,19]). However, almost all of the literature on instanton bundles has concentrated on locally-free, rank 2 bundles. Our goal here is to discuss more general instanton sheaves of arbitrary rank (see also [10]).

Proof. It is easy to see that if $\mathcal{E}$ is an instanton, then the two conditions are satisfied. Conversely, given that $c_1(\mathcal{E}) = c_3(\mathcal{E}) = 0$, we compute through Riemann–Roch:

$$-\chi\big(\mathcal{E}(-1)\big) = \chi\big(\mathcal{E}(-3)\big) = c \quad \text{and} \quad \chi\big(\mathcal{E}(-2)\big) = 0,$$

where $c = c_2(\mathcal{E})$.

The natural cohomology assumption implies that $h^0(\mathcal{E}(-1)) = h^2(\mathcal{E}(-1)) = 0$, hence $h^0(\mathcal{E}(k)) = 0$ for $k \leqslant -1$. We also conclude that either $h^1(\mathcal{E}(-1)) = c$, $h^3(\mathcal{E}(-1)) = 0$ or $h^1(\mathcal{E}(-1)) = 0$, $h^3(\mathcal{E}(-1)) = c$. It follows that

$$\chi\big(\mathcal{E}(-3)\big) = h^2\big(\mathcal{E}(-3)\big) - h^1\big(\mathcal{E}(-3)\big) - h^3\big(\mathcal{E}(-3)\big) = c > 0$$

hence $h^1(\mathcal{E}(-3)) = h^3(\mathcal{E}(-3)) = 0$ and $h^2(\mathcal{E}(-3)) = c$. Moreover, $h^3(\mathcal{E}(k)) = 0$ for $k \geqslant -3$, thus $h^1(\mathcal{E}(-1)) = c$ by the previous paragraph.

Finally, $\chi(\mathcal{E}(-2)) = h^2(\mathcal{E}(-2)) - h^1(\mathcal{E}(-2)) = 0$, so the natural cohomology assumption forces $h^2(\mathcal{E}(-2)) = h^1(\mathcal{E}(-2)) = 0$, as desired. □

We will use $[x:y:z:w]$ to denote homogeneous coordinates on $\mathbb{P}^3$; the *line at infinity* ℓ_∞ is given by $z = w = 0$. Below, we concentrate on the following class of sheaves.

Definition. A *framed instanton sheaf*, that is a pair $(\mathcal{E}, \phi)$ consisting of a torsion-free instanton sheaf $\mathcal{E}$ for which the restriction $\mathcal{E}|_{\ell_\infty}$ is trivial, plus a framing, i.e. a choice of isomorphism $\phi : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus \operatorname{rk}\mathcal{E}}$.

More on instanton sheaves in general can be found in [10].

2.2. *From ADHM data to instanton sheaves*

Let (B_{kl}, i_k, j_k) be a complex ADHM datum; combining constructions of Donaldson [6] and Nakajima [16], we define the monad:

$$V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^3}(1) \tag{13}$$

where the maps α and β are given by:

$$\alpha = \begin{pmatrix} zB_{11} + wB_{21} + x \\ zB_{12} + wB_{22} + y \\ zj_1 + wj_2 \end{pmatrix}, \tag{14}$$

$$\beta = (\, -zB_{12} - wB_{22} - y \quad zB_{11} + wB_{21} + x \quad zi_1 + wi_2 \,). \tag{15}$$

Proposition 9. $\beta\alpha = 0$ *if and only if* (B_{kl}, i_k, j_k) *satisfies the complex ADHM equations* (7)–(9).

The proof is a straightforward calculation left to the reader. It follows from Proposition 9 that $\mathcal{E} = \ker\beta / \operatorname{Im}\alpha$, the first cohomology of the monad (13), is a well-defined coherent sheaf on $\mathbb{P}^3$. We will now check that $\mathcal{E}$ is the only nontrivial cohomology of (13). It is easy to see that $GL(V)$-equivalent complex ADHM data will produce isomorphic cohomology sheaves.

Proposition 10. *α_X is injective away from a subvariety of codimension at least* 2.

In particular, α is injective as a sheaf map.

Proof. It is easy to see that α is injective on the line $\ell_\infty = \{z = w = 0\}$. So consider a point $X = [x : y : z : w] \in \mathbb{P}^3 \setminus \ell_\infty$, and take $v \in V$ such that $\alpha_X(v) = 0$, that is:

$$\begin{cases} (zB_{11} + wB_{21})v = -xv, \\ (zB_{12} + wB_{22})v = -yv, \\ (zj_1 + wj_2)v = 0. \end{cases} \tag{16}$$

Thus v is a common eigenvector of $zB_{11}+wB_{21}$ and $zB_{12}+wB_{22}$, with eigenvalues $-x$ and $-y$, respectively. Hence, for fixed z, $w \neq 0$, we conclude that α_X may fail to be injective only at points within a complete intersection curve Σ of degree 2c, given by:

$$\Sigma = \begin{Bmatrix} \det(zB_{11} + wB_{21} + x) = 0 \\ \det(zB_{12} + wB_{22} + y) = 0 \end{Bmatrix}.$$

Note that Σ does not intersect ℓ_∞. Of course, it is possible that for some points of Σ there will be no vector v satisfying the third condition in (16). In other words, α_X is injective away from a subvariety of codimension at least 2. $\square$

The following is the key result in the monad construction, and further justifies our concept of $\mathbb{C}$-stability:

Proposition 11. *β is surjective if and only if (B_{kl}, i_k, j_k) is $\mathbb{C}$-stable.*

Proof. Again, it is easy to see that β is surjective on the line $\ell_\infty = \{z = w = 0\}$. So it is enough to show that the localization of β to all points $X = [x : y : z : w] \in \mathbb{P}^3 \setminus \ell_\infty$ is surjective.

Equivalently, we argue that if (B_{kl}, i_k, j_k) is $\mathbb{C}$-stable, then the dual map $\beta_X^\dagger$ is injective for all $X \in \mathbb{P}^3 \setminus \ell_\infty$. Indeed, if $\beta_X^\dagger$ is not injective for some $X = [x : y : p_1 : p_2]$, then there is $v \in V$ such that:

$$\begin{cases} \tilde{B}_1(p_1, p_2)^\dagger v = \bar{x}v, \\ \tilde{B}_2(p_1, p_2)^\dagger v = -\bar{y}v, \\ \tilde{\imath}(p_1, p_2)^\dagger v = 0 \end{cases} \tag{17}$$

which, by duality, implies that $(\tilde{B}_1(p), \tilde{B}_2(p), \tilde{\imath}(p), \tilde{\jmath}(p))$ is not stable, where we set $p = [p_1 : p_2] \in \mathbb{P}^1$. Thus (B_{kl}, i_k, j_k) is not $\mathbb{C}$-stable.

The converse statement is now clear: if (B_{kl}, i_k, j_k) is not $\mathbb{C}$-stable, then by duality $\beta_X^\dagger$ is not injective for some $[x : y : z : w]$, hence β cannot be surjective as a sheaf map. $\square$

In order to further characterize the cohomology sheaf $\mathcal{E} = \ker\beta/\operatorname{Im}\alpha$, let $[H]$ denote the generator of $H^\bullet(\mathbb{P}^3, \mathbb{C})$, i.e. $[H] = c_1(\mathcal{O}_{\mathbb{P}^3}(1))$. We will also need the following statement [10, Proposition 4].

Proposition 12. *Consider the following exact sequence of sheaves on a regular algebraic variety V of dimension* 3:

$$0 \to \mathcal{A} \xrightarrow{\mu} \mathcal{B} \longrightarrow \mathcal{C} \to 0 \tag{18}$$

where $\mathcal{A}$ and $\mathcal{B}$ are locally-free. Then:

- *$\mathcal{C}$ is torsion-free if and only if the localized map $\mu_X : \mathcal{A}_X \to \mathcal{B}_X$ is injective away from a subset of codimension at least* 2;
- *$\mathcal{C}$ is reflexive if and only if the localized map $\mu_X : \mathcal{A}_X \to \mathcal{B}_X$ is injective at most away from finitely many points*;
- *$\mathcal{C}$ is locally-free if and only if the localized map $\mu_X : \mathcal{A}_X \to \mathcal{B}_X$ is injective for all $X \in V$.*

Proposition 13. *If (B_{kl}, i_k, j_k) is $\mathbb{C}$-stable, then the cohomology sheaf $\mathcal{E} = \ker\beta/\operatorname{Im}\alpha$ is a torsion-free instanton sheaf on $\mathbb{P}^3$, with $\operatorname{ch}(\mathcal{E}) = r - c[H]^2$. Moreover, $\mathcal{E}|_{\ell_\infty} \simeq W \otimes \mathcal{O}_{\ell_\infty}$.*

Proof. Set $\mathcal{K} = \ker\beta$, and consider the short exact sequences:

$$0 \to \mathcal{K}(k) \to \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3}(k) \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^3}(k+1) \to 0 \tag{19}$$

and

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}^3}(k-1) \xrightarrow{\alpha} \mathcal{K}(k) \longrightarrow \mathcal{E}(k) \to 0. \tag{20}$$

Passing to cohomology, one easily checks that

$$H^0\big(\mathbb{P}^3, \mathcal{E}(-1)\big) = H^1\big(\mathbb{P}^3, \mathcal{E}(-2)\big) = H^2\big(\mathbb{P}^3, \mathcal{E}(-2)\big) = H^3\big(\mathbb{P}^3, \mathcal{E}(-3)\big) = 0,$$

as desired. To compute $\operatorname{ch}(\mathcal{E})$, just notice that:

$$\operatorname{ch}(\mathcal{E}) = \operatorname{ch}(\tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3}) - \operatorname{ch}\big(V \otimes \mathcal{O}_{\mathbb{P}^3}(-1)\big) - \operatorname{ch}\big(V \otimes \mathcal{O}_{\mathbb{P}^3}(1)\big),$$

since $\mathcal{E}$ is the only nonvanishing cohomology of the monad (13).

It remains for us to show that $\mathcal{E}$ is torsion-free. First, notice that $\mathcal{K}$ is a locally-free sheaf, since β is surjective, by Proposition 11. Moreover, as it was pointed out in the proof of Proposition 10, the α_X is injective away from a subset of codimension 2 in $\mathbb{P}^3$. Applying Proposition 12 to sequence (20), we conclude that $\mathcal{E}$ must be torsion-free.

Restricting the sequence (19) to ℓ_∞, we have $\mathcal{K}|_{\ell_\infty} \simeq (V \oplus W) \otimes \mathcal{O}_{\ell_\infty}$. Then restricting sequence (20) to ℓ_∞, we conclude that $\mathcal{E}|_{\ell_\infty} \simeq W \otimes \mathcal{O}_{\ell_\infty}$; in particular, a framing on $\mathcal{E}|_{\ell_\infty}$ corresponds to a choice of basis for W. □

2.3. From instanton sheaves to complex ADHM data

Proposition 14. *Every torsion-free instanton sheaf $\mathcal{E}$ on $\mathbb{P}^3$ can be obtained as the cohomology of the linear monad*

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{21}$$

where $V = H^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}(1))$, $\tilde{W} = H^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3})$ and $V' = H^1(\mathcal{E}(-1))$.

Proof. The key ingredient of the proof is the Beilinson spectral sequence, see [17]: for any coherent sheaf F on $\mathbb{P}^3$ there exists a spectral sequence $\{E_r^{p,q}\}$ whose E_1-term is given by ($q = 0,\dots,n$ and $p = 0,-1,\dots,-3$):

$$E_1^{p,q} = H^q\big(F \otimes \Omega_{\mathbb{P}^3}^{-p}(-p)\big) \otimes \mathcal{O}_{\mathbb{P}^3}(p)$$

which converges to

$$E^i = \begin{cases} F, & \text{if } p+q=0,\\ 0, & \text{otherwise.}\end{cases}$$

As shown in Appendix A, if $\mathcal{E}$ is an instanton, then

$$H^q\big(\mathcal{E}(-1)\otimes \Omega_{\mathbb{P}^3}^{-p}(-p)\big) = 0 \quad \text{for } q \neq 1 \text{ and for } q=1,\ p \leqslant -3. \tag{22}$$

It then follows that the Beilinson spectral sequence degenerates at the E_2-term and the monad

$$\begin{aligned} 0 &\to H^1\big(\mathcal{E}(-1)\otimes \Omega_{\mathbb{P}^3}^2(2)\big)\otimes \mathcal{O}_{\mathbb{P}^3}(-2) \\ &\to H^1\big(\mathcal{E}(-1)\otimes \Omega_{\mathbb{P}^3}^1(1)\big)\otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H^1\big(\mathcal{E}(-1)\big)\otimes \mathcal{O}_{\mathbb{P}^3} \to 0\end{aligned}$$

has $\mathcal{E}(-1)$ as its cohomology. Tensoring it by $\mathcal{O}_{\mathbb{P}^3}(1)$, we conclude that $\mathcal{E}$ is the cohomology of (21), as desired. □

We are finally in a position to prove Proposition 4 by looking at the corresponding sheaves on $\mathbb{P}^3$.

Proposition 15. *There are no torsion-free instanton sheaves $\mathcal{E}$ on $\mathbb{P}^3$ with* $\mathrm{ch}(\mathcal{E}) = 1 - c[H]^2$.

Indeed, if there were $\mathbb{C}$-stable solutions of (7)–(9) for $r=1$, the monad construction would produce instanton torsion-free sheaves $\mathcal{E}$ such that $\mathrm{ch}(\mathcal{E}) = 1 - c[H]^2$. So the above result implies Proposition 4.

Proof. The proposition is a consequence of a result due to Fløystad [7], which guarantees the nonexistence of monads as in (13) for $\dim \tilde{W} = 2\dim V + 1$, and Proposition 14 above. □

Now let $\mathcal{E}$ be a torsion-free instanton sheaf on $\mathbb{P}^3$ with $\mathrm{ch}(\mathcal{E}) = r - c[H]^2$, $r \geqslant 2$, and such that $\mathcal{E}|_{\ell_\infty}$ is trivial. It remains for us to show that the monad (21) can be reduced to a $\mathbb{C}$-stable solution of the complex ADHM equations (7)–(9). We will follow an argument similar to the one used in [6,16,17] for framed instanton bundles on $\mathbb{P}^2$.

Indeed, a lengthy but straightforward cohomological calculation (see Appendix A) shows that:

$$h^1\big(\mathbb{P}^3, \mathcal{E}\otimes \Omega_{\mathbb{P}^3}^2(1)\big) = h^1\big(\mathbb{P}^3, \mathcal{E}(-1)\big) = c, \qquad h^1\big(\mathbb{P}^3, \mathcal{E}\otimes \Omega_{\mathbb{P}^3}^1\big) = c + 2r$$

and that there is a canonical isomorphism $H^1(\mathbb{P}^3, \mathcal{E}\otimes \Omega_{\mathbb{P}^3}^2(1)) \simeq H^1(\mathbb{P}^3, \mathcal{E}(-1))$. Therefore, the maps α and β in the monad (21) can be regarded as elements of $\mathrm{Hom}(V, \tilde{W}) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$

and $\mathrm{Hom}(\tilde{W}, V) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, respectively. We can then express these maps in the following manner:

$$\alpha = \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 w \quad \text{and} \quad \alpha = \beta_1 x + \beta_2 y + \beta_3 z + \beta_4 w$$

where, clearly, $\alpha_k \in \mathrm{Hom}(V, \tilde{W})$ and $\beta_k \in \mathrm{Hom}(\tilde{W}, V)$ for each $k = 1, \ldots, 4$. The condition $\beta\alpha = 0$ then implies that:

$$\beta_k \alpha_k = 0, \quad k = 1, \ldots, 4,$$
$$\beta_k \alpha_l + \beta_l \alpha_k = 0, \quad k, l = 1, \ldots, 4 \text{ and } k \neq l.$$

Restricting (21) to the line at infinity $\ell_\infty = \{z = w = 0\}$ we get:

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty}(-1) \xrightarrow{\alpha_\infty} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty} \xrightarrow{\beta_\infty} V \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty}(1) \to 0$$

where $\alpha_\infty = \alpha_1 x + \alpha_2 y$ and $\beta_\infty = \beta_1 x + \beta_2 y$. Setting $\mathcal{K} = \ker \beta$ we have:

$$0 \to V \otimes \mathcal{O}_{\ell_\infty}(-1) \xrightarrow{\alpha_\infty} \mathcal{K}|_{\ell_\infty} \longrightarrow \mathcal{E}|_{\ell_\infty} \to 0$$

from the associated long exact sequence of cohomology we conclude that $H^1(\ell_\infty, \mathcal{K}|_{\ell_\infty}) = 0$ and $H^0(\ell_\infty, \mathcal{K}|_{\ell_\infty}) \simeq H^0(\ell_\infty, \mathcal{E}|_{\ell_\infty}) \simeq \mathcal{E}_P$, for some point $P \in \ell_\infty$, since $H^p(\ell_\infty, \mathcal{O}_{\ell_\infty}(-1)) = 0$, for $p = 1, 2$, and since $\mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r}$. We set $W = H^0(\ell_\infty, \mathcal{E}|_{\ell_\infty})$; the choice of a basis for W corresponds to the choice of a trivialization for $\mathcal{E}|_{\ell_\infty}$.

Similarly, from the sequence

$$0 \to \mathcal{K}|_{\ell_\infty} \longrightarrow \tilde{W} \otimes \mathcal{O}_{\ell_\infty} \xrightarrow{\beta_\infty} V \otimes \mathcal{O}_{\ell_\infty}(1) \to 0$$

we obtain:

$$0 \to W \longrightarrow \tilde{W} \xrightarrow{\beta_\infty} V \otimes H^0\big(\ell_\infty, \mathcal{O}_{\ell_\infty}(1)\big) \to 0 \tag{23}$$

since $H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}) \simeq \mathbb{C}$ and $H^1(\ell_\infty, \mathcal{K}|_{\ell_\infty}) = 0$. Then using the identification $H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(1)) \simeq \mathbb{C}x \oplus \mathbb{C}y$ we can rewrite (23) in the following way:

$$0 \to W \longrightarrow \tilde{W} \xrightarrow{\binom{\beta_1}{\beta_2}} V \oplus V \to 0 \tag{24}$$

so that $W = \ker \beta_1 \cap \ker \beta_2$.

Applying the same argument to the dual monad:

$$0 \to V^* \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty}(-1) \xrightarrow{\beta_\infty^{\mathrm{t}}} \tilde{W}^* \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty} \xrightarrow{\alpha_\infty^{\mathrm{t}}} V^* \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty}(1) \to 0$$

we have the exact sequence:

$$0 \to H^0\big(\ell_\infty, \ker\big\{\alpha_\infty^{\mathrm{t}}\big\}\big) \longrightarrow \tilde{W}^* \xrightarrow{\binom{\alpha_1}{\alpha_2}} V^* \oplus V^*$$

which implies that $(\alpha_1\alpha_2): V \oplus V \to \tilde{W}$ is injective. Moreover, the sequence (24) splits, and we can identify $\tilde{W} \simeq V \oplus V \oplus W$.

Furthermore, notice that

$$\ker\beta_1/\operatorname{Im}\alpha_1 \simeq \mathcal{E}_{[1,0,0,0]} \simeq W \simeq \ker\beta_1 \cap \ker\beta_2.$$

Thus $\operatorname{Im}\alpha_1 \cap \ker\beta_2 = 0$, so that $\beta_1\alpha_2 = -\beta_2\alpha_1 : V \to V$ are isomorphisms.

Therefore we have:

$$\alpha_1 = \begin{pmatrix} \mathbf{1}_V \\ 0 \\ 0 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 0 \\ \mathbf{1}_V \\ 0 \end{pmatrix}, \qquad \begin{matrix} \beta_1 = (0 \quad \mathbf{1}_V \quad 0), \\ \beta_2 = (-\mathbf{1}_V \quad 0 \quad 0) \end{matrix}$$

and the condition $\beta\alpha = 0$ implies that:

$$\alpha_3 = \begin{pmatrix} B_{11} \\ B_{12} \\ j_1 \end{pmatrix}, \qquad \alpha_4 = \begin{pmatrix} B_{21} \\ B_{22} \\ j_2 \end{pmatrix}, \qquad \begin{matrix} \beta_3 = (-B_{12} \quad B_{11} \quad i_1), \\ \beta_4 = (-B_{22} \quad B_{21} \quad i_2) \end{matrix}$$

with (B_{kl}, i_k, j_k) being a complex ADHM datum satisfying the complex ADHM equations (7)–(9). The surjectivity of β implies the $\mathbb{C}$-stability of (B_{kl}, i_k, j_k), by Proposition 11. Summing up, we have proved the first part of our Main Theorem, namely that there is a 1-1 correspondence between the framed torsion-free instanton sheaves on $\mathbb{P}^3$, and $\mathbb{C}$-stable solutions of the complex ADHM equations.

2.4. *Weakly instanton sheaves*

The nonexistence of rank 1 instanton sheaves on $\mathbb{P}^3$ forces the following question: what type of rank 1 sheaves could play an analogous role? As we will see below, the answer to this question is provided by the sheaves obtained from $\mathbb{C}$-semistable ADHM data. We begin by introducing a new class of sheaves on $\mathbb{P}^3$.

Definition. A rank r torsion free sheaf $\mathcal{E}$ on $\mathbb{P}^3$ is called a weakly instanton sheaf if it satisfies the following two conditions:

- $c_1(\mathcal{E}) = 0$;
- $H^0(\mathcal{E}(-1)) = H^1(\mathcal{E}(-2)) = H^3(\mathcal{E}(-3)) = 0$.

The integer $c = c_2(\mathcal{E})$ is called the charge of $\mathcal{E}$, while the integer $d = c_3(\mathcal{E})$ is called the defect of $\mathcal{E}$.

Clearly, instanton sheaves are weakly instanton sheaves with defect $d = 0$. However, there are weakly instanton sheaves that are not instanton; for instance, the ideal sheaf of c distinct lines in $\mathbb{P}^3$ is an example of a rank one weakly instanton sheaf with charge and defect equal to c. See also Example 1 below.

Proposition 16. *A locally-free weakly instanton sheaf $\mathcal{E}$ is an instanton if and only if $d = c_3(\mathcal{E}) = 0$.*

In particular, notice that every rank 2 locally-free weakly instanton sheaf with $c_1(\mathcal{E}) = 0$ is actually an instanton. This is not true for higher rank locally-free weakly instanton sheaves, see Example 1 below.

Proof. It follows from the definition that $\chi(\mathcal{E}(-2)) = h^2(\mathcal{E}(-2)) = d$, hence $\mathcal{E}$ is an instanton if and only if $\chi(\mathcal{E}(-2)) = 0$, which occurs if and only if $c_3(\mathcal{E}) = 0$. $\square$

Now let $\tilde{X}$ be a properly $\mathbb{C}$-semistable complex ADHM datum. Considering the ADHM monad (13) with maps α and β defined as in (14) and (15), one still has that $\beta\alpha = 0$, and that α is injective as a sheaf map.

The map β however is no longer surjective. So define the sheaves:

$$K = \ker\beta, \qquad Q = \operatorname{coker}\beta,$$
$$B = \operatorname{im}\beta \quad \text{and} \quad \mathcal{E} = \ker\beta/\operatorname{im}\alpha.$$

Our goal is to show that $\mathcal{E}$ is a weakly instanton sheaf.

Proposition 17. *If the complex ADHM datum $\tilde{X}$ is $\mathbb{C}$-semistable, then sheaf $Q = \operatorname{coker}\beta$ is supported at finitely many points.*

In particular, $\operatorname{ch}(Q) = d \cdot [H]^3$, where d is the length of Q.

Proof. It is easy to see that β is surjective on the line $\ell_\infty = \{z = w = 0\}$. So it is enough to show that the localization of β to all but finitely points $X = [x : y : z : w] \in \mathbb{P}^3 \setminus \ell_\infty$ is surjective.

Equivalently, we argue that if (B_{kl}, i_k, j_k) is $\mathbb{C}$-semistable, then the dual map $\beta_X^\dagger$ is injective for all but finitely points $X \in \mathbb{P}^3 \setminus \ell_\infty$. Indeed, $\beta_X^\dagger$ is not injective for some $[x : y : p_1 : p_2]$, then there is $v \in V$ such that:

$$\begin{cases} \tilde{B}_1(p_1, p_2)^\dagger v = \bar{x}v, \\ \tilde{B}_2(p_1, p_2)^\dagger v = -\bar{y}v, \\ \tilde{\imath}(p_1, p_2)^\dagger v = 0 \end{cases} \tag{25}$$

which, by duality, implies that $(\tilde{B}_1(p_1, p_2), \tilde{B}_2(p_1, p_2), \tilde{\imath}(p_1, p_2))$ is not stable. But that can only happen for finitely many $[p_1 : p_2]$; for each of these, the conditions (25) above are only satisfied for finitely many x, y, since they are the common eigenvalues of $B_1(p_1, p_2)^\dagger$ and $B_2(p_1, p_2)^\dagger$. $\square$

Now notice that the sheaves K, B, Q and $\mathcal{E}$ fit into the following short exact sequences:

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} K \to \mathcal{E} \to 0, \tag{26}$$

$$0 \to K \to \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} B \to 0, \tag{27}$$

$$0 \to B \to V \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to Q \to 0. \tag{28}$$

A careful examination of the associated long exact sequences in cohomology will show that $H^0(\mathcal{E}(-1)) = H^1(\mathcal{E}(-2)) = H^3(\mathcal{E}(-3)) = 0$ and that $H^2(\mathcal{E}(-2)) \simeq H^0(Q(-2))$. Moreover,

$$\mathrm{ch}(\mathcal{E}) = r - c \cdot [H]^2 + d \cdot [H]^3,$$

for $r = \dim W$, $c = \dim V$ and $d = h^0(Q(-2)) = \mathrm{lenght}(Q)$. Thus $\mathcal{E}$ is indeed a weakly instanton sheaf, as desired.

To see that $\mathcal{E}$ is indeed torsion-free, notice that $\mathcal{E}|_{\ell_\infty}$ is isomorphic to $W \otimes \mathcal{O}_\ell$; in other words, $\mathcal{E}$ is a framed weakly instanton sheaf. Thus $\mathcal{E}$ is locally-free in a neighborhood of ℓ_∞, so that it must be locally-free away from a codimension 2 subvariety.

We have therefore proved:

Proposition 18. *If the complex ADHM datum $\tilde{X}$ is a $\mathbb{C}$-semistable solution of* (10)*, then first cohomology sheaf $\mathcal{E} = \ker\beta/\operatorname{im}\alpha$ of the associated ADHM monad* (21) *is a framed torsion-free weakly instanton sheaf.*

It is interesting to note that no *locally-free* weakly instanton sheaves which are not instanton can be constructed in this way. Indeed, assume that $\mathcal{E} = \ker\beta/\operatorname{im}\alpha$ constructed as above is locally-free. Then by sequence (26) $K = \ker\beta$ is also locally-free, and by sequence (27) $B = \operatorname{im}\beta$ is also locally-free. Since $Q = \operatorname{coker}\beta$ is supported on points, it follows that $\mathcal{E}xt^p(Q, \mathcal{O}_{\mathbb{P}^3}) = 0$ for $0 \leqslant p \leqslant 2$, so dualizing sequence (28) we get that $B^* \simeq V \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ and $Q = 0$. Therefore, $\mathcal{E}$ is actually an instanton sheaf.

It is not clear to the authors whether every weakly instanton sheaf on $\mathbb{P}^3$ of trivial splitting type arises in this way (through $\mathbb{C}$-semistable ADHM data). A positive answer would imply that every locally-free sheaf of trivial splitting type satisfying $h^1(\mathcal{E}(-2)) = 0$ also has $h^2(\mathcal{E}(-2)) = 0$, hence it is an instanton. The authors have not been able to construct any counter-example to this statement, but see the example below (which we owe to G. Ottaviani).

Example 1. Take the quotient bundle Q on $\mathbb{P}^3$ given by

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \to Q \to 0,$$

$$\alpha = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

(Note that α is injective as a bundle map.) Define $E = (S^3 Q)(-1)$; since the slope of Q is $1/3$, the slope of $S^3 Q$ is 1 and $c_1(E) = 0$. E is stable by Ramanan's theorem for homogeneous bundles, hence $H^0(E(-1)) = H^3(E(-3)) = 0$. From Bott's theorem for homogeneous bundles, it follows that $H^i(E(-2)) = 0$ for $i = 0, 1, 3$, but $H^2(E(-2)) \neq 0$. Thus E is a weakly instanton, but not an instanton. Note however, that E constructed as above is not of trivial splitting type.

We are finally in a position to establish our second main result, the 1-1 correspondence between weakly instanton sheaves and curves in $\mathbb{P}^3$.

Theorem 19. *There is a* 1-1 *correspondence between the following objects*:

- *rank one weakly instanton sheaves* $\mathcal{E}$;
- *ideal sheaves of curves within* $\mathbb{P}^3$.

Furthermore, $\mathcal{E}$ *restricts trivially at the line at infinity* ℓ_∞*, if and only if the corresponding curve does not intersect* ℓ_∞.

Proof. Let Σ be a curve in $\mathbb{P}^3$ and consider its ideal sheaf $\mathcal{I}_\Sigma$; it is a rank 1 torsion-free sheaf on $\mathbb{P}^3$ with $c_1(\mathcal{I}_\Sigma)=0$. Since

$$H^0\big(\mathbb{P}^3,\iota_*\mathcal{O}_\Sigma(-1)\big)=H^0\big(\Sigma,\mathcal{O}_\Sigma(-1)\big)=0$$
$$\text{and}\quad H^2\big(\mathbb{P}^3,\iota_*\mathcal{O}_\Sigma(-3)\big)=H^3\big(\mathbb{P}^3,\iota_*\mathcal{O}_\Sigma(-3)\big)=0,$$

it follows from the short exact sequence:

$$0\to\mathcal{I}_\Sigma\to\mathcal{O}_{\mathbb{P}^3}\to\iota_*\mathcal{O}_\Sigma\to 0 \tag{29}$$

that the ideal sheaf $\mathcal{I}_\Sigma$ is a rank 1 weakly instanton sheaf. Furthermore, if $\Sigma\cup\ell_\infty=\emptyset$, then $\mathcal{I}_\Sigma$ restricts trivially at ℓ_∞.

Conversely, given a rank one torsion-free weakly instanton sheaf $\mathcal{E}$ with

$$\mathrm{ch}(\mathcal{E})=1-c\cdot[H]^2+d\cdot[H]^3\quad(d\neq 0).$$

Its double-dual $\mathcal{E}^{**}$ is a rank 1 reflexive sheaf with $c_1(\mathcal{E}^{**})=c_1(\mathcal{E})=0$, thus $\mathcal{E}^{**}=\mathcal{O}_{\mathbb{P}^3}$. Thus $\mathcal{E}$ is an ideal sheaf $\mathcal{I}_\Sigma$ of a subvariety $\Sigma\subset\mathbb{P}^3$, with $\mathcal{O}_\Sigma=\mathcal{E}^{**}/\mathcal{E}$.

Since $c_1(\mathcal{O}_\Sigma)=0$, Σ must have dimension at most 1. From the exact sequence (29) we deduce that $H^0(\mathcal{O}_\Sigma(k))=0$ for $k\leqslant -2$, so Σ cannot have 0-dimensional components. Thus our rank 1 torsion-free weakly instanton sheaf $\mathcal{E}$ is the ideal sheaf of a 1-dimensional subvariety Σ within $\mathbb{P}^3$. Moreover, if $\mathcal{E}$ restricts trivially at the line at infinity ℓ_∞, then Σ does not intersect ℓ_∞. $\square$

2.5. Reflexive instanton sheaves

Reflexive sheaves on $\mathbb{P}^3$ have been extensively studied in a series of papers by Hartshorne [9], among other authors. In particular, it was shown that a rank 2 reflexive sheaf $\mathcal{F}$ on $\mathbb{P}^3$ is locally-free if and only if $c_3(\mathcal{F})=0$. Therefore, we conclude:

Proposition 20. *There are no rank* 2 *instanton sheaves on* $\mathbb{P}^3$ *which are reflexive but not locally-free.*

The situation for higher rank is quite different, though, and it is easy to construct a rank 3 instanton sheaf which is reflexive but not locally-free. Setting $r=3$ and $c=1$, consider the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1),$$

$$\alpha = \begin{pmatrix} x \\ y \\ 0 \\ 0 \\ z \end{pmatrix} \quad \text{and} \quad \beta = (\,-y \quad x \quad z \quad w \quad 0\,).$$

Again, it is easy to see that β is surjective for all $[x:y:z:w] \in \mathbb{P}^3$, while α is injective provided $x, y, z \neq 0$. It then follows from applying Proposition 12 to sequence (20) that $\mathcal{E}$ is reflexive, but not locally-free; its singularity set is just the point $[0:0:0:1] \in \mathbb{P}^3$. This example of a properly reflexive instanton sheaf corresponds to the properly $\mathbb{C}$-semiregular solution of the complex ADHM equations for $r = 3$ given in Remark 6.

Proposition 21. *A framed instanton sheaf $\mathcal{E}$ is reflexive if and only if the associated complex ADHM datum (B_{kl}, i_k, j_k) is $\mathbb{C}$-semiregular.*

It then follows from Proposition 20 that every $\mathbb{C}$-semiregular solution of the complex ADHM equations for $r = 2$ is $\mathbb{C}$-regular, as we claimed in Remark 6.

Proof. If (B_{kl}, i_k, j_k) is $\mathbb{C}$-semiregular, then $(\tilde{B}_1(p), \tilde{B}_2(p), \tilde{\imath}(p), \tilde{\jmath}(p))$ is costable for all but finitely many $p \in \mathbb{P}^1$ (see the observation following Proposition 3). Following the argument in the proof of Proposition 11, we conclude that α_X is injective for all but finitely many $X \in \mathbb{P}^3$. Thus, by Proposition 12, the cohomology sheaf $\mathcal{E}$ is reflexive.

Conversely, if $\mathcal{E}$ is reflexive then α_X is injective for all but finitely many $X \in \mathbb{P}^3$. It follows that $(\tilde{B}_1(p), \tilde{B}_2(p), \tilde{\imath}(p), \tilde{\jmath}(p))$ must be costable for all but finitely many $p \in \mathbb{P}^1$, and (B_{kl}, i_k, j_k) is $\mathbb{C}$-semiregular. □

2.6. Locally-free instanton sheaves and anti-self-dual connections

We will now describe necessary and sufficient conditions that guarantee that the cohomology sheaf of the monad (13) is locally-free. Recall that α_X denotes the localization of the map α to a point $X \in \mathbb{P}^3$.

Proposition 22. *A framed instanton sheaf $\mathcal{E}$ is locally-free if and only if the associated complex ADHM datum (B_{kl}, i_k, j_k) is $\mathbb{C}$-regular.*

Proof. Following once again the argument in the proof of Proposition 11, it is easy to see that α_X is injective for all $X \in \mathbb{P}^3$ if and only if (B_{kl}, i_k, j_k) is $\mathbb{C}$-costable. Together with Proposition 12, we obtain the desired result. □

Remark 23. As it was pointed out in Remark 6, there are solutions of the complex ADHM equations which are $\mathbb{C}$-stable but not $\mathbb{C}$-regular. Therefore, *there exist torsion-free instanton sheaves which are not locally-free*. The basic example is the cohomology $\mathcal{E}$ of the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1),$$

$$\alpha = \begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta = (\,-y \quad x \quad z \quad w\,).$$

It is easy to see that β is surjective for all $(x, y, z, w) \in \mathbb{P}^3$, while α is injective provided $x, y \neq 0$. It then follows from applying Proposition 12 to sequence (20) that $\mathcal{E}$ is torsion-free, but not locally-free. In particular, the singularity set of $\mathcal{E}$ consists of the line $\ell = \{x = y = 0\} \subset \mathbb{P}^3$.

Rank r locally-free instanton sheaves correspond, via the Penrose transform, to holomorphic vector bundles $\hat{\mathcal{E}}$ with (singular) anti-self-dual $GL(r, \mathbb{C})$-connections $\nabla : \Gamma(\hat{\mathcal{E}}) \to \Gamma(\hat{\mathcal{E}}) \otimes \Omega^1_{\mathbb{M}}$ on $\mathbb{M}$, the complexified compactified Minkowski space-time, see [21]. Recall also that $\mathbb{M}$ is just the Grassmannian of lines in $\mathbb{P}^3$.

The singularity set of the anti-self-dual $GL(r, \mathbb{C})$-connections so obtained can be characterized as follows. For a given point $x \in \mathbb{M}$, the corresponding line on $\mathbb{P}^3$ will be denoted by ℓ_x. Since the restriction of $\mathcal{E}$ to the line at infinity $\ell_\infty = \{z = w = 0\}$ is trivial, we can conclude that $\mathcal{E}|_{\ell_x} \simeq \mathcal{O}_{\ell_x}^{\oplus r}$ for generic $x \in \mathbb{M}$. The set

$$\mathcal{J}(\mathcal{E}) = \{x \in \mathbb{M} \mid \mathcal{E}|_{\ell_x} \text{ is nontrivial}\}$$

is called the *subvariety of jumping lines*.

Proposition 24. *For any framed locally-free instanton sheaf $\mathcal{E}$ on $\mathbb{P}^3$ of charge c, its set of jumping lines $\mathcal{J}(\mathcal{E}) \subset \mathbb{M}$ is a divisor of degree c.*

Proof. The rank $r = 2$ case is done in [17, p. 44]. Consider the double-fibration:

$$\mathbb{P}^3 \xleftarrow{\mu} \mathbb{F} \xrightarrow{\nu} \mathbb{M}$$

and let $\Phi : D^{\mathrm{b}}(\mathbb{P}^3) \to D^{\mathrm{b}}(\mathbb{M})$, where $D^{\mathrm{b}}(X)$ denotes the bounded derived category of coherent sheaves on a projective variety X, be the integral functor given by:

$$\Phi(\mathcal{F}) = R\nu_*\big(\mu^*\mathcal{F}\big).$$

Note that $\mathcal{J}(\mathcal{E})$ is the support of the sheaf $\Upsilon = \Phi^1(\mathcal{E}(-1)) = R^1\nu_*(\mu^*\mathcal{E}(-1))$, and that $\Phi^0(\mathcal{E}(-1)) = 0$.

Furthermore, it is not hard to see that (see e.g. [21, Theorem 7.1.4]):

$$\Phi^0\big(\mathcal{O}_{\mathbb{P}^3}(-1)\big) = \Phi^1\big(\mathcal{O}_{\mathbb{P}^3}(-1)\big) = 0, \qquad \Phi^0(\mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_{\mathbb{M}}, \qquad \Phi^1(\mathcal{O}_{\mathbb{P}^3}) = 0,$$
$$\Phi^0\big(\mathcal{O}_{\mathbb{P}^3}(-2)\big) = 0, \qquad \Phi^1\big(\mathcal{O}_{\mathbb{P}^3}(-2)\big) = (\det S_+) = \mathcal{O}_{\mathbb{M}}(-1)$$

where S_+ is a tautological bundle over $\mathbb{M}$ (seen as the Grassmannian of planes in $\mathbb{C}^4$). Applying the functor Φ to the sequences (19) and (20) twisted by $\mathcal{O}_{\mathbb{P}^3}(-1)$ we obtain:

$$0 \to \mathcal{O}_{\mathbb{M}}(-1)^{\oplus c} \to \mathcal{O}_{\mathbb{M}} \to \Upsilon \to 0,$$

which completes the proof. $\square$

Now let $U = \mathbb{M} \setminus \mathcal{J}(\mathcal{E})$ be the complement of the divisor of jumping lines. It corresponds to an open subset $\hat{U} \subset \mathbb{P}^3$ which is the union of all lines ℓ in $\mathbb{P}^3$ for which the restriction $\mathcal{E}|_\ell$ is trivial. In other words, $p \in \hat{U}$ if $\mathcal{E}$ restricts trivially to every line passing through p.

We can then apply the Penrose–Ward transform [13,21] to construct a smooth anti-self-dual $GL(r, \mathbb{C})$-valued connection on a complex vector bundle $\hat{\mathcal{E}} \to U$. Explicitly in terms of the original regular complex ADHM datum (B_{kl}, i_k, j_k) associated with $\mathcal{E}$, the bundle and the connection can be described as follows. Let $\underline{V}$ and $\underline{\tilde{W}}$ denote the trivial vector bundles over U; consider the bundle maps:

$$\underline{V} \underset{\alpha_2}{\overset{\alpha_1}{\longrightarrow}} \underline{\tilde{W}} \underset{\beta_2}{\overset{\beta_1}{\longrightarrow}} \underline{V}$$

defined as follows:

$$\alpha_1 = \begin{pmatrix} B_{11} - x_{11'} \\ B_{12} - x_{12'} \\ j_1 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} B_{21} - x_{21'} \\ B_{22} - x_{22'} \\ j_2 \end{pmatrix},$$
$$\beta_1 = (\, -B_{12} + x_{12'} \quad B_{11} - x_{11'} \quad i_1 \,),$$
$$\beta_2 = (\, -B_{22} + x_{22'} \quad B_{21} - x_{21'} \quad i_2 \,)$$

where $(x_{11}, x_{12}, x_{21}, x_{22})$ are local coordinates on $\mathbb{M}^{\mathrm{I}} = \mathbb{C}^4 \supset U$.

Proposition 25. (B_{kl}, i_k, j_k) *satisfies the complex ADHM equations* (7)–(9) *if and only if the following identities hold*:

$$\beta_1\alpha_1 = \beta_2\alpha_2 = 0, \tag{30}$$

$$\beta_2\alpha_1 + \beta_1\alpha_2 = 0. \tag{31}$$

Now we consider the maps $\vec{\alpha} : (\underline{V} \oplus \underline{V}) \to \underline{\tilde{W}}$ and $\vec{\beta} : \underline{\tilde{W}} \to (\underline{V} \oplus \underline{V})$ given by

$$\vec{\alpha} = (\, \alpha_1 \quad \alpha_2 \,) \quad \text{and} \quad \vec{\beta} = \begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix}.$$

If (B_{kl}, i_k, j_k) is a solution of (7)–(9), the identities in Proposition 25 imply that $\Xi = \vec{\beta}\vec{\alpha} = \beta_1\alpha_2\mathbf{1}_{\mathbb{C}^2}$; furthermore, Ξ is an isomorphism for each $(x_{11'}, x_{12'}, x_{21'}, x_{22'}) \in U$, and we define the map:

$$P : \underline{\tilde{W}} \to \underline{\tilde{W}},$$
$$P = \mathbf{1}_{\tilde{W}\otimes\mathfrak{M}_q^{\mathrm{I}}} - \vec{\alpha}(\Xi)^{-1}\vec{\beta}.$$

Notice that $P^2 = P$, i.e. P is a projection; the bundle $E \to U$ is then defined as

$$E = \mathrm{Im}(P) = \ker\vec{\beta} = \ker\beta_1 \cup \ker\beta_2,$$

with a connection given by the usual projection formula:

$$\nabla = P\underline{d}\iota \tag{32}$$

where $\underline{d}$ is the trivial derivative on $\underline{\tilde{W}}$ and ι is the natural inclusion. One can check that ∇ is anti-self-dual through the usual calculation, see for instance [8, Proposition 14].

The singularity locus of the connection ∇ on $\mathbb{M}^{\mathrm{I}}$ is given by $\det \Xi = 0$. Since, by construction [13, Chapter 10], the singularity locus of the instanton connection (32) must coincide with the set of jumping lines, this expression also allows us to describe subvariety of jumping lines concretely in terms of the associated complex ADHM data:

$$\mathcal{J}(\mathcal{E}) = \bigl\{\det\bigl(D \cdot (B_{11}B_{22} - B_{12}B_{21} + i_1 j_2) + z_{12}B_{21} - z_{11}B_{22} + D'\mathbf{1}_V\bigr) = 0\bigr\},$$

where $[z_{11} : z_{12} : z_{21} : z_{22} : D : D']$ are homogeneous coordinates on $\mathbb{M}$ satisfying $z_{11}z_{22} - z_{12}z_{21} = DD'$. This last expression is simply the homogenization of the affine equation $\det \Xi = 0$, using $x_{kl} = z_{kl}/D$.

It is interesting to remark that *there can be no nontrivial smooth anti-self-dual connections on* $\mathbb{M}$, since any such connection would be transformed into a bundle on $\mathbb{P}^3$ which is trivial on every line, and any such bundle is necessarily trivial [17, Theorem I.3.2.1]. In other words, *every anti-self-dual* $GL(r,\mathbb{C})$*-connection on* $\mathbb{M}$ *must be singular along a divisor within* $\mathbb{M}$.

With this in mind, we remark that $\mathcal{M}^{\mathrm{reg}}_{\mathbb{C}}(r,c)$, the open subset of $\mathcal{M}_{\mathbb{C}}(r,c)$ consisting of the $GL(V)$-orbits of $\mathbb{C}$-regular solutions of the complex ADHM equations (7)–(9), can also be interpreted as the moduli space of singular anti-self-dual $GL(r,\mathbb{C})$-connections over $\mathbb{M}$ of charge c.

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Appendix A. Cohomological calculations

We collect here the proofs for various facts used in Section 2.

Proposition 26. *Let* $\mathcal{E}$ *be a torsion-free instanton sheaf over* $\mathbb{P}^3$ *with* $\mathrm{ch}(\mathcal{E}) = r - c[H]^2$ *and such that* $\mathcal{E}|_{\ell_\infty} = \mathcal{O}_{\ell_\infty}^{\oplus r}$. *The following hold*:

- $h^1(\mathbb{P}^3, \mathcal{E}(-1)) = -\chi(\mathcal{E}(-1)) = c$;
- $h^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}) = -\chi(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}) = c + 2r$;
- $h^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}(1)) = -\chi(\mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}(1)) = c$;
- *there is a canonical isomorphism* $H^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}(1)) \simeq H^1(\mathbb{P}^3, \mathcal{E}(-1))$.

Proof. First notice that

$$\chi\bigl(\mathcal{E}(-1)\bigr) = \chi\bigl(\mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}(1)\bigr) = -c \quad \text{and} \quad \chi\bigl(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}\bigr) = -c - 2r.$$

Using the restriction sequence

$$0 \to \mathcal{E}(k-1) \to \mathcal{E}(k) \to \mathcal{E}(k)|_{\wp} \to 0 \tag{33}$$

for some hyperplane $\wp \subset \mathbb{P}^3$, one can expand the instanton condition to the following:

$$\begin{aligned} &H^0\big(\mathbb{P}^3, \mathcal{E}(k)\big) = 0, \quad \forall k \leqslant -1, \qquad && H^1\big(\mathbb{P}^3, \mathcal{E}(k)\big) = 0, \quad \forall k \leqslant -2, \\ &H^2\big(\mathbb{P}^3, \mathcal{E}(k)\big) = 0, \quad \forall k \geqslant -2, \qquad && H^3\big(\mathbb{P}^3, \mathcal{E}(k)\big) = 0, \quad \forall k \geqslant -3, \end{aligned} \tag{34}$$

which are the conditions originally used by Manin in [12]. The first statement then follows immediately.

Tensoring the Euler sequence for 1-forms by $\mathcal{E}$ we get:

$$0 \to \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3} \to \bigoplus_4 \mathcal{E}(-1) \to \mathcal{E} \to 0. \tag{35}$$

At the level of cohomology, one obtains:

$$0 \to H^0\big(\mathbb{P}^3, \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}\big) \to \bigoplus_4 H^0\big(\mathbb{P}^3, \mathcal{E}(-1)\big)$$

and since $H^0(\mathbb{P}^3, \mathcal{E}(-1)) = 0$, it follows that $H^0(\mathbb{P}^3, \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}) = 0$. Moreover, (35) also implies that:

$$H^2\big(\mathbb{P}^3, \mathcal{E}\big) \to H^3\big(\mathbb{P}^3, \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}\big) \to \bigoplus_4 H^3\big(\mathbb{P}^3, \mathcal{E}(-1)\big).$$

Since the first and third groups vanish, we obtain $H^3(\mathbb{P}^3, \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}) = 0$.

Now tensor the Euler sequence for 3-forms by $\mathcal{E}$ and obtain:

$$0 \to \mathcal{E}(-3) \to \bigoplus_4 \mathcal{E}(-2) \to \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}(1) \to 0 \tag{36}$$

since $\Omega^3_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4)$. Since $H^p(\mathbb{P}^3, \mathcal{E}(-2)) = 0$ for all p, it follows from the cohomology sequence associated with (36) that:

$$H^p\big(\mathbb{P}^3, \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}(1)\big) \simeq H^{p+1}\big(\mathbb{P}^3, \mathcal{E}(-3)\big) \tag{37}$$

thus $H^p(\mathbb{P}^3, \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}(1)) = 0$ for $p = 0, 2, 3$, which completes the proof of the third statement.

For the second statement, it only remains for us to show that $H^2(\mathbb{P}^3, \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}) = 0$. Tensoring the Euler sequence for 2-forms by $\mathcal{E}$ we obtain:

$$0 \to \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3} \to \bigoplus_6 \mathcal{E}(-2) \to \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3} \to 0. \tag{38}$$

The associated cohomology sequence yields:

$$\bigoplus_6 H^2\big(\mathbb{P}^3, \mathcal{E}(-2)\big) \to H^2\big(\mathbb{P}^3, \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}\big) \to H^3\big(\mathbb{P}^3, \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}\big) \to \bigoplus_6 H^3\big(\mathbb{P}^3, \mathcal{E}(-2)\big).$$

The first and last groups vanish, hence $H^2(\mathbb{P}^3, \mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}) \simeq H^3(\mathbb{P}^3, \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3})$. From (36) we get:

$$0 \to \mathcal{E}(-4) \to \bigoplus_4 \mathcal{E}(-3) \to \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3} \to 0 \tag{39}$$

and we conclude that $H^3(\mathbb{P}^3, \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}) = 0$ since $H^3(\mathbb{P}^3, \mathcal{E}(-3)) = 0$.

Finally, let $\wp \subset \mathbb{P}^3$ be a plane containing ℓ_∞, so that the restriction $\mathcal{E}|_\wp$ yields a torsion-free sheaf on $\wp$ which is trivial at ℓ_∞. Setting $k = -2$ in (33), we conclude that $H^1(\mathbb{P}^3, \mathcal{E}(-2)|_\wp) \simeq H^2(\mathbb{P}^3, \mathcal{E}(-3))$, since $H^1(\mathbb{P}^3, \mathcal{E}(-2)) = H^2(\mathbb{P}^3, \mathcal{E}(-2)) = 0$. Then setting $k = -1$ in (33), we get that $H^1(\mathbb{P}^3, \mathcal{E}(-1)|_\wp) \simeq H^1(\mathbb{P}^3, \mathcal{E}(-1))$ for the same reason. Together with (37), we have obtained the identifications

$$\begin{aligned} H^1\big(\mathbb{P}^3, \mathcal{E} \otimes \Omega^2_{\mathbb{P}^3}(1)\big) &\simeq H^2\big(\mathbb{P}^3, \mathcal{E}(-3)\big) \simeq H^1\big(\mathbb{P}^3, \mathcal{E}(-2)|_\wp\big) \\ &\simeq H^1\big(\mathbb{P}^3, \mathcal{E}(-1)|_\wp\big) \simeq H^1\big(\mathbb{P}^3, \mathcal{E}(-1)\big) \end{aligned}$$

where the identification $H^1(\mathbb{P}^3, \mathcal{E}(-2)|_\wp) \simeq H^1(\mathbb{P}^3, \mathcal{E}(-1)|_\wp)$ follows from [16, p. 20]. This completes the proof of the fourth statement. □

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