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A Fourier–Mukai transform for real torus bundles

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Abstract

We construct a Fourier–Mukai transform for smooth complex vector bundles E over a torus bundle $\pi : M \to B$, the vector bundles being endowed with various structures of increasing complexity. At a minimum, we consider vector bundles E with a flat partial unitary connection, that is families or deformations of flat vector bundles (or unitary local systems) on the torus T. This leads to a correspondence between such objects on M and relative skyscraper sheaves $\mathcal{S}$ supported on a spectral covering $\Sigma \hookrightarrow \hat{M}$, where $\hat{\pi} : \hat{M} \to B$ is the flat dual fiber bundle. Additional structures on (E, ∇) (flatness, anti-self-duality) will be reflected by corresponding data on the transform $(\mathcal{S}, \Sigma)$. Several variations of this construction will be presented, emphasizing the aspects of foliation theory which enter into this picture.

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1. Introduction

The construction nowadays known as the *Fourier–Mukai transform* first appeared in a seminal work by Mukai [11], where it was shown that the derived categories of sheaves on an abelian variety (e.g. a complex torus) is equivalent to the derived category of coherent sheaves on the *dual* abelian variety.

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Since then, the Fourier–Mukai transform has been generalized in different ways, and has led to a number of interesting results concerning not only the derived categories of coherent sheaves, but also the moduli spaces of stable sheaves on abelian varieties, K3 surfaces and elliptic surfaces.

This paper draws on two types of generalization of Mukai's original ideas. First, one can consider *families* of abelian varieties, and define a transform that takes (complexes of) sheaves on a family of abelian varieties to (complexes of) sheaves on the corresponding *dual* family. This has been applied with great success to the study of stable sheaves on elliptic surfaces (i.e. holomorphic families of elliptic curves parametrized by an algebraic curve), see for instance [2] and the references there. In particular, given an elliptic surface X, it can be shown that there exists a 1–1 correspondence between vector bundles on X which are stable with respect to some suitable polarization, and certain torsion sheaves (spectral data) on the relative Jacobian surface (see [6,7] for details).

On the other hand, Mukai's construction can be generalized from complex tori to *real* tori. Such a generalization, first considered by Arinkin and Polishchuk in [1] is briefly described in Section 2.

Building on previous work by Arinkin and Polishchuk [1] and by Bruzzo et al. [3,4], we consider in this paper a Fourier–Mukai transform for vector bundles with (partial) connections on families of real tori. Rather than restricting ourselves to flat connections on *Lagrangian* families of real tori as in [1,4], we study a broader class of connections on vector bundles over a (not necessarily symplectic) manifold M with the structure of a *flat* torus bundle.

After briefly reviewing the Fourier–Mukai transform for real tori, following [3], we start in Section 3 by defining a Fourier–Mukai transform for *foliated bundles*, which in our context can be viewed as *families of flat bundles* on the fibers of the torus bundle $M \to B$. This takes foliated Hermitian vector bundles over M into certain torsion sheaves on the dual fibration $\hat{M} \to B$. We then introduce the concept of *Poincaré basic connections* in Section 4, and extend our construction to include vector bundles provided with such connections. We then conclude in Section 5 by applying our techniques to three different examples: flat connections, that is unitary local systems, instantons on 4-dimensional circle fibrations, and monopoles on 3-dimensional circle fibrations.

It is a somewhat surprising fact that certain concepts and techniques from foliation theory occur quite naturally in the context of the Fourier–Mukai transform. Besides the notions of foliated bundle and Poincaré basic connection which refer to the torus fibration, that is to a foliation which is rather trivial from the point of view of foliation theory, there is also a canonical foliation on the dual fibration $\hat{M} \to B$, transverse to the fibers which has a more complicated structure. For locally trivial families of flat bundles on the fibers of the torus bundle $M \to B$, it turns out that the supports $\Sigma \hookrightarrow \hat{M}$ of the Fourier–Mukai transform are (finite unions of) leaves of this transverse foliation. This allows us to give in Section 5.1 an explicit parametrization of the representation variety $\mathcal{R}_M(n)$ of M in terms of leaves with transversal holonomy of order ℓ such that $\ell | n$.

The main motivation behind [1] and [3,4] comes from string theory and the Strominger–Yau–Zaslow approach to mirror symmetry, with the main goal of understanding Kontsevich's homological mirror symmetry conjecture. In a sense, the two main results here presented may also be relevant to the understanding of Kontsevich's conjecture. Although it seems reasonable to expect that the ideas explored in this paper might provide some interesting connections with String Theory and mirror symmetry, we do not elaborate on them, leaving such a task to mathematical physicists.

Notation. We work on the category of real C^∞-manifolds. By a vector bundle over a manifold X, we mean a C^∞ vector bundle over X. We will also identify a vector bundle with the corresponding sheaf of C^∞ sections. By the same token, a sheaf on X should be understood as a sheaf of modules over the algebra of C^∞ functions on X.

2. Local systems on tori

Let us begin by briefly recalling the Fourier–Mukai transform for real tori, as defined by Arinkin and Polishchuk [1] and Bruzzo et al. [3]; the interested reader should refer to these papers for the details of this construction.

Let T be the d-dimensional real torus, that is $T = \mathbb{R}^d/\Lambda$ for the rank d integral lattice $\Lambda \subset \mathbb{R}^d$. The associated dual torus is defined as $\hat{T} = (\mathbb{R}^d)^*/(\Lambda)^*$, where

$$(\Lambda)^* := \{z \in (\mathbb{R}^d)^* : z(y) \in \mathbb{Z}, \forall y \in \Lambda\}, \tag{2.1}$$

is the dual lattice. From the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathrm{U}(1)) \to H^1(T, \mathcal{O}_T^*) \xrightarrow{c_1} H^2(T, \mathbb{Z}) \to 0, \tag{2.2}$$

we see that, up to gauge equivalence, points in $\hat{T}$ parametrize flat unitary connections on the trivial line bundle $\underline{\mathbb{C}} = T \times \mathbb{C} \to T$, since we have

$$\hat{T} = H^1(T, \mathbb{R})/H^1(T, \mathbb{Z}) \overset{\exp}{\cong} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathrm{U}(1)) \cong \mathrm{U}(1)^d. \tag{2.3}$$

For $\xi \in \hat{T}$, $x \in \mathbb{R}^d$, $a \in \Lambda$ and $\lambda \in \mathbb{C}$, consider the equivalence relation

$$\mathbb{R}^d \times \hat{T} \times \mathbb{C} \to \mathbb{R}^d \times \hat{T} \times \mathbb{C}/\sim, \qquad (x + a, \xi, \lambda) \sim (x, \xi, \exp(\xi(a))\lambda). \tag{2.4}$$

The quotient space under '$\sim$' defines the Poincaré line bundle $P \to T \times \hat{T}$. Let p and $\hat{p}$ denote the natural projections of $T \times \hat{T}$ onto its first and second factors, respectively. In accordance with (2.4), the bundle P has the property that for $\xi \in \hat{T}$, the restriction $P|\hat{p}^{-1}(\xi) \cong L_\xi$, where the latter denotes the flat line bundle parametrized by ξ. It is straightforward to see that

$$\Omega^1_{T\times\hat{T}} = p^*\Omega^1_T \oplus \hat{p}^*\Omega^1_{\hat{T}}. \tag{2.5}$$

Corresponding to the definition of P and its above property, it is shown in [3] that there exists a canonical connection $\nabla_P : P \to P \otimes \Omega^1_{T\times\hat{T}}$, whose connection form is given by

$$\mathbb{A} = 2\pi\iota \sum_{j=1}^{d} \xi_j \,\mathrm{d}z^j, \tag{2.6}$$

where $\{z^j\}$ are (flat) coordinates on T and $\{\xi_j\}$ are dual (flat) coordinates on $\hat{T}$. The connection ∇_P splits as the sum $\nabla^r_P \oplus \nabla^t_P$, where

$$\nabla^r_P = (\mathbf{1}_P \otimes r) \circ \nabla_P, \quad \nabla^t_P = (\mathbf{1}_P \otimes t) \circ \nabla_P, \tag{2.7}$$

with natural maps $r : \Omega^1_{T\times\hat{T}} \to p^*\Omega^1_T$, and $t : \Omega^1_{T\times\hat{T}} \to \hat{p}^*\Omega^1_{\hat{T}}$.

For later purposes we shall denote the dual of any complex vector bundle E by $E^\vee$ and in particular the dual line bundle of P by $P^\vee$.

Now consider the categories $\mathbf{Sky}(\hat{T})$ and $\mathbf{Loc}(T)$ defined as follows (see [3]):

- $\mathbf{Loc}(T)$ is the category of *unitary local systems* on T. Its objects are pairs (E, ∇) consisting of a smooth complex vector bundle $E \to T$ and a flat unitary connection ∇. Morphisms are simply bundle maps compatible with the connections.
- $\mathbf{Sky}(\hat{T})$ is the category of skyscraper sheaves on $\hat{T}$ of finite length, that is, $\dim H^0(\hat{T}, S) < \infty$, for all $S \in \mathrm{Ob}(\mathbf{Sky}(\hat{T}))$.

The *Fourier–Mukai transform* is the invertible functor

$$\mathbf{F} : \mathbf{Loc}(T) \to \mathbf{Sky}(\hat{T}), \tag{2.8}$$

which we now describe. Given $(E, \nabla) \in \mathrm{Ob}(\mathbf{Loc}(T))$, we have the *relative connection*

$$\begin{aligned} &\nabla^r_E : p^*E \otimes P^\vee \to p^*E \otimes P^\vee \otimes p^*\Omega^1_T, \\ &\nabla^r_E = (\mathbf{1}_{(E\otimes P^\vee)} \otimes r) \circ (\nabla \otimes \mathbf{1}_{P^\vee} + \mathbf{1}_E \otimes \nabla_{P^\vee}), \end{aligned} \tag{2.9}$$

and *the transversal connection*

$$\begin{aligned} &\nabla^t_E : p^*E \otimes P^\vee \to p^*E \otimes P^\vee \otimes p^*\Omega^1_{\hat{T}}, \\ &\nabla^t_E = (\mathbf{1}_{(E\otimes P^\vee)} \otimes t) \circ (\nabla \otimes \mathbf{1}_{P^\vee} + \mathbf{1}_E \otimes \nabla_{P^\vee}). \end{aligned} \tag{2.10}$$

As a section of $\mathrm{End}_s(p^*E \otimes P^\vee) \otimes \Omega^2_{T\times\hat{T}}$, the commutator satisfies (see e.g. [10]):

$$\nabla^r_E\nabla^t_E + \nabla^t_E\nabla^r_E = \mathbf{1}_E \otimes \nabla^2_{P^\vee}. \tag{2.11}$$

Lemma 2.1 (Bruzzo et al. [3]). *If* $(E, \nabla) \in \mathrm{Ob}(\mathbf{Loc}(T))$, *then*:

(1) $\mathcal{R}^j\hat{p}_*(\ker \nabla^r_E) = 0$, *for* $0 \le j \le d - 1$.
(2) $S = \mathcal{R}^d\hat{p}_*(\ker \nabla^r_E) \in \mathrm{Ob}(\mathbf{Sky}(\hat{T}))$.

Moreover, $\dim H^0(\hat{T}, S) = \mathrm{rank}\, E$.

We say that $S = \mathbf{F}(E, \nabla)$ is the Fourier–Mukai transform of the local system (E, ∇).

Conversely, take $S \in \mathrm{Ob}(\mathbf{Sky}(\hat{T}))$, and let σ be the support of S. Clearly, $\hat{p}^*S \otimes P$ as a sheaf on $T \times \hat{T}$, is supported on $T \times \sigma$. Thus

$$\mathcal{R}^j p_*(\hat{p}^*S \otimes P) = 0, \quad \text{for } 0 < j \le d, \tag{2.12}$$

while $E = \mathcal{R}^0 p_*(\hat{p}^*S \otimes P)$ is a locally free sheaf of rank $\dim H^0(\hat{T}, S)$. In order to get a connection on E, consider again the relative connection:

$$\mathbf{1}_S \otimes \nabla^r_P : \hat{p}^*S \otimes P \to \hat{p}^*S \otimes P \otimes p^*\Omega^1_T. \tag{2.13}$$

Pushing it down to T, we get a connection

$$\nabla = \mathcal{R}^0 p_*(\mathbf{1}_S \otimes \nabla^r_P) : E \to E \otimes \Omega^1_T, \tag{2.14}$$

since $\mathcal{R}^0 p_*(\hat{p}^*S \otimes P \otimes p^*\Omega^1_T) = E \otimes \Omega^1_T$, by the projection formula. Since $(\nabla^r_P)^2 = 0$, we conclude that ∇ is indeed flat, hence $(E, \nabla) \in \mathrm{Ob}(\mathbf{Loc}(T))$, as desired. We use the notation $(E, \nabla) = \hat{\mathbf{F}}(S)$.

In summary, referring once more to [3], we have the following proposition.

Proposition 2.2. *The functors* $\mathbf{F}$ *and* $\hat{\mathbf{F}}$ *are inverse to each other, and yield an equivalence between the categories* $\mathbf{Loc}(T)$ *and* $\mathbf{Sky}(\hat{T})$.

3. The Fourier–Mukai transform

Let M be a smooth manifold of dimension m, which is the total space of a d-torus bundle over a $(m-d)$-dimensional connected manifold B, that is

$$T^d \hookrightarrow M \xrightarrow{\pi} B. \tag{3.1}$$

Given a point $b \in B$, we define $T_b = \pi^{-1}(b)$ to be the fiber over b, where the point $o(b)$ marks the origin of T_b. Regarded as a bundle of groups, $\pi : M \to B$ admits a discrete structure group $\mathrm{Aut}(T) \cong \mathrm{GL}(d, \mathbb{Z})$, and so the former has the structure of a flat fiber bundle and admits a 0-section $o : B \to M$. Since the fiber T is compact, this flat structure is determined by a holonomy homomorphism $\rho : \pi_1(B) \to \mathrm{Aut}(T) \cong \mathrm{GL}(d, \mathbb{Z})$ as a twisted product

$$M \cong \tilde{B} \times_\rho T. \tag{3.2}$$

Remark 3.1. For the purpose of this paper, we may weaken the structure of the fiber bundle $\pi : M \to B$ as follows. Let $\mathrm{Diff}(T, o)$ be the group of diffeomorphisms of T which fix the origin. Then $\mathrm{Diff}(T)$ is given as a crossed product $\mathrm{Diff}(T) \cong T \times_\varphi \mathrm{Diff}(T, o)$, where T acts by translations and $\mathrm{Diff}(T, o)$ acts on T in the obvious way. Moreover the canonical homomorphism $\mathrm{Diff}(T, o) \to \pi_0(\mathrm{Diff}(T, o))$ to the mapping class group is realized as a deformation retraction

$$\begin{array}{ccc} \mathrm{Diff}(T,o) & \xrightarrow{\;\widehat{}\;} & \mathrm{Aut}(\Lambda) \\ {\scriptstyle\subset}\big\uparrow & & {\scriptstyle\cong}\big\uparrow \\ \mathrm{Aut}(T) & \xrightarrow{\;\cong\;} & \mathrm{GL}(d,\mathbb{Z}) \end{array} \tag{3.3}$$

via $\varphi \mapsto \tilde{\varphi} \mapsto \hat{\varphi} = \tilde{\varphi}|\Lambda \in \mathrm{Aut}(\Lambda)$, where $\tilde{\varphi}$ is the unique equivariant lift of $\varphi \in \mathrm{Diff}(T, o)$ to $\mathrm{Diff}(\mathbb{R}^d, o)$ and $\hat{\varphi}$ coincides with the automorphism φ_* induced by φ on the fundamental group $\pi_1(T, o) \cong \Lambda$. The statement about the deformation retraction follows from the fact that any diffeomorphism (actually any homeomorphism) which fixes the lattice Λ is isotopic to the identity, and in fact the connected component $\mathrm{Diff}_e(T, o)$ is contractible to the identity; an elementary result which is stated in the 1960's thesis of John Franks (as pointed out to us by Keith Burns). This said, we may start with a fiber bundle $\pi : M \to B$ with structure group $\mathrm{Diff}(T, o)$. This still guarantees the existence of the section $o : B \to M$ and the previous holonomy homomorphism $\rho : \pi_1(B) \to \mathrm{Aut}(T)$ is now recovered as the canonical homomorphism $\pi_1(B) \to \pi_0(\mathrm{Diff}(T, o))$ associated to the fiber bundle $\pi : M \to B$. Formula (3.4) for $\pi_1(M)$ remains valid, as well as the flat structure (3.7) of the dual fiber bundle $\hat{\pi} : \hat{M} \to B$, the latter property being a consequence of the homotopy invariance of singular cohomology. In fact, the above deformation retraction implies that the structure group of a Diff(T, o)-torus bundle admits a unique reduction to $\mathrm{Aut}(T) \subset \mathrm{Diff}(T, o)$ and so $\pi : M \to B$ is still diffeomorphic to a flat fiber bundle of the form (3.2).

The fundamental group of M is determined as a crossed product

$$0 \to \Lambda = \pi_1(T) \to \pi_1(M) \cong \pi_1(T) \times_{\rho_*} \pi_1(B) \overset{\leftarrow}{\to} \pi_1(B) \to 1, \tag{3.4}$$

where ρ_* is given by the induced action of $\pi_1(B)$ on Λ via the isomorphism $\mathrm{GL}(d, \mathbb{Z}) \cong \mathrm{Aut}(\Lambda)$.

We have the exact sequence

$$0 \to T(\pi) \to TM \to \pi^* TB \to 0, \tag{3.5}$$

and the dual sequence of 1-forms

$$0 \to \pi^* \Omega^1_B \to \Omega^1_M \to \Omega^1_{M/B} \to 0. \tag{3.6}$$

Observe that a flat structure of $\pi : M \to B$ defines a splitting of the exact sequences (3.5) and (3.6).

The *dual fiber bundle* $\hat{M} \to B$ is given by the total space of $\mathcal{R}^1\pi_*\mathbb{R}/\mathcal{R}^1\pi_*\mathbb{Z}$ as a (locally constant) sheaf on B. If $\hat{\pi} : \hat{M} \to B$ is the natural projection, it is easy to see that $\hat{\pi}^{-1}(b) = \hat{T}_b$. Note that this projection also has a 0-section $\sigma_0 : B \to \hat{M}$. It follows that $\hat{\pi} : \hat{M} \to B$ is given by the flat bundle of fiber cohomologies

$$\hat{M} \cong \tilde{B} \times_{\rho^*} \hat{T}, \tag{3.7}$$

where ρ^* is the induced action of $\pi_1(B)$ on $\hat{T} = H^1(T, \mathbb{R})/H^1(T, \mathbb{Z})$. Furthermore, $\mathcal{R}^1\hat{\pi}_*\mathbb{R}/\mathcal{R}^1\hat{\pi}_*\mathbb{Z}$ coincides with M as sheaves on B, and we have $\hat{\hat{M}} \cong M$.

Let $Z = M \times_B \hat{M}$ be the fiber product, with its natural projections $p : Z \to M$ and $\hat{p} : Z \to \hat{M}$ onto the first and second factors. Clearly, $\pi \circ p = \hat{\pi} \circ \hat{p}$ and $(\pi \circ p)^{-1}(b) = T_b \times \hat{T}_b$.

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ M \arrow[dr, "\pi"'] & & \widehat{M} \arrow[dl, "\hat{\pi}"] \\ & B & \end{tikzcd} \tag{3.8}$$

It is also easy to see that $p^{-1}(x) = \hat{T}_{\pi(x)}$, for all $x \in M$ and $\hat{p}^{-1}(y) = T_{\hat{\pi}(y)}$, for all $y \in \hat{M}$.

Defining $\Omega^1_{Z/\hat{M}} = \Omega^1_Z/\hat{p}^*\Omega^1_{\hat{M}}$, recall that the Gauss–Manin connection yields a splitting of the short exact sequence

$$0 \to \hat{p}^*\Omega^1_{\hat{M}} \to \Omega^1_Z \xrightarrow{r} \Omega^1_{Z/\hat{M}} \to 0, \tag{3.9}$$

such that we have the decomposition

$$\Omega^1_Z = \hat{p}^*\Omega^1_{\hat{M}} \oplus \Omega^1_{Z/\hat{M}}. \tag{3.10}$$

From (3.6) it follows that

$$\Omega^1_{Z/\hat{M}} = p^*\Omega^1_{M/B}, \tag{3.11}$$

since $\hat{p} : Z \to \hat{M}$ is the pull-back fibration of $\pi : M \to B$ along $\hat{\pi}$.

There exists a line bundle $\mathcal{P}$ over $Z = M \times_B \hat{M}$, with the property that $\mathcal{P}|(\pi \circ p)^{-1}(b)$ is just the Poincaré line bundle $\mathcal{P}_b$ over $T_b \times \hat{T}_b$, for all $b \in B$ (see [4]). We call $\mathcal{P}$ *the relative Poincaré line bundle*. Just as in the absolute case, it has the property that for $\xi \in \hat{M}$, the restriction $\mathcal{P}|\hat{p}^{-1}(\xi) \cong L_\xi$, where the latter denotes the flat line bundle parametrized by $\xi \in \hat{T}_b \subset \hat{M}$.

There is a canonical connection on $\mathcal{P}$ which we denote by $\nabla_{\mathcal{P}}$. Following [4], we can write its connection matrix $\mathbb{A}$ in a suitable gauge on an open subset $U \times T \times \hat{T} \subset Z$ as follows:

$$\mathbb{A} = 2\pi\iota \sum_{j=1}^{d} \xi_j \, \mathrm{d}z^j, \tag{3.12}$$

where $\{z^j\}$ are (flat) coordinates on T and $\{\xi_j\}$ are dual (flat) coordinates on $\hat{T}$. In such coordinates the curvature $\mathbb{F} = \nabla^2_{\mathcal{P}}$ is then given by

$$\mathbb{F} = 2\pi\iota \sum_{j=1}^{d} \mathrm{d}\xi_j \wedge \mathrm{d}z^j. \tag{3.13}$$

In the same coordinate system, we have $\nabla^2_{\mathcal{P}^\vee} = -\mathbb{F}$.

3.1. *Transforming foliated bundles*

Let $E \to M$ be a Hermitian vector bundle of rank n. With reference to (3.5), we assume a *foliated bundle* structure on E given by a *flat partial unitary connection* [8]:

$$\overset{\circ}{\nabla}_E : E \to E \otimes \Omega^1_{M/B} = E \otimes \Omega^1_M/\pi^*\Omega^1_B, \tag{3.14}$$

satisfying $(\overset{\circ}{\nabla}_E)^2 = 0$.

The local structure of a foliated bundle $(E, \overset{\circ}{\nabla}_E)$ on M is described next.

Example 3.2. Local structure of foliated bundles on M: intuitively, a foliated bundle on $M \xrightarrow{\pi} B$ is a family of flat bundles (unitary local systems) (E_b, ∇_{E_b}) on the fibers T_b, parametrized by $b \in B$. Of course, the topology of E has to be taken into account. The local description is quite similar to that of the Poincaré line bundle in (2.4), which is of course an example of a foliated bundle. Thus for sufficiently small open sets $U \subset B$, there are isomorphisms

$$\begin{array}{ccc} U \times (\mathbb{R}^d \times_\Lambda \mathbb{C}^n) & \xrightarrow{\cong} & E \mid \pi^{-1}(U) \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \tau'} & & \Big\downarrow{\scriptstyle \tau|\pi^{-1}U} \\ U \times T & \xrightarrow{\cong} & \pi^{-1}(U)\ , \end{array} \tag{3.15}$$

where the identification on the LHS is given by

$$(b, x + a, \lambda) \sim (b, x, \exp(\xi_b(a))\lambda), \tag{3.16}$$

for $\xi = (\xi_1, \dots, \xi_n), \xi_j : U \to \hat{T}, \hat{T} \overset{\exp}{\cong} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathrm{U}(1)), b \in U, x \in \mathbb{R}^d, a \in \Lambda$ and $\lambda \in \mathbb{C}^n$. Relative to a (good) open cover $\mathcal{U}$ of B, we have coordinate changes over $U_{ik} = U_i \cap U_k$ of the form

$$(\mathrm{id}, \tilde{\varphi}_{ik}, g_{ik}) : U_{ik} \times \mathbb{R}^d \times \mathbb{C}^n \overset{\cong}{\to} U_{ik} \times \mathbb{R}^d \times \mathbb{C}^n,$$

compatible with the identifications in (3.16), that is

$$\mathrm{Adj}(g_{ik}(b)) \circ \exp(\xi_b^k(a)) = \exp(\xi_b^i(\hat{\varphi}_{ik}a)). \tag{3.17}$$

Here $\{\varphi_{ik}\}$ is the smooth 1-cocycle on $\mathcal{U}$ with values in $\mathrm{Diff}(T, o)$ describing the fiber bundle $\pi : M \to B$, $\tilde{\varphi}$ is the unique equivariant lift of $\varphi \in \mathrm{Diff}(T, o)$ to $\mathrm{Diff}(\mathbb{R}^d, o)$ and $\hat{\varphi} = \tilde{\varphi}|\Lambda$ is the induced automorphism on the lattice Λ. $\{g_{ik}\}$ is a smooth 1-cochain of local gauge transformations $g_{ik} : U_{ik} \to \mathrm{U}(n)$ on $\mathcal{U}$.

From (3.13) we see that the unitary connections $\nabla_{\mathcal{P}}$ and $\nabla_{\mathcal{P}^\vee}$ are flat along the fibers of the projection $\hat{p} : Z \to \hat{M}$ in (3.8) and induce flat partial unitary connections $\overset{\circ}{\nabla}_{\mathcal{P}}$ and $\overset{\circ}{\nabla}_{\mathcal{P}^\vee}$ on $\mathcal{P}$ and $\mathcal{P}^\vee$. Pulling E back to Z and tensoring with the dual Poincaré bundle $\mathcal{P}^\vee$, consider the flat partial connection:

$$\tilde{\nabla}_E^r = p^* \overset{\circ}{\nabla}_E \otimes \mathbf{1}_{\mathcal{P}^\vee} + \mathbf{1}_E \otimes \overset{\circ}{\nabla}_{\mathcal{P}^\vee} : p^*E \otimes \mathcal{P}^\vee \to p^*E \otimes \mathcal{P}^\vee \otimes \Omega^1_{Z/\hat{M}}. \tag{3.18}$$

Now for each $b \in B$, the pair $(E, \overset{\circ}{\nabla}_E)$ restricts to a unitary local system (E_b, ∇_{E_b}) over the fiber T_b, while the connection $\tilde{\nabla}_E^r$ restricts to the operator $\tilde{\nabla}_{E_b}^r$ induced by (2.9). Therefore,

$$\mathcal{R}^j \hat{p}_{b,*}((\ker \tilde{\nabla}_E^r)|T_b \times \hat{T}_b) \cong \mathcal{R}^j \hat{p}_{b,*}(\ker \tilde{\nabla}_{E_b}^r), \tag{3.19}$$

where $\hat{p}_b : T_b \times \hat{T}_b \to \hat{T}_b$ is the projection onto the second factor.

On the other hand, let $\iota_b, \hat{\iota}_b$ be the inclusions of T_b, $\hat{T}_b$ into M and $\hat{M}$ respectively, and consider the diagram:

$$\begin{tikzcd} T_b \times \widehat{T}_b \arrow[r, "\iota_b \times \hat{\iota}_b"] \arrow[d, "\hat{p}_b"] & Z \arrow[d, "\hat{p}"] \\ \widehat{T}_b \arrow[r, "\hat{\iota}_b"] & \widehat{M} \end{tikzcd} \tag{3.20}$$

Then the topological base change [5] yields the isomorphism (for $0 \leq j \leq d$):

$$\mathcal{R}^j \hat{p}_*(\ker \tilde{\nabla}_E^r)|\hat{T}_b \cong \mathcal{R}^j \hat{p}_{b,*}((\ker \tilde{\nabla}_E^r)|T_b \times \hat{T}_b). \tag{3.21}$$

Combining with (3.19), one obtains:

$$\mathcal{R}^j \hat{p}_*(\ker \tilde{\nabla}_E^r)|\hat{T}_b \cong \mathcal{R}^j \hat{p}_{b,*}(\ker \tilde{\nabla}_{E_b}^r). \tag{3.22}$$

It then follows from Lemma 2.1 that

$$\mathcal{R}^j \hat{p}_*(\ker \tilde{\nabla}_E^r) = 0, \quad j = 0, \dots, d-1. \tag{3.23}$$

Now we set

$$\hat{E} = \mathcal{R}^d \hat{p}_*(\ker \tilde{\nabla}^r_E), \tag{3.24}$$

with its support denoted by $\Sigma = \Sigma(E, \overset{\circ}{\nabla}_E) = \operatorname{supp} \hat{E}$. From (3.22), we have then

$$\hat{E}|\hat{T}_b \cong \mathcal{R}^d \hat{p}_{b,*}(\ker \tilde{\nabla}^r_{E_b}). \tag{3.25}$$

The following elementary lemma is useful. As it is purely local, it is valid for any foliated bundle $E \to M$.

Lemma 3.3. *For sufficiently small open sets $V \subset M$, the foliated Hermitian vector bundle $E|V$ admits $\overset{\circ}{\nabla}_E$-parallel unitary frames $s = (s_1, \dots, s_n)$, that is $\overset{\circ}{\nabla}_E s_i = 0$, $i = 1, \dots, n$. It follows that $\overset{\circ}{\nabla}_E$ is linear over the sheaf $\pi^*\mathcal{O}_B$ of basic functions and the sheaf* $\ker \overset{\circ}{\nabla}_E$ *of $\overset{\circ}{\nabla}_E$-parallel sections is locally free as a module over $\pi^*\mathcal{O}_B$, of the same rank as E.*

Proof. This can be shown easily by working in a sufficiently small Frobenius chart $V = U \times U' \subset U \times T$ over which E trivializes, choosing any unitary frame along $U \times a$, $a \in U'$ and then using parallel transport relative to the flat partial unitary connection $\overset{\circ}{\nabla}_E$ in the fiber direction U'. □

In our context, this means that $\tilde{\nabla}^r_E$ is linear with respect to $\hat{p}^*\mathcal{O}_{\hat{M}}$. In particular, the sheaf $\ker \tilde{\nabla}^r_E$ of $\tilde{\nabla}^r_E$-parallel sections in $p^*E \otimes \mathcal{P}^\vee$ is a locally free module over $\hat{p}^*\mathcal{O}_{\hat{M}}$. Further, the derived direct image $\hat{E}$ is a torsion module over $\mathcal{O}_{\hat{M}}$.

For $\Sigma = \operatorname{supp} \hat{E}$, we shall also consider the fiber product $Z_\Sigma = M \times_B \Sigma$, with $p_\Sigma : Z_\Sigma \to M$, and $\hat{p}_\Sigma : Z_\Sigma \to \Sigma$, denoting the natural projections:

$$\begin{tikzcd} & Z_\Sigma \arrow[dl, "p_\Sigma"'] \arrow[dr, "\hat{p}_\Sigma"] & \\ M \arrow[dr, "\pi"'] & & \Sigma \arrow[dl, "\hat{\pi}_\Sigma"] \\ & B & \end{tikzcd} \tag{3.26}$$

There is also the restriction to Z_Σ of the relative Poincaré line bundle $\mathcal{P}$; we denote this by $\mathcal{P}_\Sigma = \mathcal{P}|Z_\Sigma$. Let $j : \Sigma \hookrightarrow \hat{M}$ be the inclusion map, and let $\tilde{j} : Z_\Sigma \hookrightarrow Z$ be the induced inclusion.

Next we set $\mathcal{K} = \tilde{j}^*(\ker \tilde{\nabla}^r_E)$, and consider the sheaf

$$\mathcal{L} = j^*\hat{E} = \mathcal{R}^d \hat{p}_{\Sigma,*}(\mathcal{K}). \tag{3.27}$$

Proposition 3.4. *For $\hat{E}$ given by (3.24) and $\Sigma = \operatorname{supp} \hat{E}$, we have*

(1) *$\hat{E}|\hat{T}_b \in \operatorname{Ob}(\mathbf{Sky}_n(\hat{T}_b))$ for $b \in B$ and the support Σ of $\hat{E}$ is closed and transversal to all fibers $\hat{T}_b$.*

(2) *For* $\Sigma_b = \Sigma \cap \hat{T}_b = \operatorname{supp}(\hat{E}|\hat{T}_b)$, *the counting function* $|\Sigma_b|$ *satisfies* $1 \le |\Sigma_b| \le n$, $\forall b \in B$. *The sets* $U_\ell \subset B$, $\ell = 1, \dots, n$, *for which* $|\Sigma_b| \ge \ell$ *are open in B, possibly empty for* $\ell > 1$, *and satisfy* $U_n \subseteq \cdots \subseteq U_{\ell+1} \subseteq U_\ell \subseteq \cdots \subseteq U_1 = B$.

(3) *For every* $b \in B$, *there is an open neighborhood* $U \subset B$ *of* b, *such that the connected components* $\hat{\pi}_\Sigma^{-1}(U)_\xi$ of $\hat{\pi}_\Sigma^{-1}(U) \subset \Sigma$ *containing* $\xi \in \Sigma_b$ *separate the elements* $\xi \in \Sigma_b$ *and* $\hat{\pi}_\Sigma^{-1}(U)_\xi$ *can be exhausted by a finite number of smooth sections* $\sigma_i : U \to \hat{\pi}_\Sigma^{-1}(U)_\xi$, *such that* $\sigma_i(b) = \xi$. *For U sufficiently small, the number of sections needed is bounded by the rank of* $\mathcal{L}$ *at* $\xi \in \Sigma_b$.

(4) *The rank of* $\mathcal{L} \to \Sigma$ *at* $\xi \in \Sigma_b$ *is equal to the multiplicity of the irreducible representation* $\exp(\xi)$ *in the unitary local system* (E_b, ∇_{E_b}) *on* T_b, *that is the multiplicity of the trivial representation in the flat bundle* $E_b \otimes L_\xi^\vee \to T_b$.

We say that $\Sigma \hookrightarrow \hat{M}$, satisfying (1)–(3) in Lemma 3.4, is a *n-fold ramified covering of B of dimension* $m - d = \dim(B)$. A point $\xi \in \Sigma$ is called *regular* or *smooth*, if the connected component $\hat{\pi}_\Sigma^{-1}(U)_\xi$ is given by a single section $\sigma : U \cong \hat{\pi}_\Sigma^{-1}(U)_\xi$ for a sufficiently small open neighborhood U of $b = \hat{\pi}(\xi)$. The *regular* set $\Sigma_{\mathrm{reg}} \subseteq \Sigma$ is the set of regular points in Σ. Σ_{reg} is an open, dense subset of Σ, $\Sigma_{\mathrm{reg}} \hookrightarrow \hat{M}$ is a smooth submanifold and the rank of $\mathcal{L}$ is locally constant on Σ_{reg}, that is $\mathcal{L}$ is a locally free module on the connected components of Σ_{reg}. The closed, residual complement $\Sigma_{\mathrm{sing}} = \Sigma \setminus \Sigma_{\mathrm{reg}}$ is called the *branch locus* of $\hat{\pi}_\Sigma : \Sigma \to B$. We say that $\Sigma \hookrightarrow \hat{M}$ is *smooth* if the branch locus Σ_{sing} is empty. In this case we have $\Sigma_{\mathrm{reg}} = \Sigma$ and $\Sigma \hookrightarrow \hat{M}$ is a closed smooth submanifold, the rank of $\mathcal{L}$ is locally constant on Σ and the semicontinuous counting function $|\Sigma_b|$ is locally constant, hence constant on B. In particular, Σ is smooth if $U_n = B$, that is $|\Sigma_b| \equiv n$ on B, in which case $\mathcal{L}$ is a complex line bundle on Σ. If Σ is in addition connected, then Σ is a smooth n-fold covering space of B in the usual sense and we say that $\hat{\pi}_\Sigma : \Sigma \to B$ is *non-degenerate*.

Proof. Lemma 2.1 and the identification (3.22) imply that

$$\hat{E}|\hat{T}_b = \mathcal{R}^d \hat{p}_{b,*}(\ker \tilde{\nabla}^r_{E_b}) \in \mathrm{Ob}(\mathbf{Sky}_n(\hat{T}_n)). \tag{3.28}$$

Since $\Sigma_b = \operatorname{supp}(\hat{E}|\hat{T}_b)$, and $\dim H^0(\hat{T}_b, \hat{E}|\hat{T}_b) = n$, part (1) follows easily.

For $\xi \in \Sigma_b$, we have $\hat{p}^{-1}(\xi) = T_b$ and from (3.25), we see that the rank of $\mathcal{L}$ at ξ is given by the rank of the cohomology group $H^d(T_b, \ker \tilde{\nabla}^r_{E_b})$. This proves (4).

From (1), we have $1 \le |\Sigma_b| \le n$. From (4), we see that the second condition in (2) is really the semicontinuity of the number of distinct holonomy representations $\xi \in \Sigma_b$ in the bundles E_b. Thus for $b \in B$, there is a neighborhood $U_b \subset B$, such that $|\Sigma_{b'}| \ge |\Sigma_b|$, $b' \in U_b$. Then $b \in U_\ell$ implies that $U_b \subset U_\ell$ and (2) follows.

Finally, (3) is proved by using the local description of a foliated bundle in Example 3.2 and (4). In fact, the number of sections needed is equal to the number of distinct germs at b among the functions ξ_j passing through ξ in (3.16) and therefore is bounded by the rank of $\mathcal{L}$ at $\xi \in \Sigma_b$. □

In view of the above result, we say that $\hat{E}$ is a *relative skyscraper* (that is, a sheaf whose restriction $\hat{E}|\hat{T}_b$ to each fiber T_b is a skyscraper sheaf of constant finite length), $\mathcal{L}$ is the *sheaf of multiplicities* and Σ is the *spectral covering* of $(E, \overset{\circ}{\nabla}_E)$. Note that these structures are completely determined by the flat partial connection $\overset{\circ}{\nabla}_E$.

3.2. The inverse transform for relative skyscrapers

The inverse construction is considerably simpler. Our starting point is the pair $(\mathcal{S}, \Sigma)$, where $\mathcal{S}$ is a relative skyscraper of constant length n on $\hat{M}$ supported on a n-fold ramified covering $\Sigma \hookrightarrow \hat{M}$ of B of dimension $m - d = \dim(B)$.

Using the same notation as before, recall that the fiber product $Z_\Sigma = M \times_B \Sigma$, is of dimension m, and $p_\Sigma : Z_\Sigma \to M$ is an n-fold ramified covering map. Thus it is easy to see that

$$\check{\mathcal{S}} = p_{\Sigma,*}(\hat{p}^*_\Sigma \mathcal{S} \otimes \mathcal{P}_\Sigma), \tag{3.29}$$

is a locally free sheaf of rank n on M. Furthermore, the construction reveals that $\check{\mathcal{S}}$ carries a canonical flat partial connection $\overset{\circ}{\nabla}_{\check{\mathcal{S}}}$. In fact, $\hat{p}^*_\Sigma \mathcal{S}$ carries a canonical flat partial connection relative to $\hat{p}_\Sigma : Z_\Sigma \to \hat{M}$ and so does $\mathcal{P}_\Sigma$.

3.3. The main result

Motivated by the results above, let us introduce the following categories of sheaves with connections on M and $\hat{M}$.

Definition 3.5. $\mathbf{Vect}_n^{\overset{\circ}{\nabla}}(M)$ is the category of foliated Hermitian vector bundles on M endowed with a flat partial unitary connection. Objects in $\mathbf{Vect}_n^{\overset{\circ}{\nabla}}(M)$ are pairs $(E, \overset{\circ}{\nabla}_E)$ consisting of a Hermitian vector bundle E of rank n and a flat partial unitary connection $\overset{\circ}{\nabla}_E$. Morphisms are bundle maps compatible with such connections.

Definition 3.6. $\mathbf{RelSky}_n(\hat{M})$ is the category of *relative skyscrapers* on $\hat{M}$. Objects in $\mathbf{RelSky}_n(\hat{M})$ are pairs $(\mathcal{S}, \Sigma)$ consisting of a relative skyscraper $\mathcal{S}$ of constant length n on $\hat{M}$, supported on a n-fold ramified covering $\Sigma \hookrightarrow \hat{M}$ of B of dimension $m - d = \dim(B)$. Morphisms are sheaf maps of $\mathcal{O}_{\hat{M}}$-modules.

The constructions in Sections 3.1 and 3.2 define additive covariant functors

$$\mathbf{F} : \mathbf{Vect}_n^{\overset{\circ}{\nabla}}(M) \to \mathbf{RelSky}_n(\hat{M}), \qquad \hat{\mathbf{F}} : \mathbf{RelSky}_n(\hat{M}) \to \mathbf{Vect}_n^{\overset{\circ}{\nabla}}(M). \tag{3.30}$$

For limits in the appropriate sense, let

$$\mathbf{Vect}^{\overset{\circ}{\nabla}}(M) = \lim_n \mathbf{Vect}_n^{\overset{\circ}{\nabla}}(M) \quad \text{and} \quad \mathbf{RelSky}(\hat{M}) = \lim_n \mathbf{RelSky}_n(\hat{M}). \tag{3.31}$$

With these definitions in place, we can state our main result.

Theorem 3.7. *The Fourier–Mukai transform* $\mathbf{F}$ *defines an additive natural equivalence of categories*

$$\mathbf{F} : \mathbf{Vect}^{\overset{\circ}{\nabla}}(M) \xrightarrow{\cong} \mathbf{RelSky}(\hat{M}). \tag{3.32}$$

Proof. We claim that $\mathbf{F}$ and $\hat{\mathbf{F}}$ are adjoint functors which in fact define an equivalence of categories. From the construction of $\mathbf{F}$ and $\hat{\mathbf{F}}$, there exist natural transformations

$$\phi_{\mathcal{S}} : \mathcal{S} \to \mathbf{F} \circ \hat{\mathbf{F}}(\mathcal{S}), \quad \mathcal{S} \in \mathrm{Ob}(\mathbf{RelSky}_n(\hat{M})), \tag{3.33}$$

and

$$\psi_E : \hat{\mathbf{F}} \circ \mathbf{F}(E) \to E, \quad E \in \mathrm{Ob}(\mathbf{Vect}_n^{\overset{\circ}{\nabla}}(M)). \tag{3.34}$$

These natural transformations define adjunction maps

$$\begin{aligned} &\Phi : \mathrm{Morph}_{\mathbf{V}}(\hat{\mathbf{F}}(\mathcal{S}), E) \to \mathrm{Morph}_{\mathbf{R}}(\mathcal{S}, \mathbf{F}(E)), \\ &\Psi : \mathrm{Morph}_{\mathbf{R}}(\mathcal{S}, \mathbf{F}(E)) \to \mathrm{Morph}_{\mathbf{V}}(\hat{\mathbf{F}}(\mathcal{S}), E), \end{aligned} \tag{3.35}$$

where $\mathrm{Morph}_{\mathbf{V}}$ and $\mathrm{Morph}_{\mathbf{R}}$ denote morphisms in $\mathbf{Vect}_n^{\overset{\circ}{\nabla}}(M)$ and $\mathbf{RelSky}_n(\hat{M})$, respectively. Explicitly, for $f : \hat{\mathbf{F}}(\mathcal{S}) \to E$, we have by naturality

$$\Phi(f) = \mathbf{F}(f) \circ \phi_{\mathcal{S}}, \tag{3.36}$$

so that $\phi_{\mathcal{S}}$ determines Φ. Likewise, for $g : \mathcal{S} \to \mathbf{F}(E)$, we have by naturality

$$\Psi(g) = \psi_E \circ \hat{\mathbf{F}}(g), \tag{3.37}$$

so that ψ_E determines Ψ as well. The natural transformations (3.33), (3.34) correspond then to $\phi_{\mathcal{S}} = \Phi(\mathbf{1}_{\hat{\mathbf{F}}(\mathcal{S})})$ and $\psi_E = \Psi(\mathbf{1}_{\mathbf{F}(E)})$, respectively. The fact that the adjunction maps Φ and Ψ are inverses of each other, is equivalent to the compositions

$$\begin{aligned} &\mathbf{F}(E) \xrightarrow{\phi_{\mathbf{F}(E)}} \mathbf{F} \circ \hat{\mathbf{F}} \circ (\mathbf{F}(E)) = \mathbf{F} \circ (\hat{\mathbf{F}} \circ \mathbf{F}(E)) \xrightarrow{\mathbf{F}(\psi_E)} \mathbf{F}(E), \\ &\hat{\mathbf{F}}(\mathcal{S}) \xrightarrow{\hat{\mathbf{F}}(\phi_{\mathcal{S}})} \hat{\mathbf{F}} \circ (\mathbf{F} \circ \hat{\mathbf{F}}(\mathcal{S})) = \hat{\mathbf{F}} \circ \mathbf{F} \circ (\hat{\mathbf{F}}(\mathcal{S})) \xrightarrow{\psi_{\hat{\mathbf{F}}(\mathcal{S})}} \hat{\mathbf{F}}(\mathcal{S}), \end{aligned} \tag{3.38}$$

resulting in the identities of $\mathbf{F}(E)$ and $\hat{\mathbf{F}}(\mathcal{S})$, respectively.

The construction has the further property that it is compatible with localization relative to open subsets $U \subset B$, that is, the restrictions to $\pi^{-1}(U)$ and $\hat{\pi}^{-1}(U)$. Moreover, we observe that the restriction of $\mathbf{F}$ and $\hat{\mathbf{F}}$ to the fibers of M and $\hat{M}$ at $b \in B$ respectively, coincides with the functors

$$\mathbf{F}_b : \mathbf{Loc}(T_b) \to \mathbf{Sky}(\hat{T}_b), \qquad \hat{\mathbf{F}}_b : \mathbf{Sky}(\hat{T}_b) \to \mathbf{Loc}(T_b), \tag{3.39}$$

for each $b \in B$. It follows from [3,10] that $\phi_{\mathcal{S}_b} : \mathbf{1}_{\hat{T}_b} \cong \mathbf{F}_b \circ \hat{\mathbf{F}}_b$ and $\psi_{E_b} : \hat{\mathbf{F}}_b \circ \mathbf{F}_b \cong \mathbf{1}_{T_b}$. From this we conclude that $\phi_{\mathcal{S}}$ and ψ_E are indeed isomorphisms. □

Let $V \in \mathbf{Vect}_n(B)$, where $\mathbf{Vect}_n(B)$ is the category of complex vector bundles of rank n over B. Then $\pi^* V$ carries a canonical flat partial connection $\overset{\circ}{\nabla}_{\pi^* V}$, so that $(\pi^* V, \overset{\circ}{\nabla}_{\pi^* V})$ is an object in $\mathbf{Vect}_n^{\overset{\circ}{\nabla}}(M)$, while $\hat{\pi}_0^* V = \hat{\pi}_{\Sigma_0}^* V$ is an object in $\mathbf{RelSky}_n(\hat{M})$, supported on the 0-section $\Sigma_0 = \sigma_0(B) \subset \hat{M}$. The construction of $\mathbf{F}$ is compatible with these pull-backs, that is we have a commutative diagram

$$\begin{array}{ccc} \mathbf{Vect}_n^{\overset{\circ}{\nabla}}(M) & \xrightarrow{\mathbf{F}} & \mathbf{RelSky}_n(\widehat{M}) \\ {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \hat{\pi}_0^*}\uparrow \\ \mathbf{Vect}_n(B) & \xrightarrow{=} & \mathbf{Vect}_n(B). \end{array} \tag{3.40}$$

Moreover, the Fourier–Mukai transform $\mathbf{F}$ has a module property with respect to $\mathbf{Vect}^{\nabla}(B)$.

Corollary 3.8. *For $(E, \overset{\circ}{\nabla}_E) \in \mathbf{Vect}^{\overset{\circ}{\nabla}}(M)$ and $V \in \mathbf{Vect}(B)$, the Fourier–Mukai transform $\mathbf{F}$ satisfies*

$$\mathbf{F}((\pi^* V, \overset{\circ}{\nabla}_{\pi^* V}) \otimes (E, \overset{\circ}{\nabla}_E)) \cong \hat{\pi}_\Sigma^* V \otimes \mathbf{F}(E, \overset{\circ}{\nabla}_E), \tag{3.41}$$

where Σ is the support of $\mathbf{F}(E, \overset{\circ}{\nabla}_E)$.

4. The Fourier–Mukai transform for vector bundles with Poincaré basic connections

4.1. Transforming bundles with Poincaré basic connections

Let $E \to M$ be a foliated Hermitian vector bundle of rank n, and let $\nabla_E : E \to E \otimes \Omega_M^1$ be a unitary connection on E. We say that ∇_E is *adapted* to the foliated structure on E, if ∇_E induces the flat partial connection $\overset{\circ}{\nabla}_E : E \to E \otimes \Omega_{M/B}^1$ via the canonical map $\Omega_M^1 \to \Omega_{M/B}^1$ in (3.6). The existence of adapted connections follows from an elementary partition of unity argument.

At this point, it is also useful to introduce the bigrading on the DeRham algebra Ω_M^* determined by a splitting of the exact sequence (3.5), respectively (3.6):

$$\Omega_M^{u,v} = \Omega_M^{u,0} \otimes \Omega_M^{0,v} = \pi^* \Omega_B^u \otimes \Omega_{M/B}^v. \tag{4.1}$$

u is called the *transversal or basic* degree and v is called the *fiber* degree.

Consider now the adapted connection

$$\tilde{\nabla}_E = p^* \nabla_E \otimes \mathbf{1}_{\mathcal{P}^\vee} + \mathbf{1}_E \otimes \nabla_{\mathcal{P}^\vee} : p^* E \otimes \mathcal{P}^\vee \to p^* E \otimes \mathcal{P}^\vee \otimes \Omega_Z^1, \tag{4.2}$$

on $p^* E \otimes \mathcal{P}^\vee$. With respect to a corresponding splitting of (3.9), we have $\tilde{\nabla}_E = \nabla_E^r \oplus \nabla_E^t$, where

$$\nabla_E^r = (\mathbf{1}_{E \otimes \mathcal{P}^\vee} \otimes r) \circ \tilde{\nabla}_E : p^* E \otimes \mathcal{P}^\vee \to p^* E \otimes \mathcal{P}^\vee \otimes \Omega_{Z/\hat{M}}^1, \tag{4.3}$$

is the relative connection, and

$$\nabla_E^t = (\mathbf{1}_{E \otimes \mathcal{P}^\vee} \otimes t) \circ \tilde{\nabla}_E : p^* E \otimes \mathcal{P}^\vee \to p^* E \otimes \mathcal{P}^\vee \otimes \hat{p}^* \Omega_{\hat{M}}^1, \tag{4.4}$$

is the transversal connection, that is the components of type $(0, 1)$ and $(1, 0)$ of ∇_E respectively.

In the sequel, we always view the curvature ∇_E^2 of ∇_E as a 2-form with values in the adjoint bundle $\mathrm{End}_s(E)$ of skew-hermitian endomorphisms of E.

Lemma 4.1. *The type-decomposition of the curvature* $\tilde{\nabla}_E^2$ *is given by*

$$(\tilde{\nabla}_E^2)^{0,2} = p^*((\nabla_E^2)^{0,2}) \otimes \mathbf{1}_{\mathcal{P}^\vee} = 0, \tag{4.5}$$

$$(\tilde{\nabla}_E^2)^{2,0} = p^*((\nabla_E^2)^{2,0}) \otimes \mathbf{1}_{\mathcal{P}^\vee} = (\nabla_E^t)^2, \tag{4.6}$$

and

$$(\tilde{\nabla}_E^2)^{1,1} = p^*((\nabla_E^2)^{1,1}) \otimes \mathbf{1}_{\mathcal{P}^\vee} - \mathbf{1}_E \otimes \mathbb{F} = \varXi, \tag{4.7}$$

where the operator $\varXi$ *is given by the commutator*

$$\varXi = \nabla_E^r \circ \nabla_E^t - \nabla_E^t \circ \nabla_E^r : p^*E \otimes \mathcal{P}^\vee \to p^*E \otimes \mathcal{P}^\vee \otimes \Omega_Z^{1,1}. \tag{4.8}$$

Henceforth we adopt the usual sign rule which equips the extension of the transversal operator ∇_E^t to forms of higher degree with a sign $(-1)^v$ on forms of type (u, v).

Proof. Firstly from (4.2) and the decomposition (4.3), (4.4) we have

$$\tilde{\nabla}_E = p^*\nabla_E \otimes \mathbf{1}_{\mathcal{P}^\vee} + \mathbf{1}_E \otimes \nabla_{\mathcal{P}^\vee} = \nabla_E^r + \nabla_E^t. \tag{4.9}$$

Computing the curvature operator $\tilde{\nabla}_E^2$ in two ways, we obtain

$$\begin{aligned} \tilde{\nabla}_E^2 &= (p^*\nabla_E)^2 \otimes \mathbf{1}_{\mathcal{P}^\vee} + \mathbf{1}_E \otimes \nabla_{\mathcal{P}^\vee}^2 = p^*(\nabla_E^2) \otimes \mathbf{1}_{\mathcal{P}^\vee} - \mathbf{1}_E \otimes \mathbb{F} = (\nabla_E^r \pm \nabla_E^t)^2 \\ &= (\nabla_E^r)^2 + (\nabla_E^t)^2 + \varXi. \end{aligned} \tag{4.10}$$

Since ∇_E is adapted to the foliated structure $\overset{\circ}{\nabla}_E$ on E, we have $(\nabla_E^2)^{0,2} = 0$. Since $\tilde{\nabla}_E$ is adapted to the foliated structure on $p^*E \otimes \mathcal{P}^\vee$ relative to $\hat{p} : Z \to \hat{M}$, we have $(\tilde{\nabla}_E^2)^{0,2} = (\nabla_E^r)^2 = 0$. The curvature $\mathbb{F}$ of the relative Poincaré bundle $\mathcal{P}$ is of type $(1, 1)$ by (3.13) and $(\nabla_E^t)^2$ and $\varXi$ are of type $(2, 0)$ and $(1, 1)$ respectively by definition. Thus the assertions (4.5)–(4.7) follow from (4.10). We use (3.11), to conclude that the pull-back p^* preserves the curvature types. □

We need to recall a few facts about *basic* connections in the foliated Hermitian vector bundle $(E, \overset{\circ}{\nabla}_E)$ [8]. Note that all the statements below are of local nature.

Lemma 4.2. *For any adapted connection* ∇_E, *the following conditions are equivalent*:

(1) *The contraction* $i_X\nabla_E^2 = 0$, *for all vector fields* X *in* $T(\pi)$;
(2) *The mixed component* $(\nabla_E^2)^{1,1}$ *of* ∇_E *vanishes*;
(3) *The curvature* ∇_E^2 *coincides with the basic component* $(\nabla_E^2)^{2,0}$, *that is* $\nabla_E^2 = (\nabla_E^2)^{2,0}$;
(4) *For any* π*-projectable transversal vector field* $\tilde{Y}$, *the operator* $\nabla_{\tilde{Y}}$ *preserves the sheaf* $\ker \overset{\circ}{\nabla}_E$ *and depends only on* $Y = \pi_*\tilde{Y}$.

The following condition is a consequence of the above properties:

(5) *For* π*-projectable transversal vector fields* $\tilde{Y}, \tilde{Y}'$, *the curvature* $\nabla_E^2(\tilde{Y}, \tilde{Y}')$ *preserves* $\ker \overset{\circ}{\nabla}_E$ *and depends only on* $Y = \pi_*\tilde{Y}$, $Y' = \pi_*\tilde{Y}'$.

Proof. Since $(\nabla_E^2)^{0,2} = (\overset{\circ}{\nabla}_E)^2 = 0$, the equivalence of (1)–(3) is immediate, so we elaborate only on conditions (4) and (5). The mixed component $(\nabla_E^2)^{1,1}$ is characterized by the formula

$$(\nabla_E^2)^{1,1}(X, Y)(s) = \nabla_E^2(X, \tilde{Y})(s) = \overset{\circ}{\nabla}_X(\nabla_{\tilde{Y}} s) - \nabla_{\tilde{Y}}(\overset{\circ}{\nabla}_X s) - \overset{\circ}{\nabla}_{[X,\tilde{Y}]} s, \tag{4.11}$$

for vector fields X in $T(\pi)$ and π-projectable transversal vector fields $\tilde{Y}$. Thus for $s \in \ker \overset{\circ}{\nabla}$, we have from (2)

$$(\nabla_E^2)^{1,1}(X, Y)(s) = \overset{\circ}{\nabla}_X(\nabla_{\tilde{Y}} s) = \overset{\circ}{\nabla}_X(\nabla_Y s) \equiv 0, \tag{4.12}$$

since $\pi_*[X, \tilde{Y}] = [\pi_* X, \pi_* \tilde{Y}] = [0, Y] = 0$. The implication (4) $\Rightarrow$ (2) follows from (4.12), using Lemma 3.3.

Likewise, the vector fields $\tilde{Y}, \tilde{Y}'$ satisfy $\pi_*[\tilde{Y}, \tilde{Y}'] = [\pi_* \tilde{Y}, \pi_* \tilde{Y}'] = [Y, Y']$ and we have from (4) for $s \in \ker \overset{\circ}{\nabla}$:

$$\begin{aligned}(\nabla_E^2)^{2,0}(Y, Y')(s) &= \nabla_E^2(\tilde{Y}, \tilde{Y}')(s) = \nabla_{\tilde{Y}}(\nabla_{\tilde{Y}'} s) - \nabla_{\tilde{Y}'}(\nabla_{\tilde{Y}} s) - \nabla_{[\tilde{Y},\tilde{Y}']} s \\ &= \nabla_Y(\nabla_{Y'} s) - \nabla_{Y'}(\nabla_Y s) - \nabla_{[Y,Y']} s.\end{aligned} \tag{4.13}$$

Thus (4) implies (5). □

We say that ∇_E is a *basic* connection, if any of the equivalent conditions in Lemma 4.2 holds. In general, a foliated vector bundle $(E, \overset{\circ}{\nabla}_E)$ does not admit basic connections. In our context, the following example describes essentially the class of foliated bundles which do admit basic connections.

Example 4.3. Locally trivial families of flat bundles: as in Example 3.2, we view a foliated vector bundle $(E, \overset{\circ}{\nabla}_E)$ as a family of flat bundles on the fibers T_b, parametrized by $b \in B$. We say that this family is *locally trivial* if there exists a flat bundle (E_0, ∇_0) on the torus T, determined by a holonomy homomorphism $\xi = (\xi_1, \dots, \xi_n) \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathrm{U}(n))$, and a (good) open cover $\mathcal{U}$ of B such that for every $U \in \mathcal{U}$, there are isomorphisms of foliated vector bundles as indicated in the following diagram, similar to (3.15).

$$\begin{array}{ccccc} U \times (\mathbb{R}^d \times_\Lambda \mathbb{C}^n) & \xrightarrow{\cong} & U \times E_0 & \xrightarrow{\cong} & E \mid \pi^{-1}(U) \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \tau'} & & \Big\downarrow{\scriptstyle \mathrm{id} \times \tau_0} & & \Big\downarrow{\scriptstyle \tau|\pi^{-1}U} \\ U \times T & \xrightarrow{=} & U \times T & \xrightarrow{\cong} & \pi^{-1}(U)\ . \end{array} \tag{4.14}$$

On overlaps U_{ik} in $\mathcal{U}$, the coordinate changes on the LHS are given by (3.17), except that now the holonomy homomorphisms ξ^i are independent of $b \in U_i$.

In the case of locally trivial families, much more can be said about the spectral covering Σ of $(E, \overset{\circ}{\nabla}_E)$. In short, we claim that Σ is a finite union of *leaves*, that is maximal integral manifolds, of the transverse foliation $\hat{\mathcal{F}}$ on $\hat{\pi} : \hat{M} \to B$ determined by (3.7), the leaves of $\hat{\mathcal{F}}$ being holonomy coverings over B. Locally over U, $\Sigma_U = \hat{\pi}_\Sigma^{-1}(U)$ is given by a finite number of constant sections of the corresponding trivialization $U \times \hat{T} \to U$ of $\hat{M} \to B$ and we have $\Sigma_{\mathrm{reg}} = \Sigma$, that is all points of Σ are regular and the branch locus is empty.

Therefore $\Sigma \hookrightarrow \hat{M}$ is a smooth submanifold and the rank of the multiplicity sheaf $\mathcal{L}$ is locally constant on Σ, hence constant on the connected components of Σ. The above local properties of Σ_U imply that the connected components of the spectral covering Σ are integral manifolds of the transverse foliation $\hat{\mathcal{F}}$. The structure of Σ_U shows also that $\hat{\pi}_\Sigma : \Sigma \to B$ satisfies the unique path-lifting property. Therefore any path in the leaf $\hat{\mathcal{F}}_\xi$ through $\xi \in \Sigma$, starting at ξ must already be in the connected component of Σ containing ξ. Thus the connected components of Σ are maximal integral manifolds of $\hat{\mathcal{F}}$. These leaves are closed in $\hat{M}$, since they intersect the complete transversals $\hat{T}_b$ in $\leq n$ points. More precisely, the sets $\Sigma_b \subset \hat{T}_b$ are invariant under the action of $\pi_1(B, b)$ on $\hat{T}_b$ determined by $\rho^* : \pi_1(B) \to \mathrm{Aut}(\hat{T}) \cong \mathrm{Aut}(\Lambda)$, with the orbits and their multiplicities corresponding to the component leaves of Σ and the rank of $\mathcal{L}$ on these components respectively. This allows us to decompose $(\mathcal{L}, \Sigma)$, respectively $(E, \overset{\circ}{\nabla}_E)$ according to its leaf components.

We abbreviate the above properties of Σ by saying that $\Sigma \hookrightarrow \hat{M}$ is *locally constant*. From the local formula (3.13) for the curvature $\mathbb{F}$ of the Poincaré bundle $\mathcal{P}$, we see that $\mathbb{F}|Z_{\Sigma_U} = 0$. Obviously, ∇_0 extends to a basic, in fact a flat connection ∇_U on $U \times E_0$ and these connections can be patched together to a basic connection ∇_E on E via a partition of unity on B subordinate to the cover $\mathcal{U}$. A special case of locally trivial families of flat bundles on the fibers is of course given by flat bundles (E, ∇_E) on the total space M (compare Section 5.1), in which case the bundle is determined by a global holonomy homomorphism $\tilde{\rho} : \pi_1(M) \to \mathrm{U}(n)$, so that the representation on Λ is determined by restriction, that is by the diagram

$$\begin{array}{ccc} \Lambda = \pi_1(T) & \xrightarrow{\xi} & \mathrm{U}(1)^n \\ \Big\downarrow{\scriptstyle \iota_*} & & \Big\downarrow{\scriptstyle i} \\ \pi_1(M) & \xrightarrow{\tilde{\rho}} & \mathrm{U}(n) \end{array} \quad . \tag{4.15}$$

In order to understand the interplay between the obstruction for the existence of a basic connection and the behavior of the curvature term $\mathbb{F}|Z_\Sigma$, we next look at the case of foliated complex line bundles.

Example 4.4. Suppose that $(E, \overset{\circ}{\nabla}_E)$ is a foliated complex line bundle on M. In this case, the spectral covering $\Sigma \hookrightarrow \hat{M}$ is a section σ of $\hat{M} \to B$, the multiplicity sheaf $\mathcal{L}$ on Σ is a sheaf of rank 1 and $p_\Sigma : Z_\Sigma \to M$ is a diffeomorphism. Thus by Theorem 3.7 we have the following.

$$p_\Sigma^* E \cong \hat{p}_\Sigma^* \mathcal{L} \otimes \mathcal{P}_\Sigma. \tag{4.16}$$

It follows that the connection $\nabla_{\mathcal{P}}$ and any connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$ induce an adapted connection ∇_E on E, such that $p_\Sigma^*((\nabla_E^2)^{1,1}) = \mathbb{F}|Z_\Sigma$ (compare Section 4.2). This connection ∇_E is basic, if (and only if) the spectral section σ is locally constant, that is $(E, \overset{\circ}{\nabla}_E)$ is a locally trivial family of flat line bundles. This follows from Example 4.3.

For foliated line bundles, the functor $\mathbf{F} : E \mapsto \mathcal{L}$ has the following multiplicative property. Given $E = E_1 \otimes E_2$, the spectral sections are related by $\sigma = \sigma_1 + \sigma_2$ and we denote by

$p_i : \Sigma \to \Sigma_i$ the canonical projection. Then a direct calculation from $p_\Sigma^* E \cong \hat{p}_\Sigma^* \mathcal{L} \otimes \mathcal{P}_\Sigma$, $p_{\Sigma_i}^* E_i \cong \hat{p}_{\Sigma_i}^* \mathcal{L}_i \otimes \mathcal{P}_{\Sigma_i}$ yields the product formula on Σ, respectively Z_Σ

$$\mathcal{L} \cong p_1^* \mathcal{L}_1 \otimes p_2^* \mathcal{L}_2, \qquad \mathcal{P}_\Sigma \cong (\mathbf{1} \times p_1)^* \mathcal{P}_{\Sigma_1} \otimes (\mathbf{1} \times p_2)^* \mathcal{P}_{\Sigma_2}. \tag{4.17}$$

These examples motivate the following definition.

Definition 4.5. The adapted connection ∇_E is *Poincaré basic* if the $\hat{p}_\Sigma$-adapted connection $\tilde{\nabla}_E$ on $p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee$ on Z_Σ is basic, that is from (4.7) in Lemma 4.1

$$\tilde{j}^* (\tilde{\nabla}_E^2)^{1,1} = p_\Sigma^* (\nabla_E^2)^{1,1} - \mathbb{F} | Z_\Sigma = 0. \tag{4.18}$$

Here we view the scalar form $\mathbb{F}$ as a form with values in the center of $p_\Sigma^* \operatorname{End}_s(E)$, using the canonical isomorphism of foliated bundles $\operatorname{End}_s(p^* E \otimes \mathcal{P}^\vee) \cong p^* \operatorname{End}_s(E)$.

For our purposes, it is actually sufficient that the curvature $\tilde{j}^* (\tilde{\nabla}_E^2)^{1,1}$ vanishes on the subsheaf $\mathcal{K} = \tilde{j}^* (\ker \nabla_E^r)$ defined earlier, that is we have $\tilde{j}^* (\tilde{\nabla}_E^2)^{1,1} | \mathcal{K} = 0$ or equivalently

$$p_\Sigma^* (\nabla_E^2)^{1,1} | \mathcal{K} = \mathbb{F} | Z_\Sigma. \tag{4.19}$$

This corresponds to the equivalent condition (4) in Lemma 4.2 applied to the connection $\tilde{\nabla}_E$.

We now proceed to construct a connection $\nabla_\mathcal{L} : \mathcal{L} \to \mathcal{L} \otimes \Omega_\Sigma^1$, given a Poincaré basic connection ∇_E on $E \to M$. From (4.7) we see that

$$\Xi | Z_\Sigma = 0 : p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \to p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \otimes \Omega_{Z_\Sigma}^{1,1}. \tag{4.20}$$

Therefore the diagram below with exact rows is commutative (up to sign)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \arrow[r, "\tilde{j}^* \nabla_E^r"] \arrow[d, "\tilde{j}^* \nabla_E^t"] & p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \otimes \Omega^1_{Z_\Sigma/\Sigma} \arrow[r, "\tilde{j}^* \nabla_E^r"] \arrow[d, "\tilde{j}^* \nabla_E^t"] & p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \otimes \Omega^{0,2}_{Z_\Sigma} \arrow[d, "\tilde{j}^* \nabla_E^t"] \\
0 \arrow[r] & \mathcal{K} \otimes \hat{p}_\Sigma^* \Omega_\Sigma^1 \arrow[r] \arrow[d] & p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \otimes \hat{p}_\Sigma^* \Omega_\Sigma^1 \arrow[r, "\tilde{j}^* \nabla_E^r"] \arrow[d, "\tilde{j}^* \nabla_E^t"] & p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \otimes \Omega^{1,1}_{Z_\Sigma} \arrow[r, "\tilde{j}^* \nabla_E^r"] \arrow[d, "\tilde{j}^* \nabla_E^t"] & p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \otimes \Omega^{1,2}_{Z_\Sigma} \arrow[d, "\tilde{j}^* \nabla_E^t"] \\
0 \arrow[r] & \mathcal{K} \otimes \hat{p}_\Sigma^* \Omega_\Sigma^2 \arrow[r] & p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \otimes \hat{p}_\Sigma^* \Omega_\Sigma^2 \arrow[r, "\tilde{j}^* \nabla_E^r"] & p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \otimes \Omega^{2,1}_{Z_\Sigma} \arrow[r, "\tilde{j}^* \nabla_E^r"] & p_\Sigma^* E \otimes \mathcal{P}_\Sigma^\vee \otimes \Omega^{2,2}_{Z_\Sigma}
\end{tikzcd} \tag{4.21}$$

and the restriction of $\tilde{j}^* \nabla_E^t$ to the subsheaf $\mathcal{K}$ induces a connection

$$\nabla_E^{\ker} : \mathcal{K} \to \mathcal{K} \otimes \hat{p}_\Sigma^* \Omega_\Sigma^1. \tag{4.22}$$

Alternatively, we may use the condition (4.19) to arrive at the same conclusion. Recalling that $\mathcal{L} = \mathcal{R}^d \hat{p}_{\Sigma,*}(\mathcal{K})$ and using the projection formula, this leads to a connection

$$\nabla_\mathcal{L} = \mathcal{R}^d \hat{p}_{\Sigma,*}(\nabla_E^{\ker}) : \mathcal{L} \to \mathcal{L} \otimes \Omega_\Sigma^1. \tag{4.23}$$

For later reference, we compute the curvature of the transformed connection $\nabla_\mathcal{L}$.

Lemma 4.6. *The curvature of the connection* $\nabla_{\mathcal{L}}$ *is given by*

$$\nabla_{\mathcal{L}}^2 = \mathcal{R}^d \hat{p}_{\Sigma,*}(\nabla_E^{\ker})^2 = \mathcal{R}^d \hat{p}_{\Sigma,*}(p_\Sigma^*(\nabla_E^2)^{2,0}|\mathcal{K}). \tag{4.24}$$

Proof. Since $\tilde{j}^*(\tilde{\nabla}_E^2)^{0,2} = \tilde{j}^*(\tilde{\nabla}_E^2)^{1,1} = 0$ by assumption, the curvature term $\tilde{j}^*\tilde{\nabla}_E^2 = \tilde{j}^*(\tilde{\nabla}_E^2)^{2,0} = p_\Sigma^*(\nabla_E^2)^{2,0}$ leaves the sheaf $\mathcal{K} \to Z_\Sigma$ invariant by Lemma 4.2, (5) and we have from (4.6)

$$\begin{aligned}\nabla_{\mathcal{L}}^2 &= \mathcal{R}^d \hat{p}_{\Sigma,*}(\nabla_E^{\ker})^2 = \mathcal{R}^d \hat{p}_{\Sigma,*}(\tilde{j}^*\nabla_E^t|\mathcal{K})^2 = \mathcal{R}^d \hat{p}_{\Sigma,*}(\tilde{j}^*(\nabla_E^t)^2|\mathcal{K})\\ &= \mathcal{R}^d \hat{p}_{\Sigma,*}(\tilde{j}^*(p^*(\nabla_E^2)^{2,0})|\mathcal{K}) = \mathcal{R}^d \hat{p}_{\Sigma,*}(p_\Sigma^*(\nabla_E^2)^{2,0}|\mathcal{K}).\end{aligned}$$

In this calculation we also used diagram (4.21). □

Recall that the pair (E, ∇_E) is said to be *reducible* if there are bundles with connections (E_1, ∇_{E_1}) and (E_2, ∇_{E_2}) such that $E = E_1 \oplus E_2$ and $\nabla_E = \nabla_{E_1} \oplus \nabla_{E_2}$. The pair (E, ∇_E) is said to be *irreducible* if it is not reducible.

Lemma 4.7. *If* $(E, \nabla_E) = (E_1, \nabla_{E_1}) \oplus (E_2, \nabla_{E_2})$, *then*

$$(\mathcal{L}, \nabla_{\mathcal{L}}, \Sigma) = (\mathcal{L}_1, \nabla_{\mathcal{L}_1}, \Sigma_1) \oplus (\mathcal{L}_2, \nabla_{\mathcal{L}_2}, \Sigma_2), \tag{4.25}$$

where $\Sigma = \Sigma_1 \cup \Sigma_2$.

Proof. The statement is clear from the definitions. Indeed, we have $\ker \nabla_E = \ker \nabla_{E_1} \oplus \ker \nabla_{E_2}$, so that $\hat{E} = \hat{E}_1 \oplus \hat{E}_2$ by (3.24) and therefore $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$. Since $\nabla_E^{\ker}$ also splits as a direct sum, it follows from (4.23) that $\nabla_{\mathcal{L}} = \nabla_{\mathcal{L}_1} \oplus \nabla_{\mathcal{L}_2}$. As for supports, we note that $\Sigma = \Sigma(E, \overset{\circ}{\nabla}_E) = \Sigma_1 \cup \Sigma_2$, where $\Sigma_i = \Sigma(E_i, \overset{\circ}{\nabla}_{E_i})$. □

Definition 4.8. The triple $(\mathcal{L}, \nabla_{\mathcal{L}}, \Sigma)$ is called the *Fourier–Mukai transform* of (E, ∇_E), and $\Sigma = \Sigma(E, \overset{\circ}{\nabla}_E) = \operatorname{supp} \hat{E}$ is called the *spectral covering* associated with the underlying foliated structure $(E, \overset{\circ}{\nabla}_E)$.

4.2. The inverse transform for connections

Given $(\mathcal{S}, \Sigma)$ as in Section 3.2, let $\nabla_{\mathcal{S}}$ be a connection on $\mathcal{S}$. In order to obtain a connection on $\hat{\mathbf{F}}(\mathcal{S}) = \check{\mathcal{S}}$, recall from (3.29) that $\check{\mathcal{S}}$ is defined by $\check{\mathcal{S}} = p_{\Sigma,*}(\hat{p}_\Sigma^*\mathcal{S} \otimes \mathcal{P}_\Sigma)$. Consider the connection

$$\tilde{\nabla}_{\mathcal{S}} = \hat{p}_\Sigma^*\nabla_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{P}_\Sigma} + \mathbf{1}_{\mathcal{S}} \otimes \nabla_{\mathcal{P}_\Sigma} : \hat{p}_\Sigma^*\mathcal{S} \otimes \mathcal{P}_\Sigma \to \hat{p}_\Sigma^*\mathcal{S} \otimes \mathcal{P}_\Sigma \otimes \Omega^1_{Z_\Sigma}. \tag{4.26}$$

Since $\Omega^1_{Z_\Sigma} = p_\Sigma^*\Omega^1_M$, it follows from the projection formula that

$$p_{\Sigma,*}(\hat{p}_\Sigma^*\mathcal{S} \otimes \mathcal{P}_\Sigma \otimes \Omega^1_{Z_\Sigma}) = \check{\mathcal{S}} \otimes \Omega^1_M. \tag{4.27}$$

Thus we define the connection $\nabla_{\check{\mathcal{S}}}$ on $\check{\mathcal{S}}$, adapted to the flat partial connection $\overset{\circ}{\nabla}_{\check{\mathcal{S}}}$ in Section 3.2, by

$$\nabla_{\check{\mathcal{S}}} = p_{\Sigma,*}\tilde{\nabla}_{\mathcal{S}} : \check{\mathcal{S}} \to \check{\mathcal{S}} \otimes \Omega^1_M. \tag{4.28}$$

The curvature $\nabla^2_{\check{\mathcal{S}}}$ of $\nabla_{\check{\mathcal{S}}}$ is computed next, from the formula

$$\tilde{\nabla}^2_{\mathcal{S}} = \hat{p}^*_{\Sigma}\nabla^2_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{P}_\Sigma} + \mathbf{1}_{\mathcal{S}} \otimes \mathbb{F}|Z_{\Sigma}. \tag{4.29}$$

Lemma 4.9. *The curvature* $\nabla^2_{\check{\mathcal{S}}}$ *of the connection* $\nabla_{\check{\mathcal{S}}}$ *is determined by*

$$(\nabla^2_{\check{\mathcal{S}}})^{0,2} = 0, \qquad (\nabla^2_{\check{\mathcal{S}}})^{2,0} = p_{\Sigma,*}(\hat{p}^*_{\Sigma}\nabla^2_{\mathcal{S}}), \tag{4.30}$$

and

$$(\nabla^2_{\check{\mathcal{S}}})^{1,1} = p_{\Sigma,*}(\nabla^2_{\mathcal{P}_\Sigma}) = p_{\Sigma,*}(\mathbb{F}|Z_{\Sigma}). \tag{4.31}$$

This implies that the connection $\nabla_{\check{\mathcal{S}}}$ *is Poincaré basic, since the pull-back* $\hat{p}^*_{\Sigma}\nabla_{\mathcal{S}}$ *is* $\hat{p}_{\Sigma}$*-basic.*

The triple $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma)$ is reducible if there are triples $(\mathcal{S}_1, \nabla_{\mathcal{S}_1}, \Sigma_1)$ and $(\mathcal{S}_2, \nabla_{\mathcal{S}_2}, \Sigma_2)$, such that we have $\Sigma = \Sigma_1 \cup \Sigma_2$ and $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma) = (\mathcal{S}_1, \nabla_{\mathcal{S}_1}, \Sigma_1) \oplus (\mathcal{S}_2, \nabla_{\mathcal{S}_2}, \Sigma_2)$. The triple $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma)$ is said to be irreducible if it is not reducible. Moreover, an $(m-d)$-dimensional smooth submanifold $\Sigma \hookrightarrow \hat{M}$, which is an n-fold covering of B transversal to all fibers, is said to be *proper* if the trivial local system $(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma)$, consisting of the trivial line bundle on Σ with the trivial flat connection, is irreducible. Clearly, if Σ is proper, then it is connected.

Lemma 4.10. *If* $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma) = (\mathcal{S}_1, \nabla_{\mathcal{S}_1}, \Sigma_1) \oplus (\mathcal{S}_2, \nabla_{\mathcal{S}_2}, \Sigma_2)$, *then*

$$(\check{\mathcal{S}}, \nabla_{\check{\mathcal{S}}}) = (\check{\mathcal{S}}_1, \nabla_{\check{\mathcal{S}}_1}) \oplus (\check{\mathcal{S}}_2, \nabla_{\check{\mathcal{S}}_2}). \tag{4.32}$$

Moreover, if $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma)$ *is irreducible with smooth support* Σ, *then* $\Sigma = \operatorname{supp}\mathcal{S}$ *is proper.*

Proof. The first statement follows easily from the definitions of the previous paragraph. Now if $\Sigma = \operatorname{supp}\mathcal{S}$ is not proper, then $(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma)$ splits as the sum $(\mathcal{S}_1, \nabla_{\mathcal{S}_1}, \Sigma_1) \oplus (\mathcal{S}_2, \nabla_{\mathcal{S}_2}, \Sigma_2)$, where $\Sigma = \Sigma_1 \cup \Sigma_2$, $\Sigma_i = \operatorname{supp}\mathcal{S}_i$. Thus

$$\begin{aligned}(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma) &= (\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma) \otimes (\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma)\\ &= (\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma) \otimes (\mathcal{S}_1, \nabla_{\mathcal{S}_1}, \Sigma_1) \oplus (\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma) \otimes (\mathcal{S}_2, \nabla_{\mathcal{S}_2}, \Sigma_2)\\ &= (\mathcal{S}\otimes\mathcal{S}_1, \nabla_{\mathcal{S}\otimes\mathcal{S}_1}, \Sigma_1) \oplus (\mathcal{S}\otimes\mathcal{S}_2, \nabla_{\mathcal{S}\otimes\mathcal{S}_2}, \Sigma_2),\end{aligned}$$

and $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma)$ is reducible. □

4.3. *The main theorem for bundles with Poincaré basic connections*

Definition 4.11. $\mathbf{Vect}^{\nabla}_n(M)$ is the category of foliated Hermitian vector bundles on M endowed with a Poincaré basic unitary connection. Objects in $\mathbf{Vect}^{\nabla}_n(M)$ are pairs (E, ∇_E) consisting of a foliated Hermitian vector bundle E of rank n and a Poincaré basic unitary connection ∇_E. Morphisms are bundle maps compatible with the connections.

Definition 4.12. $\mathbf{Spec}^{\nabla}_n(\hat{M})$ is the category of *spectral data* on $\hat{M}$. Objects in $\mathbf{Spec}^{\nabla}_n(\hat{M})$ are triples $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma)$, such that the pair $(\mathcal{S}, \Sigma)$ is an object in $\mathbf{RelSky}_n(\hat{M})$ and $\nabla_{\mathcal{S}}$ is a connection on $\mathcal{S}|_{\Sigma}$. Morphisms are sheaf maps of $\mathcal{O}_{\hat{M}}$-modules compatible with the connections.

Theorem 4.13. *The Fourier–Mukai transform* $\mathbf{F}$ *defines an additive natural equivalence of categories*

$$\mathbf{F} : \mathbf{Vect}^{\nabla}(M) \xrightarrow{\cong} \mathbf{Spec}^{\nabla}(\hat{M}). \tag{4.33}$$

Proof. In view of the natural isomorphisms $\phi_{\mathcal{S}} : \mathcal{S} \xrightarrow{\cong} \mathbf{F} \circ \hat{\mathbf{F}}(\mathcal{S})$, $\psi_E : \hat{\mathbf{F}} \circ \mathbf{F}(E) \xrightarrow{\cong} E$ in the proof of Theorem 3.7, it suffices to show that we have gauge equivalences $\phi_{\mathcal{S}}^* \nabla_{\mathcal{L}_{\check{\mathcal{S}}}} = \nabla_{\mathcal{S}}$ and $\psi_E^* \nabla_E = \nabla_{\check{\mathcal{L}}}$. We comment only on the proof for the second gauge equivalence. In fact, our constructions of $\mathbf{F}$ in Section 4.1 and $\hat{\mathbf{F}}$ in Section 4.2 show that ∇_E corresponds to $p_{\Sigma,*}(\hat{p}_{\Sigma}^* \tilde{\nabla}_{\mathcal{L}})$. □

Let $\mathbf{Vect}_n^{\nabla}(B)$ be the category of complex vector bundles V of rank n over B with unitary connection ∇_V and fix a pair $(V, \nabla_V) \in \mathbf{Vect}_n^{\nabla}(B)$. Then $\pi^*(V, \nabla_V) = (\pi^* V, \pi^* \nabla_V)$ is an object in $\mathbf{Vect}_n^{\nabla}(M)$, while $\hat{\pi}_0^*(V, \nabla_V)$ is an object in $\mathbf{Spec}_n^{\nabla}(\hat{M})$ supported on the 0-section $\Sigma_0 = \sigma_0(B) \subset \hat{M}$. The construction of $\mathbf{F}$ is again compatible with these pull-backs, that is we have the commutative diagram similar to

$$\begin{array}{ccc} \mathbf{Vect}_n^{\nabla}(M) & \xrightarrow{\mathbf{F}} & \mathbf{Spec}_n^{\nabla}(\widehat{M}) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \hat{\pi}_0^*} \\ \mathbf{Vect}_n^{\nabla}(B) & \xrightarrow{=} & \mathbf{Vect}_n^{\nabla}(B). \end{array} \tag{4.34}$$

Moreover, Corollary 3.8 remains valid for $\mathbf{F}$ on $\mathbf{Vect}^{\nabla}(M)$.

Corollary 4.14. *For* $(E, \nabla_E) \in \mathbf{Vect}^{\nabla}(M)$ *and* $(V, \nabla_V) \in \mathbf{Vect}^{\nabla}(B)$, *the Fourier–Mukai transform* $\mathbf{F}$ *satisfies*

$$\mathbf{F}(\pi^*(V, \nabla_V) \otimes (E, \nabla_E)) \cong \hat{\pi}_{\Sigma}^*(V, \nabla_V) \otimes \mathbf{F}(E, \nabla_E), \tag{4.35}$$

where Σ *is the support of* $\mathbf{F}(E, \nabla_E)$.

Corollary 4.15. $(E, \nabla_E) \in \mathbf{Vect}_n^{\nabla}(M)$ *is of the form* $(E, \nabla_E) = \pi^*(V, \nabla_V)$, *for* $(V, \nabla_V) \in \mathbf{Vect}_n^{\nabla}(B)$, *if and only if the support of the Fourier–Mukai transform* $\mathbf{F}(E, \nabla_E)$ *is the* 0-*section* $\Sigma_0 = \sigma_0(B)$ *of* $\hat{p} : \hat{M} \to B$.

As a consequence of Lemmas 4.7 and 4.10 and Theorem 4.13, we have the following.

Corollary 4.16. *The pair* (E, ∇_E) *is irreducible, if and only if its transform* $\mathbf{F}(E, \nabla_E)$ *is irreducible.*

For complex vector bundles (E, ∇_E) with unitary connection ∇_E, there is a well-known reduction theorem [9, Chapter II, Theorem 7.1] based on the decomposition of the holonomy group in $\mathrm{U}(n)$ into irreducible components. Our construction shows that this defines a decomposition of $(E, \nabla_E) \in \mathbf{Vect}^{\nabla}(M)$ into irreducible components. From Theorem 4.13 and Corollary 4.16, we obtain a similar decomposition of $\mathbf{F}(E, \nabla_E)$ in $\mathbf{Spec}^{\nabla}(\hat{M})$. In the smooth case, the irreducible pairs $(E, \nabla_E) \in \mathbf{Vect}_n^{\nabla}(M)$ are characterized as follows.

Proposition 4.17. *Suppose that the spectral covering Σ of $(E, \nabla_E) \in \mathbf{Vect}_n^{\nabla}(M)$ is smooth. Then the pair (E, ∇_E) is irreducible, if and only if its transform $(\mathcal{L}, \nabla_{\mathcal{L}}, \Sigma) = \mathbf{F}(E, \nabla_E)$ satisfies the following conditions: Σ is connected, $|\Sigma_b| \equiv \ell$ on B for some $\ell|n$, and $(\mathcal{L}, \nabla_{\mathcal{L}})$ is a vector bundle of rank $k = n/\ell$ with irreducible holonomy. The covering $\hat{\pi}_\Sigma : \Sigma \to B$ is non-degenerate exactly for $\ell = n$, $k = 1$. In addition, any smooth, connected spectral manifold $\Sigma \hookrightarrow \hat{M}$ is proper.*

Proof. This is a consequence of Corollary 4.16 and the remarks following Proposition 3.4. □

From Example 4.3, we have the following characterization of locally trivial families $(E, \overset{\circ}{\nabla}_E)$ of flat bundles along the fibers.

Corollary 4.18. *The Fourier–Mukai transform $\mathbf{F}$ defines an equivalence between pairs $(E, \nabla_E) \in \mathbf{Vect}_n^{\nabla}(M)$, such that $(E, \overset{\circ}{\nabla}_E)$ is a locally trivial family of flat bundles along the fibers and ∇_E is basic; and spectral data $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma) \in \mathbf{Spec}_n^{\nabla}(\hat{M})$, such that the spectral covering $\Sigma \subset \hat{M}$ is locally constant and $\mathcal{S}$ has locally constant rank on Σ.*

Combining this with Proposition 4.17, we obtain in addition the following.

Corollary 4.19. *$(E, \nabla_E) \in \mathbf{Vect}_n^{\nabla}(M)$ as in Corollary 4.18 is irreducible, if and only if the spectral covering Σ is a leaf of the transversal foliation $\hat{\mathcal{F}}$ in (3.17), $|\Sigma_b| \equiv \ell$ on B for some $\ell|n$, $\pi_1(B, b)$ acts transitively on $\Sigma_b = \Sigma \cap \hat{T}_b$ and $(\mathcal{L}, \nabla_{\mathcal{L}})$ is a vector bundle of rank $k = n/\ell$* with irreducible holonomy.

From Example 4.4, in particular formula (4.17), we have the following characterization of foliated line bundles on M.

Corollary 4.20. *The Fourier–Mukai transform $\mathbf{F}$ defines a multiplicative equivalence between pairs $(E, \nabla_E) \in \mathbf{Vect}_1^{\nabla}(M)$ and spectral data $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma) \in \mathbf{Spec}_1^{\nabla}(\hat{M})$, where $\mathcal{S}$ is a complex line bundle with connection $\nabla_{\mathcal{S}}$ on the spectral section $\Sigma = \sigma(B)$.*

5. Applications and examples

Let us now apply our theorems to a few interesting examples. We are particularly interested in seeing how differential conditions on the connection ∇_E are transformed.

5.1. Local systems and the representation variety

As a first example, we now look at the action of the Fourier–Mukai transform in the subcategory $\mathbf{Loc}_n(M)$ of unitary local systems of rank n on M. So let $E \to M$ be a complex Hermitian vector bundle of rank n and take ∇_E to be a flat unitary connection on E.

Lemma 5.1. *If ∇_E is flat, then the spectral covering $\Sigma \hookrightarrow \hat{M}$ of $(E, \overset{\circ}{\nabla}_E)$ is locally constant, the rank of $\mathcal{L}$ is locally constant on Σ and the transform $\nabla_{\mathcal{L}}$ is also flat.*

Proof. This follows from (4.24), observing that a flat connection is basic with locally constant spectral covering $\Sigma \hookrightarrow \hat{M}$ and hence $\mathbb{F}|Z_\Sigma = 0$ (compare Example 4.3). □

Next, we argue that the inverse transform also preserves flatness, provided the spectral covering $\Sigma \hookrightarrow \hat{M}$ is locally constant.

Lemma 5.2. *If the spectral covering $\Sigma \hookrightarrow \hat{M}$ is locally constant and $\nabla_{\mathcal{S}}$ is flat, then its transform $\nabla_{\check{\mathcal{S}}}$ is also flat.*

Proof. Firstly, the assumption on the spectral covering implies that $\nabla^2_{\mathcal{P}}|Z_\Sigma = \mathbb{F}|Z_\Sigma = 0$. Since $\nabla_{\mathcal{S}}$ is flat, the Lemma follows from (4.29) or Lemma 4.9. □

With these facts in mind, we introduce the following definition.

Definition 5.3. $\mathbf{SpecLoc}_n(\hat{M})$ is the full subcategory of $\mathbf{Spec}_n^{\nabla}(\hat{M})$ consisting of those objects $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma)$ such that the spectral covering $\Sigma \hookrightarrow \hat{M}$ and the rank of $\mathcal{S}$ on Σ are locally constant, and $\nabla_{\mathcal{S}}$ is flat.

As a consequence of Theorem 4.13 and Lemmas 5.1 and 5.2, we obtain the following theorem.

Theorem 5.4. *The Fourier–Mukai transform* $\mathbf{F}$ *defines a natural equivalence of categories*

$$\mathbf{F} : \mathbf{Loc}_n(M) \xrightarrow{\cong} \mathbf{SpecLoc}_n(\hat{M}).$$

For unitary local systems on M, the decomposition into irreducible components in Section 4.3 applies mutatis mutandis and we may sharpen Corollary 4.19 accordingly, using Theorem 5.4.

Corollary 5.5. *$(E, \nabla_E) \in \mathbf{Loc}_n(M)$ is irreducible, if and only if the spectral covering Σ is a leaf of the transversal foliation $\hat{\mathcal{F}}$ in (3.7), $|\Sigma_b| \equiv \ell$ on B for some $\ell|n$, $\pi_1(B, b)$ acts transitively on Σ_b and $(\mathcal{L}, \nabla_{\mathcal{L}})$ is an irreducible flat vector bundle of rank $k = n/\ell$, that is an irreducible $\mathrm{U}(k)$-local system on Σ.*

Now let $\mathcal{R}_M(n)$ denote the moduli space of irreducible unitary local systems of rank $n \geq 1$ on M. Recall that $\mathcal{R}_M(n)$ coincides with the *representation variety* of $\pi_1(M)$, that is, the set of all irreducible representations $\pi_1(M) \to \mathrm{U}(n)$ modulo conjugation. Let also $\mathcal{S}(n)$ denote the set of all connected, locally constant $(m-d)$-dimensional submanifolds $\Sigma \hookrightarrow \hat{M}$ (modulo isomorphisms), such that the trivial local system $(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma)$ is a relative skyscraper of length ℓ for $\ell|n$, that is

$$(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma) \in \mathbf{SpecLoc}_\ell(\hat{M}). \tag{5.1}$$

Then (5.1) implies that $|\Sigma_b| \equiv \ell$ on B, since the multiplicity of this system is 1. From Example 4.3 and Corollary 5.5, we see that $\Sigma \hookrightarrow \hat{M}$ is a (proper) ℓ-sheeted leaf of the transverse foliation $\hat{\mathcal{F}}$ on $\hat{\pi} : \hat{M} \to B$ determined by (3.7), with transitive transversal holonomy of order ℓ. For given $\ell | n$, we denote the corresponding subset of $\mathcal{S}(n)$ by $\mathcal{S}(n)^\ell$. We will see that the *generic* elements of $\mathcal{S}(n)$ are those in $\mathcal{S}(n)^n$ and we proceed with an explicit parametrization of these moduli spaces.

The transverse foliation $\hat{\mathcal{F}}$ on $\hat{\pi} : \hat{M} \to B$ provides the link between our geometric setup and the representation theory and we refer again to Example 4.3 and Corollary 5.5 for the discussion to follow. Recall that the leaves of $\hat{\mathcal{F}}$ are the images $\hat{\mathcal{F}}_\xi$ of the level sets $\tilde{B} \times \{\xi\}$ in (3.7) and are therefore covering spaces over B of the form $\hat{\mathcal{F}}_\xi \cong \tilde{B}/\Gamma_{\rho,\xi}$, where $\Gamma_{\rho,\xi} \subset \pi_1(B, b)$ is the isotropy group at $\xi \in \hat{T}_b$ under the action corresponding to $\rho^* : \pi_1(B) \to \mathrm{Aut}(\hat{T})$. Here we fix a basepoint $b \in B$ once and for all. It is now clear that the structure of the spaces $\mathcal{S}(n)$, respectively $\mathcal{S}(n)^\ell$ may be described in terms of the transversal holonomy groupoid on the complete transversal $\hat{T}$ and the leaf space of $\hat{\mathcal{F}}$. The leaf space of $\hat{\mathcal{F}}$ is the quotient $\pi_1(B) \setminus \hat{T}$, which may behave quite badly. But for our purposes, we need only consider the invariant subspace $\hat{T}_{\text{fin}} \subset \hat{T}$ defined by the $\xi \in \hat{T}$ satisfying $[\Gamma_{\rho,\xi} : \pi_1(B)] < \infty$, that is the leaves with finite transversal holonomy, on which the $\pi_1(B)$-orbits are finite by definition. $\hat{T}_{\text{fin}}$ has an invariant relatively closed *stratification* $\hat{T}_{n-1} \subset \hat{T}_n \subset \cdots$, given by the points $\xi \in \hat{T}_{\text{fin}}$ satisfying $[\Gamma_{\rho,\xi} : \pi_1(B)] \leq n$. The main stratum in $\hat{T}_n$ is then given by the invariant relatively open set $\hat{T}^n \subset \hat{T}_n$ of those points ξ for which $[\Gamma_{\rho,\xi} : \pi_1(B)] = n$. Here we have tacitly used the 'semicontinuity' for the isotropy groups of a smooth group action, that is $[\Gamma_{\rho,\xi} : \pi_1(B)] \geq n$ is an open condition and hence $[\Gamma_{\rho,\xi} : \pi_1(B)] \leq n$ is a closed condition.

For the generic $\Sigma \in \mathcal{S}(n)^n$, we have $\Sigma = \hat{\mathcal{F}}_\xi \cong \tilde{B}/\Gamma_{\rho,\xi}$, where the isotropy group $\Gamma_{\rho,\xi} \subset \pi_1(B, b)$ has index n and $\Sigma_b = \Sigma \cap \hat{T}_b$ corresponds to an orbit (of order n) in the main stratum $\hat{T}^n$. For $\ell < n$, $\ell | n$ and $\Sigma \in \mathcal{S}(n)^\ell$, we have $\Sigma = \hat{\mathcal{F}}_\xi \cong \tilde{B}/\Gamma_{\rho,\xi}$, where the isotropy group $\Gamma_{\rho,\xi} \subset \pi_1(B, b)$ has index ℓ and Σ_b corresponds to the orbit (of order ℓ) of a limit element of $\hat{T}^n$ in $\hat{T}^\ell \subset \hat{T}_{n-1} = \hat{T}_n \setminus \hat{T}^n$.

Theorem 5.6. *The space $\mathcal{S}(n)$ of spectral manifolds associated to irreducible* $\mathrm{U}(n)$*-representations in $\mathcal{R}_M(n)$ is of the form*

$$\mathcal{S}(n) = \mathcal{S}(n)^n \cup \left(\bigcup_{\ell < n}^{\ell | n} \mathcal{S}(n)^\ell \right), \tag{5.2}$$

and is parametrized (*up to automorphisms*) *by*

$$\pi_1(B) \setminus \left\{ \hat{T}^n \cup \left(\bigcup_{\ell < n}^{\ell | n} \hat{T}^\ell \right) \right\} \subset \pi_1(B) \setminus \hat{T}_n, \tag{5.3}$$

that is the space of leaves of $\hat{\mathcal{F}}$ with finite transversal holonomy of order ℓ with $\ell | n$. The mapping

$$\Psi(n) : \mathcal{R}_M(n) \to \mathcal{S}(n), \tag{5.4}$$

defined by $\Psi(n)(E, \nabla_E) = \operatorname{supp} \mathbf{F}(E, \nabla_E) = \Sigma$, *has the following properties*:

(1) *The generic part* $\Psi(n)^n = \Psi(n)|\mathcal{R}_M(n)^n : \mathcal{R}_M(n)^n \to \mathcal{S}(n)^n$ *is surjective, where* $\mathcal{R}_M(n)^n = \Psi(n)^{-1}\mathcal{S}(n)^n$ *is the space of irreducible representations for which the induced fiber holonomy representations* $\{\xi_1, \dots, \xi_n\}$, $\xi_j \in \hat{T} \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathrm{U}(1))$ *of* Λ *consist of n distinct elements. The fiber* $\Psi(n)^{-1}(\Sigma)$ *over the generic elements* $\Sigma \in \mathcal{S}(n)^n$ *corresponds exactly to the* $\mathrm{U}(1)$-*local systems on* Σ *under the functor* $\mathbf{F}$.
(2) *The fiber* $\Psi(n)^{-1}(\Sigma)$ *for* $\Sigma \in \mathcal{S}(n)^\ell$, $\ell < n$, *corresponds to* (*equivalence classes of*) *irreducible* $\mathrm{U}(k)$-*local systems on* Σ *for* $k = n/\ell$ *under the functor* $\mathbf{F}$.

An extreme situation occurs in the context of Corollary 4.15. In this case, the irreducible flat vector bundle (E, ∇_E) is the pull-back of an irreducible flat vector bundle (V, ∇_V) on the base B, determined by an irreducible $\mathrm{U}(n)$-representation of $\pi_1(B)$. We have $\ell = 1$, $k = n$, Σ is the 0-section $\Sigma_0 = \sigma_0(B)$ of $\hat{\pi} : \hat{M} \to B$ and the corresponding orbit is given by the origin $0 \in \hat{T}^1$.

Proof. The structure of $\mathcal{S}(n)$ follows from the description preceding the theorem. From Corollary 5.5 we see that $\Psi(n)(E, \nabla_E) = \operatorname{supp} \mathbf{F}(E, \nabla_E)$ belongs to $\mathcal{S}(n)$. To see that $\Psi(n)^n$ is surjective on $\mathcal{R}_M(n)^n$, take $\Sigma \in \mathcal{S}(n)^n$ and let $(\underline{\mathbb{C}}, \underline{\mathrm{d}})$ be the trivial $\mathrm{U}(1)$-local system on Σ. Since Σ is proper, $(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma)$ is irreducible and $(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma) \in \mathbf{SpecLoc}_n(\hat{M})$ implies that $\Sigma \in \mathcal{S}(n)^n$ satisfies $|\Sigma_b| \equiv n$. Then it follows from Corollary 4.16 and Theorem 5.4 that $\hat{\mathbf{F}}(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma)$ is an irreducible $\mathrm{U}(n)$-local system on M and defines a point in $\mathcal{R}_M(n)^n$ mapping to Σ under $\Psi(n)^n$. The statements about the fibers of $\Psi(n)$ in (1) and (2) also follow from Corollary 5.5, that is $(E, \nabla_E) \in \Psi(n)^{-1}(\Sigma)$, if and only if $\mathbf{F}(E, \nabla_E)$ is an irreducible $\mathrm{U}(k)$-local system on $\Sigma \in \mathcal{S}(n)^\ell$, for $\ell | n$, $k = n/\ell$. □

For the corresponding irreducible $\mathrm{U}(n)$-representations, we recall that the induced fiber holonomy representations $\{\xi_1, \dots, \xi_n\}$, $\xi_j \in \hat{T} \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathrm{U}(1))$ of Λ consist generically of n distinct elements, that is they are orbits of order n of the action $\rho^* : \pi_1(B) \to \mathrm{Aut}(\hat{T}) \cong \mathrm{Aut}(\Lambda)$. How does one then describe the irreducible $\mathrm{U}(n)$-representation of $\pi_1(M)$ in $\Psi(n)^{-1}(\Sigma) \subset \mathcal{R}_M(n)^n$ associated to a $\mathrm{U}(1)$-local system $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma)$ for $\Sigma \in \mathcal{S}(n)^n$, or more generally to any irreducible $\mathrm{U}(k)$-local system $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma)$ for $\Sigma \in \mathcal{S}(n)^\ell$ via $\hat{\mathbf{F}}$? We claim that the irreducible $\mathrm{U}(n)$-representations of the crossed product (3.4) for $\pi_1(M)$ are obtained by the *induced representation* from irreducible $\mathrm{U}(k)$-representations on a subgroup of index ℓ with $\ell | n$ in $\pi_1(M)$ to the full group. In fact for $k = n/\ell$, any $\mathrm{U}(k)$-local system $\mathcal{S}$ on Σ is determined by a homomorphism $\eta : \pi_1(\Sigma, \xi) \cong \Gamma_{\rho,\xi} \to \mathrm{U}(k)$, which together with the above datum $\xi \in \hat{T}^\ell$ defines an irreducible unitary representation

$$(\exp(\xi), \eta) : \Lambda \times_{\rho_*} \Gamma_{\rho,\xi} \to \mathrm{U}(k), \tag{5.5}$$

that is a $\mathrm{U}(k)$-representation on a subgroup of index ℓ in the crossed product $\pi_1(M) = \Lambda \times_{\rho_*} \pi_1(B)$. We need to verify $\exp(\xi)(\rho_*(\gamma)(a)) = \eta(\gamma)\exp(\xi)(a)\eta(\gamma)^{-1} = \exp(\xi)(a)$, $a \in \Lambda$, $\gamma \in \Gamma_{\rho,\xi}$, which is obvious, since $\mathrm{U}(1)$ is identified with the center of $\mathrm{U}(k)$ and $\Gamma_{\rho,\xi}$ fixes ξ under ρ^*.

Looking at the construction of the inverse Fourier–Mukai transform $\hat{\mathbf{F}}$, in particular the push-down operation $p_{\Sigma,*}$ for the ℓ-fold covering map $p_\Sigma : Z_\Sigma \to M$, we see that the irreducible $\mathrm{U}(n)$-representation of $\pi_1(M)$ in Theorem 5.6, given by $\hat{\mathbf{F}}(\mathcal{S}_\eta, \nabla_{\mathcal{S}_\eta}, \Sigma)$ of the $\mathrm{U}(k)$-local system $(\mathcal{S}_\eta, \nabla_{\mathcal{S}_\eta})$ on Σ defined by η, is in fact the induced representation of $(\exp(\xi), \eta)$.

For the generic case $\ell = n$, $k = 1$, that is $(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma) \in \mathbf{SpecLoc}_n(\hat{M})$, the index of $\Gamma_{\rho,\xi} \subset \pi_1(B)$ is n and the unitary local systems on Σ are 1-dimensional. It then follows also that the irreducible $\mathrm{U}(n)$-representation of $\pi_1(M)$ associated to $(E_\Sigma, \nabla_{E_\Sigma}) = \hat{\mathbf{F}}(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma)$ corresponds to the induced representation of $(\exp(\xi), \mathbf{1})$, that is the trivial representation $\eta = \mathbf{1}$ of $\Gamma_{\rho,\xi}$.

In summary, the following theorem is the algebraic version of Theorem 5.6, given purely in terms of representation theory.

Theorem 5.7. *For the torus bundle $\pi : M \to B$ in (3.1), the representation variety $\mathcal{R}_M(n)$ of the fundamental group $\pi_1(M)$, given by the crossed product (3.4) with respect to the action $\rho : \pi_1(B) \to \mathrm{Aut}(T) \cong \mathrm{GL}(d, \mathbb{Z})$, is parametrized by the following data*:

(1) *Elements $[\xi] \in \pi_1(B) \setminus \hat{T}^\ell$, for $\ell | n$, that is orbits of order ℓ in the dual torus $\hat{T}$ under the induced action $\rho^* : \pi_1(B) \to \mathrm{Aut}(\hat{T})$.*
(2) *Irreducible unitary representations $\eta : \Gamma_{\rho,\xi} \to \mathrm{U}(k)$ of the isotropy group $\Gamma_{\rho,\xi} \subset \pi_1(B)$ of index ℓ at $\xi \in \hat{T}^\ell$, for $k = n/\ell$.*

These data determine an irreducible $\mathrm{U}(n)$-representation of $\pi_1(M)$ by the induced representation of $(\exp(\xi), \eta) : \Lambda \times_{\rho_*} \Gamma_{\rho,\xi} \to \mathrm{U}(k)$ from the subgroup of index ℓ in the crossed product $\pi_1(M)$.

This induction corresponds to the functor $\hat{\mathbf{F}}$. The generic case occurs for $\xi \in \pi_1(B) \setminus \hat{T}^n$, that is $\ell = n$, $k = 1$ and the induction process yields the elements in $\mathcal{R}_M(n)^n$ in this case.

So far, we have not made any assumptions which guarantee non-trivial examples, but there is no doubt that there are many such situations, e.g. when $\rho : \pi_1(B) \to \mathrm{Aut}(T) \cong \mathrm{GL}(d, \mathbb{Z})$ is surjective, $\rho : \pi_1(B_g) \to \mathrm{Aut}(T^2) \cong \mathrm{GL}(2, \mathbb{Z})$, where B_g is an oriented surface of genus $g > 1$, or $\pi_1(B)$ finite, etc.

Remark 5.8. Degeneracy properties for the variety $\mathcal{S}(n)$ and the representation variety $\mathcal{R}_M(n)$:

(1) The parametrization (5.3) of $\mathcal{S}(n)$ is not closed in $\pi_1(B) \setminus \hat{T}_n$. At the limit points in (5.3) corresponding to $\mathcal{S}(n)^\ell$, $\ell < n$, the action of $\pi_1(B)$ is still transitive, even though the orbit degenerates and the corresponding representations of $\pi_1(M)$ are still irreducible. We denote by $\overline{\mathcal{S}(n)}$ the set of all $(m-d)$-dimensional submanifolds $\Sigma \hookrightarrow \hat{M}$ which are a finite union of leaves of the foliation $\hat{\mathcal{F}}$, whose transversal holonomy is of order $\leq n$ (modulo transversal automorphisms of $\hat{\mathcal{F}}$). In a precise sense, $\overline{\mathcal{S}(n)}$ corresponds to the closure of $\pi_1(B) \setminus \hat{T}^n$ in the orbit space $\pi_1(B) \setminus \hat{T}_n$ of leaves with finite holonomy of order $\leq n$. The generic elements $\mathcal{S}(n)^n$ correspond to the open, dense subset $\pi_1(B) \setminus \hat{T}^n \subset \overline{\pi_1(B) \setminus \hat{T}^n} \subset \pi_1(B) \setminus \hat{T}_n$.
(2) At a limit point of $\pi_1(B) \setminus \hat{T}^n$ in $\pi_1(B) \setminus \hat{T}_{n-1} = \pi_1(B) \setminus (\hat{T}_n \setminus \hat{T}^n)$, the action of the holonomy group will generally fail to be transitive, the orbit structure degenerates, the index drops and we end up with a finite number of orbits, say k_i times an orbit

of order ℓ_i, such that $\sum_i k_i \ell_i = n$. Geometrically, this means that the n-fold covering (leaf) $\hat{\pi}_\Sigma : \Sigma \to B$ collapses under this limiting process to ℓ_i-fold coverings (leaves) $\hat{\pi}_{\Sigma_i} : \Sigma_i \to B$ of multiplicity k_i, satisfying the above relation. The 'degeneracy' condition for $\Sigma \in \mathcal{S}(n) \subset \overline{\mathcal{S}(n)}$ at the limit is then of the form

$$(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma) \mapsto \bigoplus_i k_i(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma_i),$$

$$k_i(\underline{\mathbb{C}}, \underline{\mathrm{d}}, \Sigma_i) = (\underline{\mathbb{C}}^{k_i}, \underline{\mathrm{d}}, \Sigma_i) \in \mathbf{SpecLoc}_{\ell_i}(\hat{M}), \quad \sum_i k_i \ell_i = n. \tag{5.6}$$

(3) The limit degeneracy of the spectral covering Σ corresponds of course to the degeneracy of the representations $\{\xi_1, \ldots, \xi_n\}$ of Λ. In fact, with multiple holonomy representations in the limit, the action of the holonomy group will generally fail to be transitive, the orbit structure will degenerate, the index will fall and the representation will decompose into sums of k_i times an irreducible representations of rank ℓ_i, such that $\sum_i k_i \ell_i = n$.

(4) The space $\mathcal{R}_M(n)$ of irreducible $\mathrm{U}(n)$-representations may be completed as well by adding (sums of) irreducible representations of lower rank, as described above. This is similar to the completion of stable bundles to include semistable bundles. Then $\Psi(n)$ extends by continuity to a surjective mapping

$$\overline{\Psi(n)} : \overline{\mathcal{R}_M(n)} \to \overline{\mathcal{S}(n)}. \tag{5.7}$$

Theorem 5.6 (1) remains valid on the generic (open dense) subset $\mathcal{R}_M(n)^n$, but on the boundary of $\overline{\mathcal{S}(n)}$ the structure of the fibers of $\overline{\Psi(n)}$ is more complicated, in accordance with Theorem 5.6 (2) and the previous remarks.

Remark 5.9. Structure of the representation variety $\mathcal{R}_M(n)$: Theorem 5.6 (1) means that the representation variety of a torus bundle $\pi : M \to B$ resembles generically an *integrable system*, that is a fibration by abelian groups. It would be very interesting to determine the conditions under which $\mathcal{R}_M(n)$ is a symplectic manifold, with the fibers of $\Psi(n) : \mathcal{R}_M(n) \to \mathcal{S}(n)$ being Lagrangian over $\mathcal{S}(n)^n$.

5.2. *Instantons on T^1-fibered 4-manifolds*

Here we consider the case $m = 4$, $d = 1$, and take $g_M = g_{T(\pi)} \oplus \pi^* g_B$ to be a bundle-like Riemannian metric on M with respect to the fiber space (3.1) and the exact sequence (3.5). Assuming M to be oriented, g_M induces a splitting of the bundle of 2-forms on M into self-dual (SD) and anti-self-dual (ASD) 2-forms under the Hodge operator $*$:

$$\Omega^2_M = \Omega^+_M \oplus \Omega^-_M. \tag{5.8}$$

From (4.1) we also have the decomposition

$$\Omega^2_M \cong \Omega^{2,0}_M \oplus \Omega^{1,1}_M = \pi^*\Omega^2_B \oplus \pi^*\Omega^1_B \otimes \Omega^1_{M/B}. \tag{5.9}$$

Since g_M is bundle-like, the Hodge operator exchanges the summands in (5.9) and therefore a 2-form $\omega = (\omega^{2,0}, \omega^{1,1})$ satisfies $*\omega = \pm\omega$ if and only if

$$*\omega = (*\omega^{1,1}, *\omega^{2,0}) = \pm(\omega^{2,0}, \omega^{1,1}) = \pm\omega, \tag{5.10}$$

that is $*\omega^{1,1} = \pm\omega^{2,0}$ or equivalently $*\omega^{2,0} = \pm\omega^{1,1}$, so that the projections $\Omega^2_M \to \Omega^{2,0}_M$ and $\Omega^2_M \to \Omega^{1,1}_M$ induce isomorphisms

$$\Omega^{\pm}_M \xrightarrow{\cong} \Omega^{2,0}_M, \qquad \Omega^{\pm}_M \xrightarrow{\cong} \Omega^{1,1}_M.$$

Given a Hermitian vector bundle E with unitary connection ∇_E over M, recall that ∇_E is said to be SD (respectively ASD) if its curvature ∇^2_E is SD (respectively ASD) as a $\mathrm{End}_s(E)$-valued 2-form, that is

$$*\nabla^2_E = \pm\nabla^2_E, \tag{5.11}$$

which by (5.10) is equivalent to

$$(\nabla^2_E)^{2,0} = \pm * (\nabla^2_E)^{1,1}. \tag{5.12}$$

Now Z_Σ is 4-dimensional and the metric $p^*_\Sigma g_M$ prescribes a Riemannian ramified covering $(Z_\Sigma, p^*_\Sigma g_M) \to (M, g_M)$. The pull-back p^*_Σ defines decompositions like (5.8) and (5.9) on Z_Σ relative to the pull-back fiber bundle $\hat{p}_\Sigma : Z_\Sigma \to \hat{M}$. Using (4.6), we see that the (A)SD Eq. (5.12) on Z_Σ can be written as

$$\tilde{j}^*(\tilde{\nabla}^2_E)^{2,0} = p^*_\Sigma(\nabla^2_E)^{2,0} = \pm p^*_\Sigma(*(\nabla^2_E)^{1,1}) = \pm *_\Sigma\, p^*_\Sigma(\nabla^2_E)^{1,1}. \tag{5.13}$$

Suppose now that the adapted unitary connection ∇_E is in addition Poincaré basic. Then we have from (5.13) and (4.18)

$$\tilde{j}^*(\tilde{\nabla}^2_E)^{2,0} = p^*_\Sigma(\nabla^2_E)^{2,0} = \pm *_\Sigma (\mathbb{F}|Z_\Sigma). \tag{5.14}$$

In the following lemma we use the functor $\mathbf{F} : \mathbf{Vect}^\nabla_n(M) \to \mathbf{Spec}^\nabla_n(\hat{M})$ to transform an instanton (E, ∇_E) on M to the corresponding spectral data $(\mathcal{L}, \nabla_\mathcal{L}, \Sigma)$.

Lemma 5.10. *Suppose that the Poincaré basic unitary connection* ∇_E *is* (A)SD. *Then we have*

$$p^*_\Sigma(\nabla^2_E)^{2,0} = \pm *_\Sigma (\mathbb{F}|Z_\Sigma). \tag{5.15}$$

Further, the scalar $(2,0)$*-form* $\hat{\omega} = *_\Sigma(\mathbb{F}|Z_\Sigma)$ *is harmonic and in particular* $\hat{p}_\Sigma$*-basic, that is* $\hat{\omega} = \hat{p}^*_\Sigma\omega$. *The curvature* $\nabla^2_\mathcal{L}$ *of the transformed connection* $\nabla_\mathcal{L}$ *is then given by*

$$\nabla^2_\mathcal{L} = \pm R^1\hat{p}_{\Sigma,*}(\hat{\omega}) = \pm\omega. \tag{5.16}$$

Proof. We need to show that $\hat{\omega}$ is harmonic. Since $\tilde{j}^*(\tilde{\nabla}^2_E)^{0,2} = \tilde{j}^*(\tilde{\nabla}^2_E)^{1,1} = 0$ by assumption, we have $\tilde{j}^*\tilde{\nabla}^2_E = \tilde{j}^*(\tilde{\nabla}^2_E)^{2,0} = p^*_\Sigma(\nabla^2_E)^{2,0}$. Computing traces and using (5.14), we obtain

$$\tilde{j}^*\,\mathrm{Tr}\,\tilde{\nabla}^2_E = \pm n *_\Sigma (\mathbb{F}|Z_\Sigma). \tag{5.17}$$

Since the first Chern polynomial $\mathrm{Tr}\,\tilde{\nabla}^2_E$ is closed, we see from (5.17) that

$$\mathrm{d}\hat{\omega} = \mathrm{d} *_\Sigma (\mathbb{F}|Z_\Sigma) = 0. \tag{5.18}$$

As $\mathrm{d}\mathbb{F} = \mathrm{d}\nabla^2_\mathcal{P} = 0$ from (3.13), it follows that $\mathbb{F}|Z_\Sigma$ must be a *harmonic* 2-form with respect to the bundle-like metric $g_\Sigma = p^*_\Sigma g_M$ on Z_Σ. Since $\hat{\omega} = *_\Sigma(\mathbb{F}|Z_\Sigma)$ is of type $(2,0)$, we have also $i_X\hat{\omega} = 0$, for any vector field X in $T(\hat{p}_\Sigma)$ and therefore $L_X\hat{\omega} = i_X \mathrm{d}\hat{\omega} = 0$; in other words, $\hat{\omega}$ is in addition a $\hat{p}_\Sigma$-basic form and so $\hat{\omega} = \hat{p}_\Sigma^*\omega$, for a unique closed 2-form ω on Σ. Eq. (5.16) follows then from (4.24). In fact, we have $\nabla_{\mathcal{L}}^2 = \pm R^1\hat{p}_{\Sigma,*}(\hat{\omega}) = \pm R^1\hat{p}_{\Sigma,*}(\hat{p}_\Sigma^*\omega) = \pm\omega$. □

Note that the curvature term $p_\Sigma^*(\nabla_E^2)^{2,0}$ must be independent of the choice of the connection ∇_E, since $\mathbb{F}|Z_\Sigma$ depends only on the foliated structure $(E, \overset{\circ}{\nabla}_E)$. Moreover, the curvature term $p_\Sigma^*(\nabla_E^2)^{2,0}$ must also be scalar-valued (i.e. assume values in the center of $p_\Sigma^*\,\mathrm{End}_s(E)$), for $\mathbb{F}$ is scalar-valued.

Theorem 5.11. *Assume that there exists a bundle-like metric g_M with respect to which the form $\hat{\omega} = *_\Sigma(\mathbb{F}|Z_\Sigma)$ is harmonic. The functors* $\mathbf{F}$ *and* $\hat{\mathbf{F}}$ *induce an equivalence between the following objects*:

(1) *Foliated Hermitian vector bundles $(E, \nabla_E) \in \mathbf{Vect}_n^\nabla(M)$ with Poincaré basic unitary connections ∇_E, satisfying the* (A)SD-Eq. (5.12).
(2) *Relative skyscrapers $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma)$ in $\mathbf{Spec}_n^\nabla(\hat{M})$, such that the curvature $\nabla_{\mathcal{S}}^2$ of the connection $\nabla_{\mathcal{S}}$ satisfies*

$$\nabla_{\mathcal{S}}^2 = \pm\omega. \tag{5.19}$$

The harmonicity condition (5.18) for the curvature $\mathbb{F}|Z_\Sigma$ of the connection $\nabla_{\mathcal{P}}$ on $\mathcal{P}_\Sigma$ depends only on the foliated structure $(E, \overset{\circ}{\nabla}_E)$ and the bundle-like metric g_M and is therefore an a priori obstruction for the existence of Poincaré basic instantons, that is solutions of Eq. (5.12), respectively (5.15).

Proof. This follows from combining Lemma 5.10 with Theorem 4.13. □

Finally, we analyze the properties of the parameter spaces for the various structures for a *fixed* foliated vector bundle $(E, \overset{\circ}{\nabla}_E)$. For two adapted connections ∇_E, ∇'_E, we have $\nabla'_E = \nabla_E + \varphi$, where $\varphi \in C^\infty(M, \mathrm{End}_s(E) \otimes \Omega_M^{1,0})$, that is the adapted connections form an affine space modeled on the linear space $C^\infty(M, \mathrm{End}_s(E) \otimes \Omega_M^{1,0})$. For $\nabla'_E = \nabla_E + \varphi$, we have also

$$(\nabla'^2_E)^{1,1} = (\nabla_E^2)^{1,1} + \overset{\circ}{\nabla}_E(\varphi). \tag{5.20}$$

Therefore if ∇_E is *basic*, then the curvature term $(\nabla_E^2)^{1,1}$ vanishes and $\nabla'_E = \nabla_E + \varphi$ is basic if and only if $\overset{\circ}{\nabla}_E(\varphi) = 0$. Thus the space of basic connections is either empty or else an affine space modeled on the linear space of $\overset{\circ}{\nabla}$-parallel sections in $\mathrm{End}_s(E) \otimes \Omega_M^{1,0}$.

Now if ∇_E is *Poincaré basic*, then the curvature term $\tilde{j}^*(\tilde{\nabla}_E^2)^{1,1} = 0$ and $p_\Sigma^*(\nabla_E^2)^{1,1}$ is fixed by (4.18). Then $\nabla'_E = \nabla_E + \varphi$ is Poincaré basic, if and only if $p_\Sigma^*\overset{\circ}{\nabla}_E(\varphi) = 0$ on Z_Σ which is equivalent to $\overset{\circ}{\nabla}_E(\varphi) = 0$. Thus the space of Poincaré basic connections on E is an affine space modeled also on the linear space of $\overset{\circ}{\nabla}$-parallel sections in $\mathrm{End}_s(E) \otimes \Omega_M^{1,0}$.

For the instantons in this Section 5.2, the curvature term $\tilde{j}^*(\tilde{\nabla}_E^2)^{2,0}$ is also fixed by (5.14) and $\nabla'_E = \nabla_E + \varphi$ satisfies the instanton Eq. (5.14), if and only if in addition to the previous condition, the parameter φ satisfies the quadratic PDE

$$\nabla_E(\varphi) + \tfrac{1}{2}[\varphi, \varphi] = 0. \tag{5.21}$$

Here the Lie bracket is taken in the adjoint bundle $\mathrm{End}_s(E)$. Note that the expressions in (5.21) are of type (2, 0), since we already have $\overset{\circ}{\nabla}_E(\varphi) = 0$ from (2).

5.3. *Monopoles on T^1-fibered 3-manifolds*

We keep the assumptions and notation of the previous Section 5.2, but now we take $m = 3$, $d = 1$. The Hodge operator given by the bundle-like metric g_M transforms now

$$* : \Omega_M^{2,0} \xrightarrow{\cong} \Omega_M^{0,1}, \qquad * : \Omega_M^{1,1} \xrightarrow{\cong} \Omega_M^{1,0}.$$

Given a Hermitian vector bundle E with unitary connection ∇_E over M, we consider the corresponding connection form A on the unitary frame bundle $F_U(E)$ with curvature form F_A.

The (A)SD equation is now replaced by the *monopole* equation relative to a Higgs field ϕ in $\mathrm{End}_s(E)$.

$$*F_A = D_A\phi = \mathrm{d}\phi + [A, \phi], \tag{5.22}$$

or in terms of the corresponding unitary connection ∇_E

$$*\nabla_E^2 = \nabla_E(\phi). \tag{5.23}$$

The type-decomposition of (5.22), respectively (5.23) is then given by

$$*F_A^{2,0} = D_A^{0,1}\phi = d^{0,1}\phi + [A^{0,1}, \phi], \qquad *F_A^{1,1} = D_A^{1,0}\phi = d^{1,0}\phi + [A^{1,0}, \phi], \tag{5.24}$$

or in terms of the corresponding unitary connection ∇_E

$$*(\nabla_E^2)^{2,0} = \overset{\circ}{\nabla}_E(\phi), \qquad *(\nabla_E^2)^{1,1} = \nabla_E^{1,0}(\phi). \tag{5.25}$$

Now Z_Σ is 3-dimensional and the metric $p_\Sigma^* g_M$ prescribes a Riemannian ramified covering $(Z_\Sigma, p_\Sigma^* g_M) \to (M, g_M)$. From (5.25) it follows that the monopole equations (5.25) for the pair (A, ϕ), respectively (∇_E, ϕ) on M are now expressed on Z_Σ by

$$*_\Sigma p_\Sigma(\nabla_E^2)^{2,0} = p_\Sigma^* \overset{\circ}{\nabla}_E(\phi), \qquad *_\Sigma p_\Sigma^*(\nabla_E^2)^{1,1} = \nabla_E^t(p_\Sigma^*\phi) = p_\Sigma^* \nabla_E^{1,0}(\phi). \tag{5.26}$$

In order to proceed with the reduction to Σ, we need to assume that the Higgs field ϕ is parallel along the fibers, that is $\overset{\circ}{\nabla}_E(\phi) = 0$. By (5.25), this is equivalent to $(\nabla_E^2)^{2,0} = 0$. Therefore we restrict attention to special solutions of the monopole equations (5.25), namely

$$(\nabla_E^2)^{2,0} = 0, \qquad \overset{\circ}{\nabla}_E(\phi) = 0, \qquad *(\nabla_E^2)^{1,1} = \nabla_E^{1,0}(\phi). \tag{5.27}$$

Suppose again that the adapted unitary connection ∇_E is in addition Poincaré basic. Then we have $\tilde{j}^*(\tilde{\nabla}_E^2)^{1,1} = p_\Sigma^*(\nabla_E^2)^{1,1} - \mathbb{F}|Z_\Sigma = 0$ and therefore the connection $\tilde{j}^*\tilde{\nabla}_E$ must be flat by Lemma 4.1. On Z_Σ, the monopole equations (5.27) are now given by

$$(\nabla_E^2)^{2,0} = 0, \qquad \overset{\circ}{\nabla}_E(\phi) = 0, \qquad \nabla_E^t(p_\Sigma^*\phi) = p_\Sigma^*\nabla_E^{1,0}(\phi) = *_\Sigma(\mathbb{F}|Z_\Sigma). \tag{5.28}$$

Observe that the second equation is of type $(1,0)$. From (5.28) and the vanishing of the commutator Ξ in (4.20), it follows that the scalar $(1,0)$-form $\hat{\omega} = *_\Sigma(\mathbb{F}|Z_\Sigma)$ is closed along the fibers, that is

$$d_{\overset{\circ}{\nabla}}\hat{\omega} = d_{\overset{\circ}{\nabla}}(*_\Sigma(\mathbb{F}|Z_\Sigma)) = 0. \tag{5.29}$$

From the first equation in (5.27) and (4.6), we see that $(\nabla_E^t)^2 = 0$ and therefore (5.28) implies that

$$\nabla_E^t\hat{\omega} = \nabla_E^t(*_\Sigma(\mathbb{F}|Z_\Sigma)) = 0. \tag{5.30}$$

As $\hat{\omega}$ is scalar-valued, (5.29) and (5.30) are equivalent to

$$d *_\Sigma (\mathbb{F}|Z_\Sigma)) = \mathrm{d}\hat{\omega} = \nabla_E\hat{\omega} = 0. \tag{5.31}$$

It follows that the $(1,1)$-form $\mathbb{F}|Z_\Sigma$ is *harmonic*, since $\mathrm{d}\mathbb{F} = 0$ from (3.13).

As was the case for instantons in Section 5.2, the harmonicity of the curvature $\mathbb{F}|Z_\Sigma$ of the connection $\nabla_{\mathcal{P}}$ on $\mathcal{P}_\Sigma$ is a necessary condition for the existence of solutions of (5.27), depending only on the foliated structure $(E, \overset{\circ}{\nabla}_E)$ and the bundle-like metric g_M.

In the following Lemma we use again the functor $\mathbf{F} : \mathbf{Vect}_n^\nabla(M) \to \mathbf{Spec}_n^\nabla(\hat{M})$ to transform a monopole (E, ∇_E, ϕ) on M to the corresponding spectral data $(\mathcal{L}, \nabla_\mathcal{L}, \Sigma)$.

Lemma 5.12. *Suppose that the Poincaré basic unitary connection ∇_E and the Higgs field ϕ satisfy the monopole Eq. (5.27). Then the transformed connection $\nabla_\mathcal{L}$ is flat, that is $\nabla_\mathcal{L}^2 = 0$. Further, the scalar $(1,0)$-form $\hat{\omega} = *_\Sigma(\mathbb{F}|Z_\Sigma)$ is harmonic and in particular $\hat{p}_\Sigma$-basic, that is $\hat{\omega} = \hat{p}_\Sigma^*\omega$. Setting $\bar{\phi} = R^1\hat{p}_{\Sigma,*}(\phi_\mathcal{K})$, the second equation in (5.28) transforms then into the equation*

$$\nabla_\mathcal{L}\bar{\phi} = \nabla_\mathcal{L}R^1\hat{p}_{\Sigma,*}(\phi_\mathcal{K}) = R^1\hat{p}_{\Sigma,*}(\hat{\omega}) = \omega. \tag{5.32}$$

Proof. Recall that $\nabla_\mathcal{L}$ is defined in (4.23) by

$$\nabla_\mathcal{L} = \mathcal{R}^1\hat{p}_{\Sigma,*}(\nabla_E^{\ker}) : \mathcal{L} \to \mathcal{L} \otimes \Omega_\Sigma^1,$$

where $\mathcal{L} = \mathcal{R}^1\hat{p}_{\Sigma,*}(\mathcal{K})$. The flatness of $\nabla_\mathcal{L}$ follows from (4.24), since $(\nabla_E^2)^{2,0} = 0$. From $\overset{\circ}{\nabla}_E(\phi) = 0$ it follows that $p_\Sigma^*\phi$ preserves the sheaf $\mathcal{K}$. Thus ϕ induces an endomorphism $\phi_\mathcal{K} = p_\Sigma^*\phi|\mathcal{K} : \mathcal{K} \to \mathcal{K}$ and ∇_E^t determines the homomorphism $\nabla_E^{\ker}\phi_\mathcal{K} : \mathcal{K} \to \mathcal{K} \otimes \hat{p}_\Sigma^*\Omega_\Sigma^1$, by the usual formula $\nabla_E^{\ker}(\phi_\mathcal{K}) = \nabla_E^{\ker} \circ \phi_\mathcal{K} - (\phi_\mathcal{K} \otimes \mathbf{1}) \circ \nabla_E^{\ker}$. Using (5.29), we may now rewrite the second equation in (5.28) as $\nabla_E^{\ker}(\phi_\mathcal{K}) = \hat{\omega}$. Thus by (4.23), this transforms into the equation

$$\nabla_\mathcal{L}R^1\hat{p}_{\Sigma,*}(\phi_\mathcal{K}) = R^1\hat{p}_{\Sigma,*}(\hat{\omega}). \tag{5.33}$$

As already noted, the form $\hat{\omega}$ is harmonic by (5.31). Since $\hat{\omega}$ is of type $(1,0)$, we have also $i_X\hat{\omega} = 0$, for any vector field X in $T(\hat{p}_\Sigma)$ and therefore from (5.29) $L_X\hat{\omega} = i_X\,\mathrm{d}\hat{\omega} =$

$i_X \, d_{\overset{\circ}{\nabla}} \hat{\omega} = 0$; in other words, $\hat{\omega}$ is in addition a $\hat{p}_\Sigma$-basic form and so $\hat{\omega} = \hat{p}_\Sigma^* \omega$, for a unique closed 2-form ω on Σ. Hence $\nabla_{\mathcal{L}} R^1 \hat{p}_{\Sigma,*}(\phi_{\mathcal{K}}) = \omega$ from (5.33). The fact that ω, hence $\hat{\omega} = \hat{p}_\Sigma^* \omega$ are closed follows of course also from the flatness of $\nabla_{\mathcal{L}}$, since we have $0 = \nabla_{\mathcal{L}}^2 R^1 \hat{p}_{\Sigma,*}(\phi_{\mathcal{K}}) = d\omega$. □

Theorem 5.13. *The functors* **F** *and* $\hat{\mathbf{F}}$ *induce an equivalence between the following objects*:

(1) *Foliated Hermitian vector bundles* $(E, \nabla_E) \in \mathbf{Vect}_n^\nabla(M)$ *with Poincaré basic unitary connections* ∇_E *and* $\overset{\circ}{\nabla}$*-parallel Higgs fields* ϕ *in* $\mathrm{End}_s(E)$, *satisfying the monopole equations* (5.27).
(2) *Relative skyscrapers* $(\mathcal{S}, \nabla_{\mathcal{S}}, \Sigma) \in \mathbf{Spec}_n^\nabla(\hat{M})$, *such that* $\nabla_{\mathcal{S}}$ *is flat, and Higgs fields* $\phi_{\mathcal{S}}$ *in* $\mathrm{End}(\mathcal{S})$, *satisfying*

$$\nabla_{\mathcal{S}} \phi_{\mathcal{S}} = \omega. \tag{5.34}$$

Proof. This follows from Lemma 5.12 combined with Theorem 4.13. □

To conclude, let us analyze the properties of the parameter space of monopoles (∇_E, ϕ) for a *fixed* foliated vector bundle $(E, \overset{\circ}{\nabla}_E)$. For two Poincaré basic connections ∇_E, ∇'_E, we have again $\nabla'_E = \nabla_E + \varphi$, where $\varphi \in C^\infty(M, \mathrm{End}_s(E) \otimes \Omega_M^{1,0})$, satisfies $\overset{\circ}{\nabla}_E(\varphi) = 0$ as in the remarks at the end of Section 5.2. For the monopoles in this Section 5.3, the curvature term $\tilde{j}^*(\tilde{\nabla}_E^2) = 0$. Hence $\nabla'_E = \nabla_E + \varphi$ satisfies the monopole equations (5.27), if and only if φ satisfies the quadratic PDE

$$\nabla_E(\varphi) + \tfrac{1}{2}[\varphi, \varphi] = 0. \tag{5.35}$$

Note that the expressions in (5.35) are of type (2, 0), since we already have $\overset{\circ}{\nabla}_E(\varphi) = 0$. Thus the parameter space for the monopoles is the same as for the instantons (compare (5.21)). The monopole equations (5.27) are linear in the Higgs fields ϕ. Therefore ϕ' also satisfies (5.27), respectively (5.28), if and only if $\phi' = \phi + \psi$, where $\psi \in C^\infty(M, \mathrm{End}_s(E))$ satisfies

$$\nabla_E(\psi) = 0. \tag{5.36}$$

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Appendix A. DeRham complexes along the fibers

Throughout this paper we made use of the fiberwise DeRham complex relative to a fiber bundle $\pi : M \to B$. This complex is well known from foliation theory; in our context is extensively used in [3,4].

For any foliated vector bundle $(E, \overset{\circ}{\nabla}_E)$ on M, there is a fiberwise DeRham complex of sheaves

$$0 \to \ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B \xrightarrow{\varepsilon} E \otimes \Omega^{u,0}_M \xrightarrow{d_{\overset{\circ}{\nabla}}} E \otimes \Omega^{u,1}_M \xrightarrow{d_{\overset{\circ}{\nabla}}} E \otimes \Omega^{u,2}_M \xrightarrow{d_{\overset{\circ}{\nabla}}} \cdots \xrightarrow{d_{\overset{\circ}{\nabla}}} E \otimes \Omega^{u,d}_M, \tag{A.1}$$

which is a fine resolution of the sheaf of $\overset{\circ}{\nabla}$-parallel sections in $E \otimes \Omega^{u,0}_M$. Therefore the sheaves $E \otimes \Omega^{u,*}_M$ are p_*-acyclic and also $\Gamma(M,)$-acyclic for the global section functor $\Gamma(M,)$, that is the derived direct images $\mathcal{R}^j \pi_*(E \otimes \Omega^{u,*}_M) = 0$, $j > 0$. It follows that the higher direct images of $\ker(\overset{\circ}{\nabla}_E) \otimes \pi^* \Omega^u_B$ can be computed from the fine resolution (A.1) and the projection formula by

$$\mathcal{R}^j \pi_*(\ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B) \cong \mathcal{R}^j \pi_*(\ker \overset{\circ}{\nabla}_E) \otimes \Omega^u_B \cong \mathcal{H}^j(\pi_*(E \otimes \Omega^{0,*}, d_{\overset{\circ}{\nabla}})) \otimes \Omega^u_B. \tag{A.2}$$

Likewise, the global fiberwise cohomology is given by

$$H^{u,j}_\pi(M, E) \equiv H^j(M, \ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B) \cong H^j(\Gamma(M, E \otimes \Omega^{u,*}), d_{\overset{\circ}{\nabla}}). \tag{A.3}$$

The two cohomologies are linked by the convergent Leray spectral sequence

$$E^{i,j}_2 = H^i(B, \mathcal{R}^j \pi_*(\ker \overset{\circ}{\nabla}_E) \otimes \Omega^u_B) \Rightarrow H^{i+j}(M, \ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B), \tag{A.4}$$

with edge homomorphisms

$$\begin{aligned} E^{j,0}_2 &= H^j(B, \pi_*(\ker \overset{\circ}{\nabla}_E) \otimes \Omega^u_B) \to H^j(M, \ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B) \to E^{0,j}_2 \\ &= R^j \pi_*(\ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B), \end{aligned} \tag{A.5}$$

where we set $R^j \pi_*(\cdot) = \Gamma(B, \mathcal{R}^j \pi_*(\cdot))$ for the global sections in $\mathcal{R}^j \pi_*(\cdot)$. In our context of torus fiber bundles, we encounter vanishing conditions, leading to degeneracy conditions for the spectral sequence. If $\mathcal{R}^j \pi_*(\ker \overset{\circ}{\nabla}_E) = 0, \quad 0 < j \le d$, the non-zero terms are determined by edge isomorphisms

$$E^{j,0}_2 = H^j(B, \pi_*(\ker \overset{\circ}{\nabla}_E) \otimes \Omega^u_B) \xrightarrow{\cong} H^j(M, \ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B), \quad j \ge 0. \tag{A.6}$$

If $\mathcal{R}^j \pi_*(\ker \overset{\circ}{\nabla}_E) = 0, 0 \le j < d$, the non-zero terms are determined by edge isomorphisms

$$H^{d+j}(M, \ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B) \xrightarrow{\cong} E^{j,d}_2 = H^j(B, \mathcal{R}^d \pi_*(\ker \overset{\circ}{\nabla}_E) \otimes \Omega^u_B), \quad j \ge 0. \tag{A.7}$$

In particular, we have for $j = 0$:

$$\begin{aligned} H^d(M, \ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B) \xrightarrow{\cong} E^{0,d}_2 &= \Gamma(B, \mathcal{R}^d \pi_*(\ker \overset{\circ}{\nabla}_E) \otimes \Omega^u_B) \\ &= R^d \pi_*(\ker \overset{\circ}{\nabla}_E \otimes \pi^* \Omega^u_B). \end{aligned} \tag{A.8}$$

The previous discussion of basic connections in Section 4.1 could have been formulated in terms of this fiberwise resolution (and its global cohomology) with coefficients in the foliated adjoint vector bundle $\mathrm{End}_s(E)$. Specifically, the mixed curvature term $(\nabla_E^2)^{1,1}$ of an adapted connection ∇_E for $(E, \overset{\circ}{\nabla}_E)$ satisfies $d_{\overset{\circ}{\nabla}}(\nabla_E^2)^{1,1} = 0$ and for $\nabla'_E = \nabla_E + \varphi$, $\varphi \in \Gamma(M, \mathrm{End}_s(E) \otimes \Omega_M^{1,0})$, we have from

$$(\nabla_E'^2)^{1,1} = (\nabla_E^2)^{1,1} + \overset{\circ}{\nabla}_E(\varphi) = (\nabla_E^2)^{1,1} + d_{\overset{\circ}{\nabla}}\varphi.$$

Thus $(\nabla_E^2)^{1,1}$ defines a cohomology class

$$a(E, \overset{\circ}{\nabla}_E) = [(\nabla_E^2)^{1,1}] \in H_\pi^{1,1}(M, \mathrm{End}_s(E)) = H^1(\Gamma(M, \mathrm{End}_s(E) \otimes \Omega^{1,*}, d_{\overset{\circ}{\nabla}})), \tag{A.9}$$

depending only on the foliated vector bundle $(E, \overset{\circ}{\nabla}_E)$. This class is very similar to the Atiyah class in the theory of holomorphic vector bundles, where it obstructs the existence of a complex analytic connection. By construction, the class $a(E, \overset{\circ}{\nabla}_E)$ is exactly the obstruction to the existence of a basic connection for $(E, \overset{\circ}{\nabla}_E)$.

In Sections 3 and 4, the resolution (A.1) is implicitly used with respect to the pull-back fiber bundle $\hat{p} : Z \to \hat{M}$, the fiberwise derivative ∇_E^r and its restriction to $\hat{p}_\Sigma : Z_\Sigma \to \Sigma$.

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