Description of some ground states by Puiseux techniques

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Abstract

Let (Σ_G^+, σ) be a one-sided transitive subshift of finite type, where symbols are given by a finite spin set S, and admissible transitions are represented by an irreducible directed graph $G \subset S \times S$. Let $H : \Sigma_G^+ \to \mathbb{R}$ be a locally constant function (that corresponds with a local observable which makes finite-range interactions). Given $\beta > 0$, let $\mu_{\beta H}$ be the Gibbs-equilibrium probability measure associated with the observable $-\beta H$. It is known, by using abstract considerations, that $\{\mu_{\beta H}\}_{\beta>0}$ converges as $\beta \to +\infty$ to a H-minimizing probability measure $\mu_{\min}^H$ called zero-temperature Gibbs measure. For weighted graphs with a small number of vertices, we describe here an algorithm (similar to the Puiseux algorithm) that gives the explicit form of $\mu_{\min}^H$ on the set of ground-state configurations.

Keywords: zero-temperature Gibbs measures, ground-state configurations, Puiseux algorithm.

1 Introduction

The purpose of this article is to present, for one-dimensional lattice-gas models, for specific class of nearest-neighbor interactions H, rigorous results on the convergence of the Gibbs measure $\mu_{\beta H}$ as the temperature $T = \beta^{-1}$ of the system goes to zero. The limit measures thus obtained are called zero-temperature Gibbs measures. For most part of the article, the dynamical system is represented by a one-dimensional lattice, or more generally by a transitive subshift of finite type (Σ_G^+, σ), in which some edges may not follow a given edge, or equivalently in which some hardcore exclusions apply. The exclusion rule is given by an irreducible finite directed graph $G \subset S \times S$. The set S of vertices of G represents the possible states of the system at each site. We say that the interaction energy function H

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has infinite range if it depends on the whole configuration; H is then assumed to be Hölder. We say H has finite range if it depends only on two adjacent sites. Actually, finite range here means nearest neighbor, but it is well known that, by passing to a higher block presentation, one can translate general finite-range models into nearest-neighbor models with more spin states.

Our first goal in section 2 is to improve results on the convergence of Gibbs measures for a certain class of infinite-range interaction energy functions H. We use there the language of ergodic optimization theory and dynamical system theory. More precisely, we prove in theorem 16 the convergence as $\beta \to +\infty$ of a dual notion $V_{\beta H}$, that we call "Gibbs potential", under the hypothesis that the set of ground-state configurations (or H-minimizing non-wandering set, definition 6) $\Omega(H)$ admits a unique irreducible component of maximal entropy. The Gibbs potential may be seen as an approximate effective potential at positive temperature following Chou and Griffiths works [12, 19].

Our second aim is to understand the zero-temperature phase diagram for finite-range interaction energy functions. It is known [8, 10, 23] that, for finite-spin finite-range models in one dimension, the family of Gibbs measures $\{\mu_{\beta H}\}_\beta$ converges to a unique invariant probability measure called zero-temperature Gibbs measure. We present in section 3 the beginning of an algorithm, valid for any weighted directed graph, that describes precisely all possible zero-temperature Gibbs measures. We collect all proofs both for general subshift of finite type systems and for weighted directed graphs in sections 4 and 5. We discuss in section 6 the complete phase diagram for all nonsymmetric complete graphs on 3 symbols. We discuss in section 7 the complete phase diagram of zero-temperature Gibbs measures for the BEG model: a specific model well studied in solid state physics.

We close this introduction by detailing the different phase diagrams we obtain in the case of the one-dimensional Blume-Emery-Griffiths model. The BEG model was initially developed in order to understand the phase transition of mixed systems with two isotopes He^3 and He^4 (see [6]). In particular, it exhibits a tricritical point, separating a regime of first-order transitions from a regime of second-order transitions. Our purpose in this introduction is to describe the zero-temperature phase diagram of the one-dimensional BEG model at the level of ground states. For the one-dimensional Ising models, Georgii [18] gives a complete discussion of the zero-temperature Gibbs measures. There are also examples of zero-temperature Gibbs measures for more than one dimension (see, for instance, the case of the bidimensional Blume-Capel model in [9]).

We consider a one-dimensional spin system with a nearest-neighbor interaction given by the Hamiltonian

$$H(x) = -J \sum_{\langle i,j \rangle} x_i x_j - K \sum_{\langle i,j \rangle} x_i^2 x_j^2 + \Delta \sum_i x_i^2,$$

where $x_i \in S = \{-1, 0, +1\}$ represents a possible state at the site i.

For each positive temperature $T = \beta^{-1}$, there exists a unique *translation-invariant Gibbs measure*, or simply Gibbs measure, $\mu_{\beta H}$, obtained for instance by the Ruelle transfer operator method. We first write H in terms of a unique energy

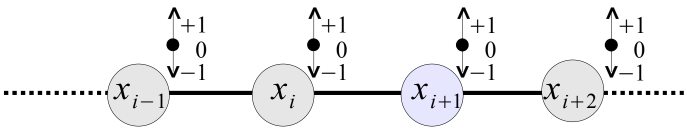


Figure 1: The schematic Blume-Emery-Griffiths model.

function per site H_0, that is, $H = \sum_{i \in \mathbb{Z}} H_0(x_i, x_{i+1})$, where

$$H_0(x,y) = -Jxy - Kx^2y^2 + \frac{\Delta}{2}(x^2 + y^2).$$

In the BEG model, a site having a state ± 1 represents an atom He^4, a site having a state 0 represents He^3. The constant J is supposed to be positive for ferromagnetic systems and negative for antiferromagnetic systems. The constant K takes into account the isotopic interaction, Δ may be interpreted as a chemical potential. An external magnetic field could be added and would give an additional term $h \sum_i x_i$ in the Hamiltonian. We do not consider this term in this introduction. Even so, we emphasize that the algorithm to be described applies without changes in all these cases, ferromagnetic or antiferromagnetic, with or without external magnetic field.

The Ruelle transfer operator method tells us that the Gibbs measure $\mu_{\beta H}$ at temperature $T = \beta^{-1}$ is a Markov chain (π_β, Q_β) on the finite state space S, defined by an irreducible transition matrix $\big[Q_\beta(x,y)\big]_{x,y \in S}$ and a stationary probability vector $\big[\pi_\beta(x)\big]_{x \in S}$,

$$Q_\beta(x,y) := \frac{\Phi_\beta(y)}{\Phi_\beta(x)} \exp\big[-\beta(H_0(y,x) - \bar{H}_\beta)\big], \ \pi_\beta(x) := \frac{\Phi_\beta^*(x)\Phi_\beta(x)}{\sum_{y \in S} \Phi_\beta^*(y)\Phi_\beta(y)}.$$

The factor $\exp(-\beta \bar{H}_\beta)$ denotes the maximal eigenvalue of the transfer operator $\mathcal{L}_\beta$, where $\mathcal{L}_\beta$ may be described here by a matrix indexed by $S \times S$,

$$\mathcal{L}_\beta = [\mathcal{L}_\beta(x,y)]_{x,y \in S}\,, \quad \mathcal{L}_\beta(x,y) = \exp(-\beta H_0(x,y)).$$

The two vectors $[\Phi_\beta(x)]_{x \in S}$ and $[\Phi_\beta^*(x)]_{x \in S}$ denote the left and right eigenvector of $\mathcal{L}_\beta$

$$\sum_{y \in S} \mathcal{L}_\beta(x,y)\Phi_\beta^*(y) = e^{-\beta \bar{H}_\beta}\Phi_\beta^*(x), \quad \sum_{x \in S} \Phi_\beta(x)\mathcal{L}_\beta(x,y) = e^{-\beta \bar{H}_\beta}\Phi_\beta(y),$$

normalized by $\sum_{x \in S} \Phi_\beta(x) = \sum_{x \in S} \Phi_\beta^*(x) = 1$, $\Phi_\beta(x) > 0$, $\Phi_\beta^*(x) > 0$. Notice that in the definition of $Q_\beta(x,y)$, the order of (x,y) has been interchanged in $H_0(y,x)$. The normalizing factor $F = \bar{H}_\beta$ is sometimes called in the physics literature the *free energy*.

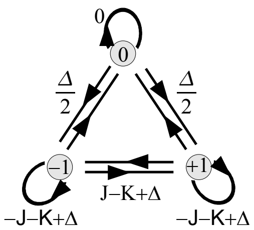
We shall see that $\bar{H}_\beta \to \bar{H}$ as $\beta \to +\infty$, where $\bar{H}$ (see definition 5) represents the *ground-state energy density of the chain* (or the minimizing ergodic value of

H in the language of ergodic optimization theory). In order to understand the convergence of $\mu_{\beta H}$, we rewrite the problem in a framework of bifurcation of singular matrices.

In the BEG model, by numbering the state space $S = \{s_1, s_2, s_3\}$, $s_1 = -1$, $s_2 = 0$ and $s_3 = +1$, and by changing the parameter β to $\epsilon = \exp(-\beta)$, we are left to study a singular perturbation of a one-parameter family of matrices $M_\epsilon = [A(x,y)\epsilon^{a(x,y)}]$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} -J-K+\Delta & \frac{1}{2}\Delta & J-K+\Delta \\ \frac{1}{2}\Delta & 0 & \frac{1}{2}\Delta \\ J-K+\Delta & \frac{1}{2}\Delta & -J-K+\Delta \end{bmatrix}.$$

We summarize the set of possible interactions between two consecutive sites x_i and x_{i+1} by a (directed) graph $G \subset S \times S$ weighted by the principal exponent $a(x,y)$ as explained in figure 2. We also indicate in this figure the mean of a along all simple cycles.



Mean of a along simple cycles:

cycles of order 1	0, $(-J-K+\Delta)$
cycles of order 2	$\frac{1}{2}\Delta$, $(J-K+\Delta)$
cycles of order 3	$\frac{1}{3}(J-K+2\Delta)$

Figure 2: Graph of interactions and determination of minimizing cycles (a cycle of minimizing mean) in the BEG model.

We shall show that $\mu_{\beta H}$ converges to a unique measure μ^H_{min}, called *zero-temperature Gibbs measure*, which has the structure of a Markov chain characterized by an initial law π_∞ and a transition matrix Q_∞. The two figures 3 and 4 describe the structure of this Markov chain with respect to (J,K) for $\Delta > 0$ fixed.

Each region of the plane (J,K) represents a *limit phase*: each box indicates the initial law, the transition matrix and the beginning of the Puiseux series expansion of the free energy F. The three bidimensional regions correspond to the case where all parameters 0, $\frac{1}{2}\Delta$, $-J-K+\Delta$, $J-K+\Delta$ and $\frac{1}{3}(J-K+2\Delta)$ are distinct: a generic case without degeneracy. For instance, when $J-K+\Delta < 0$ and $J < 0$, corresponding to the upper left part of the phase diagram, the smallest parameter is $J-K+\Delta$ and μ^H_{min} is equal to the uniform distribution on the configuration $\cdots -1, +1, -1, +1, \cdots$, or more precisely, because we fix an origin, it is equal to a periodic probability measure of period 2:

$$\mu^H_{min} = \frac{1}{2}\delta_{<\cdots+1|-1+1\cdots>} + \frac{1}{2}\delta_{<\cdots-1|+1-1\cdots>}.$$

The zero-temperature Gibbs measure is pure (or ergodic) and made of atoms with alternate spins ± 1. We show that the initial law π_β, the maximal eigenvalue $e^{-\beta \bar{H}_\beta}$

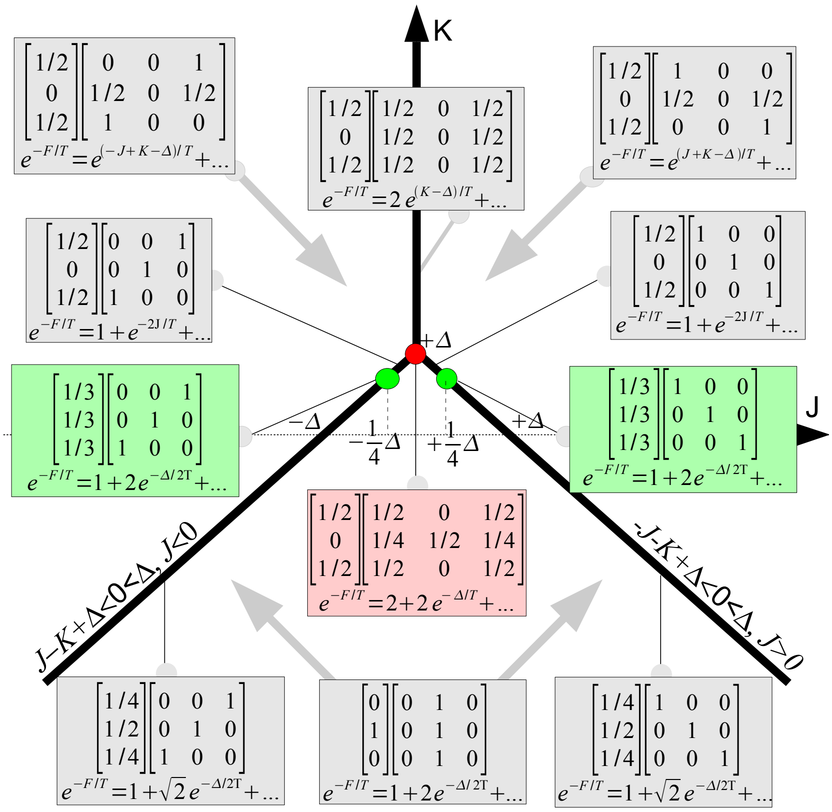


Figure 3: Phase diagram of the BEG model at zero temperature for $\Delta > 0$. The Markov chain structure (π_∞, Q_∞) at zero temperature and the Puiseux series expansion of the free energy $F = \bar{H}_\beta$ is shown for each phase.

and the transition matrix Q_β admit expansions of the following forms

$$\pi_\beta \sim \begin{bmatrix} 1/2 \\ 2e^{-2\beta(-J+K-\Delta/2)} \\ 1/2 \end{bmatrix} \to \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \quad \lambda_\beta = e^{-\beta\bar{H}_\beta} \sim e^{-\beta(J-K+\Delta)},$$

$$Q_\beta \sim \begin{bmatrix} e^{2\beta J} & 2e^{-2\beta(-J+K-\Delta/2)} & 1 \\ 1/2 & e^{-\beta(-J+K-\Delta)} & 1/2 \\ 1 & 2e^{-2\beta(-J+K-\Delta/2)} & e^{2\beta J} \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}.$$

We notice that, in the region $J-K+\Delta > 0$ and $-J-K+\Delta > 0$, independently of the sign of J, the zero-temperature Gibbs measure is pure with only the presence of He^3. We show in all cases that $e^{-\beta\bar{H}_\beta} \sim \bar{\alpha}e^{-\beta\bar{H}}$, where $\ln(\bar{\alpha})$ represents the *zero-temperature entropy* (or topological entropy) of the set of *ground-state configurations* (definition 6). We see in figure 3 that $\ln(\bar{\alpha}) > 0$ when $J = 0$ and $K \geq \Delta$, that is, when the set of ground-state configurations is strongly degenerate (coexistence of at least two adjacent minimizing cycles, figure 4).

The results we present here are essentially one-dimensional as they rely fundamentally on the existence of a transfer operator. We use the language of ergodic

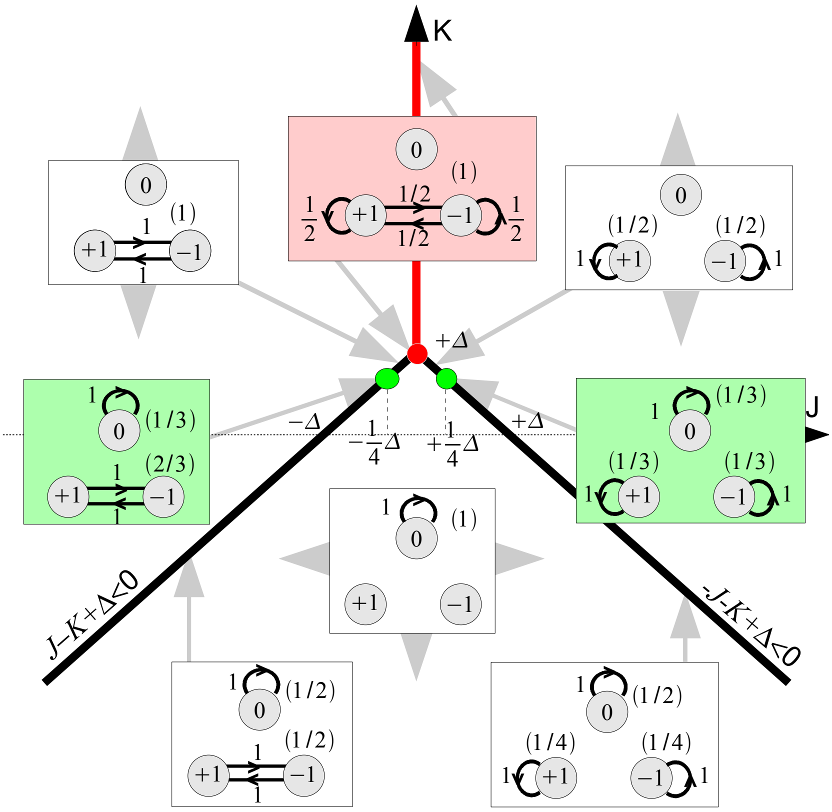


Figure 4: Phase diagram of the BEG model at zero temperature for $\Delta > 0$. Numbers in parentheses indicate the weight of each indecomposable (ergodic) Markov chain which contributes to the zero-temperature Gibbs measure.

optimization in dynamical system in order to better describe the set of ground-state configurations and the set of zero-temperature Gibbs measures. For infinite-range Hamiltonians, we point out a general condition in sections 2 and 4 that implies the uniqueness of the zero-temperature Gibbs measure. For finite-range Hamiltonians, we explain in sections 3, 5, 6 and 7 a complete algorithm that describes the phase diagram of the unique zero-temperature Gibbs measure.

We thank the referee for her/his careful reading and the references [1, 14, 28].

2 A dynamical system approach

We consider a one-sided transitive *subshift of finite type* (Σ_G^+, σ), where S is a finite set of vertices (or states) and $G \subset S \times S$ is an irreducible directed graph representing the admissible transitions (or hardcore exclusions) from one vertex to another. A point in Σ_G^+, called *configuration*, represents a complete state of half of a chain of atoms compatible with the transitions given by the graph G,

$$\Sigma_G^+ = \big\{ x = (x_k)_{k \ge 0} \in S^{\mathbb{N}} \ : \ (x_k, x_{k+1}) \in G, \ \forall\, k \in \mathbb{N} \big\}.$$

Recall that Σ_G^+ is a compact metric space equipped with the distance $d(x,y)=1$ if $x_0 \neq y_0$ and $d(x,y)=(\frac{1}{2})^n$ if $x_0=y_0,\ldots,x_{n-1}=y_{n-1}$ and $x_n \neq y_n$. The *left shift map* $\sigma : \Sigma_G^+ \to \Sigma_G^+$ plays the role of the space translation,

$$\sigma(x_0,x_1,x_2,\ldots)=(x_1,x_2,x_3,\ldots), \quad \forall\, x \in \Sigma_G^+.$$

We prefer to work on the one-sided model instead of the two-sided one in order to use the transfer operator. The two models are mathematically identical but are restricted solely to one-dimensional problems.

We consider, in this one-dimensional setting, a unique interaction energy function $H : \Sigma_G^+ \to \mathbb{R}$, and assume that H is Hölder or, in other words, that H has infinite range. The Gibbs measure at positive temperature $T=\beta^{-1}$, that we recall below, will be denoted $\mu_{\beta H}$. More generally, we allow each transition to have a weight that measures the strength of the constraint. In order to do that, we consider also another Hölder map $E : \Sigma_G^+ \to \mathbb{R}$. We shall introduce the corresponding Gibbs measure $\mu_{E+\beta H}$. The transitivity of (Σ_G^+,σ) (or the irreducibility of G) guarantees the uniqueness of the Gibbs measure $\mu_{E+\beta H}$.

We will use the notation $x \overset{G}{\to} y$ to indicate an admissible transition $(x,y)\in G$ between two vertices $x,y \in S$ and $x_0 \overset{G}{\to} x_1 \overset{G}{\to} x_2 \overset{G}{\to} \ldots \overset{G}{\to} x_{n-1}$ to indicate an admissible path. Let $C_n(x)=[x_0,\ldots,x_{n-1}]$ be the set of configurations $x' \in \Sigma_G^+$ whose first n symbols are prescribed $x'_0=x_0$, $x'_1=x_1$, $\ldots$, $x'_{n-1}=x_{n-1}$; we also say that $C_n(x)$ is a *cylinder* of length n. Let $\mathcal{C}_n(G)=\{C_n(x) \ : \ x\in\Sigma_G^+\}$ be the set of all cylinders of length n. Let us recall Ruelle's definition of the pressure of an observable Ψ (which shall be seen as $E+\beta H$).

Definition 1. *Let $\Psi : \Sigma_G^+ \to \mathbb{R}$ be a continuous observable. We call pressure of Ψ and topological entropy*

$$\mathrm{Pres}(\Psi) := \max\Big\{\mathrm{Ent}(\mu) - \int \Psi \, d\mu \ : \ \mu \in \mathcal{M}(\Sigma_G^+,\sigma)\Big\},$$
$$\mathrm{Ent}(\Sigma_G^+) := \max\big\{\mathrm{Ent}(\mu) \ : \ \mu \in \mathcal{M}(\Sigma_G^+,\sigma)\big\},$$

where $\mathcal{M}(\Sigma_G^+,\sigma)$ denotes the set of σ-invariant Borel probability measures on Σ_G^+, and $\mathrm{Ent}(\mu)$ denotes the Kolmogorov-Sinai entropy of σ with respect to μ,

$$\mathrm{Ent}(\mu) := \lim_{n\to+\infty} \frac{1}{n} \sum_{C_n \in \mathcal{C}_n(G)} -\mu[C_n] \ln \mu[C_n].$$

More generally, for any σ-invariant Borel probability measure μ or σ-invariant compact set Ω, we call relative pressure with respect to μ or Ω, respectively,

$$\mathrm{Pres}(\Psi,\mu) := \mathrm{Ent}(\mu) - \int \Psi \, d\mu,$$
$$\mathrm{Pres}_\Omega(\Psi) := \max\Big\{\mathrm{Pres}(\Psi,\mu) \ : \ \mu \in \mathcal{M}(\Sigma_G^+,\sigma) \ \textit{and} \ \mathrm{supp}(\mu) \subset \Omega\Big\}.$$

We say that $\mu \in \mathcal{M}(\Sigma_G^+,\sigma)$ has relative maximal pressure in Ω for Ψ if

$$\mathrm{Pres}_\Omega(\Psi) = \mathrm{Pres}(\Psi,\mu) \quad \textit{and} \quad \mathrm{supp}(\mu)\subset\Omega.$$

Definition 2. *We call Gibbs measure associated with Ψ a σ-invariant Borel probability measure μ_Ψ on Σ_G^+ satisfying*

$$\mu_\Psi[C_n(x)] \asymp \exp\Big(-\sum_{k=0}^{n-1} \big[\Psi \circ \sigma^k(x) + \operatorname{Pres}(\Psi)\big]\Big), \quad \forall\, x \in \Sigma_G^+, \quad \forall\, n \geq 1.$$

The notation $a_n(x) \asymp b_n(x)$ is a shorthand for $C^{-1}a_n(x) \leq b_n(x) \leq Ca_n(x)$ for some constant $C > 0$ independent of n and x.

Notice that this definition is a typical dynamical system definition or Sinai-Ruelle-Bowen definition, in contrast to the Dobrushin-Lanford-Ruelle definition of Gibbs measures, as given in [18], which immediately works in higher dimensions. For more details on a SRB definition of Gibbs measures in higher dimensions, we refer the reader to [22].

It is known that, for any given Hölder observable $\Psi : \Sigma_G^+ \to \mathbb{R}$, there exists a unique Gibbs measure μ_Ψ, which is also the unique σ-invariant Borel probability measure with maximal pressure:

$$\operatorname{Pres}(\Psi) = \operatorname{Pres}(\Psi, \mu_\Psi) > \operatorname{Pres}(\Psi, \mu), \quad \forall\, \mu \in \mathcal{M}(\Sigma_G^+, \sigma) \setminus \{\mu_\Psi\}.$$

For $E, H : \Sigma_G^+ \to \mathbb{R}$ Hölder observables, we are interesting in the convergence (with respect to the weak* topology) of $\mu_{E+\beta H}$ as β tends to $+\infty$ (or as $T = \beta^{-1} \to 0$).

Question 3. *What are the possible weak* limits of $\mu_{E+\beta H}$ as β tends to $+\infty$? Is there a unique limit ? How can one characterize them in an effective way ?*

We collect in this section several general facts for arbitrarily Hölder H. We shall show in the next section how to improve these results when H has finite range. We begin by adopting a terminology proposed in the appendix B of [14].

Definition 4. *We call zero-temperature Gibbs measure any weak* limit of $\mu_{E+\beta H}$ as β tends to $+\infty$.*

An immediate observation tells us that a zero-temperature Gibbs measure is a minimizing measure in the following sense.

Definition 5. *Let $H : \Sigma_G^+ \to \mathbb{R}$ be a continuous observable. We call minimizing ergodic value of H (or ground-state energy density) the quantity*

$$\bar{H} := \min\Big\{ \int H(x)\, d\mu(x) \;:\; \mu \in \mathcal{M}(\Sigma_G^+, \sigma)\Big\}.$$

We call minimizing measure any σ-invariant Borel probability measure μ_{min} realizing the minimum in the previous equality $\int H(x)\, d\mu_{min}(x) = \bar{H}$. The set of H-minimizing measures is denoted by $\mathcal{M}_{min}(\Sigma_G^+, \sigma, H)$.

From Aizenman and Lieb work [1], it is known that in general dimensions any zero-temperature limit of Gibbs measures has maximal entropy. Hence it is not a surprise that here a zero-temperature Gibbs measure μ has maximal pressure $\operatorname{Pres}(E, \mu)$ (or maximal entropy $\operatorname{Ent}(\mu)$ for $E = 0$) among all minimizing measures. In order to explain this fact, it is convenient to introduce a set $\Omega(H)$ that plays the role of the set of ground-state configurations but which is called the set of *H-minimizing non-wandering configurations* in ergodic optimization.

Definition 6. *Let $H : \Sigma_G^+ \to \mathbb{R}$ be a continuous observable. We define the set of H-minimizing non-wandering configurations (or ground-state configurations) by*

$$\Omega(H) := \Big\{ x \in \Sigma_G^+ : \forall\, \epsilon > 0,\ \exists\, n \geq 1,\ \exists\, z \in \Sigma_G^+ \ \ s.\ t. \\ d(x,z) < \epsilon,\ d(x,\sigma^n(z)) < \epsilon \ and\ \Big| \sum_{k=0}^{n-1} [H \circ \sigma^k(z) - \bar{H}] \Big| < \epsilon \Big\}.$$

It is easy to show that $\Omega(H)$ is compact and completely σ-invariant, $\sigma(\Omega(H)) = \Omega(H)$. We recognize $\Omega(H)$ as the set of ground-state configurations in the following sense. If H is Hölder, there exists a Hölder function $V : \Sigma_G^+ \to \mathbb{R}$ (a calibrated sub-action as in proposition 13) such that

$$\begin{cases} \sum_{k=0}^{n-1} H \circ \sigma^k(x) = n\bar{H} + V \circ \sigma^n(x) - V(x), & \forall\, x \in \Omega(H), \quad \forall\, n \geq 1, \\ \sum_{k=0}^{n-1} H \circ \sigma^k(y) \geq n\bar{H} + V \circ \sigma^n(y) - V(y), & \forall\, y \in \Sigma_G^+, \quad \forall\, n \geq 1. \end{cases}$$

Therefore, up to a coboundary $\Delta(x,y) := V(x) - V(y)$, the energy $\sum_{k=0}^{n-1} H \circ \sigma^k(x)$ of the configuration $x \in \Omega(H)$ on n consecutive sites can only increase

$$\sum_{k=0}^{n-1} H \circ \sigma^k(y) - \sum_{k=0}^{n-1} H \circ \sigma^k(x) \geq \Delta(\sigma^n(y), \sigma^n(x)) - \Delta(y,x), \quad \forall\, y \in \Sigma_G^+.$$

Moreover, it follows from the result of Radin and Schulman [28] that, for finite-range interactions, the set of ground-state configurations always has periodic configurations. Actually, it is easy to show that, when H has finite range, $\Omega(H)$ is exactly the closure of its periodic configurations.

We state in the following proposition that $\Omega(H)$ contains the support of all minimizing measures and that any invariant measure whose support belongs to $\Omega(H)$ is minimizing.

Proposition 7. *[13] Let $H : \Sigma_G^+ \to \mathbb{R}$ be a Hölder observable. A σ-invariant Borel probability measure μ is H-minimizing if, and only if, its support* $\mathrm{supp}(\mu)$ *is included into $\Omega(H)$:*

$$\mathcal{M}_{min}(\Sigma_G^+, \sigma, H) = \{\mu \in \mathcal{M}(\Sigma_G^+, \sigma) \ : \ \mathrm{supp}(\mu) \subset \Omega(H)\}.$$

From the previous discussion, a zero-temperature Gibbs measure is minimizing and has a support included in $\Omega(H)$. There may exist several minimizing measures even for a finite-range interaction H (see section 3). The next proposition states that, by freezing the system, the Gibbs measures accumulate on minimizing measures satisfying a zero-temperature variational principle. Similar results have been obtained in other contexts (see, for instance, [4] or [20, 25]).

Proposition 8. *[13, 23] Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be Hölder observables. Then any zero-temperature Gibbs measure μ_∞ is H-minimizing and has a support in $\Omega(H)$. In addition, μ_∞ achieves the maximum of the pressure among all invariant measures in $\Omega(H)$; if $E = 0$, μ_∞ achieves the maximum of the entropy in $\Omega(H)$. More pecisely,*

1. $\mathrm{Pres}(E, \mu_\infty) = \mathrm{Pres}_{\Omega(H)}(E)$ *and* $\int H \, d\mu_\infty = \bar{H}$. *If* $\Omega(H)$ *supports a unique measure* μ_{min} *with maximal pressure* $P_{\Omega(H)}(E)$, *then* $\{\mu_{E+\beta H}\}_\beta$ *converges to* μ_{min}.

2. *Let be* $\bar{H}_\beta := -\frac{1}{\beta} P(E + \beta H)$. *Then* $\beta(\bar{H} - \bar{H}_\beta) \to \mathrm{Pres}_{\Omega(H)}(E)$ *as* $\beta \to +\infty$. *In the physics literature,* $\bar{H}_\beta$ *is called the free energy and converges to* $\bar{H}$ *with speed* $\frac{1}{\beta}$.

3. *As* $\beta \to +\infty$, $\int H \, d\mu_{E+\beta H} \to \bar{H}$ *and* $\mathrm{Pres}(E, \mu_{E+\beta H}) \to \mathrm{Pres}_{\Omega(H)}(E)$. *In the particular case* $E = 0$, $\mathrm{Ent}(\mu_{\beta H}) \to \mathrm{Ent}(\Omega(H))$.

Notice that for a generic interaction energy function, $\{\mu_{E+\beta H}\}_\beta$ converges to a unique minimizing measure as $\beta \to +\infty$.

Proposition 9. *[13] For any* $\alpha > 0$, *the set of* α*-Hölder* H *admitting a unique* H*-minimizing probability measure is generic in* C^α. *Thus* $\{\mu_{E+\beta H}\}_\beta$ *converges to a unique* μ_∞ *for generic* α*-Hölder* H.

The uniqueness of the zero-temperature Gibbs measure, which is the content of the previous proposition, holds for generic continuous interactions too. It is also important to keep in mind that there are examples of Hölder interactions for which the convergence $\{\mu_{E+\beta H}\}_\beta$ does not hold (see [11]).

Gibbs measures have a different functional characterization in terms of the Ruelle transfer operator. They are also called *equilibrium measures.*

Definition 10. *We call Ruelle transfer operator associated with a Hölder observable* $\Psi : \Sigma_G^+ \to \mathbb{R}$ *the operator* $\mathcal{L}_\Psi$ *acting on Hölder functions* $f : \Sigma_G^+ \to \mathbb{R}$ *as follows*

$$\mathcal{L}_\Psi f(x) = \sum_{y \,:\, \sigma(y) = x} e^{-\Psi(y)} f(y), \quad \forall \, x \in \Sigma_G^+,$$

where the summation is taken among all preimages of x *by* σ.

It is well known that, by extending the standard Perron-Frobenius theory for nonnegative matrices, the Ruelle transfer operator $\mathcal{L}_\Psi$ admits similar "right and left eigenvectors" that we recall in the following proposition.

Proposition 11. *[7, 27, 29] Let* $\Psi : \Sigma_G^+ \to \mathbb{R}$ *be a Hölder observable. Then there exist a unique left eigenmeasure, or Borel probability measure* ν_Ψ *on* Σ_G^+, *a unique normalized right eigenfunction, or positive Hölder function* $\Phi_\Psi : \Sigma_G^+ \to \mathbb{R}$, *such that*

$$\mathcal{L}_\Psi^* \nu_\Psi = e^{\mathrm{Pres}(\Psi)} \nu_\Psi, \quad \mathcal{L}_\Psi \Phi_\Psi = e^{\mathrm{Pres}(\Psi)} \Phi_\Psi \quad \textit{and} \quad \int \Phi_\Psi \, d\nu_\Psi = 1.$$

Moreover, $\mu_\Psi := \Phi_\Psi \nu_\Psi$ *is a Gibbs measure and the unique* σ*-invariant probability that maximizes the pressure for* Ψ *among all* σ*-invariant probabilities. We call* $V_\Psi := -\frac{1}{\beta} \ln \Phi_\Psi$ *the Gibbs potential associated with* Ψ.

The Gibbs potential $V_{E+\beta H} = -\frac{1}{\beta}\ln\Phi_{E+\beta H}$ plays the role, at positive temperature, of the *effective potential* introduced by W. Chou and R. B. Griffiths in [12, 19] to study ground states in the Frenkel-Kontorova model. We shall see below in proposition 13 and theorem 16 that indeed, in some cases, the Gibbs potential converges to an effective potential as $\beta \to +\infty$. We have seen in proposition 8 that $\bar{H}_\beta = -\frac{1}{\beta}P(E+\beta H)$ converges to $\bar{H}$ and that any weak* limit of $\{\mu_{E+\beta H}\}_\beta$ is H-minimizing. It would be interesting to obtain similar characterizations for limit points of $\{V_{E+\beta H}\}_\beta$ or $\{\nu_{E+\beta H}\}_\beta$. The first result in that direction is that any limit point of $V_{E+\beta H}$ is a *calibrated sub-action*:

Definition 12. *Let $H : \Sigma_G^+ \to \mathbb{R}$ be a continuous observable. We call sub-action with respect to H any continuous function $V : \Sigma_G^+ \to \mathbb{R}$ such that*

$$V\circ\sigma(x) - V(x) \le H(x) - \bar{H}, \quad \forall\, x \in \Sigma_G^+.$$

We call calibrated sub-action any sub-action V which in addition satisfies

$$V(y) = \min\big\{V(x) + H(x) - \bar{H} \ : \ x \in \Sigma_G^+,\ \sigma(x) = y\big\}, \quad \forall\, y \in \Sigma_G^+.$$

Similarly to proposition 29 of [13], we obtain easily the following proposition.

Proposition 13. *Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be Hölder observables. Let $\Phi_{E+\beta H} := \exp(-\beta V_{E+\beta H})$ be the right eigenfunction of $\mathcal{L}_{E+\beta H}$. Then $\{V_{E+\beta H}\}_\beta$ is uniformly bounded and has a uniform Hölder norm. Moreover, any accumulation function of $\{V_{E+\beta H}\}_\beta$ is a calibrated sub-action with respect to H.*

If $\Omega(H)$ supports a unique probability measure $\mu^H_{\min}$ with relative maximal pressure $\mathrm{Pres}_{\Omega(H)}(E)$, then $\mu_{E+\beta H} \to \mu^H_{\min}$ although $\mathcal{M}_{\min}(\Sigma_G^+, \sigma, H)$ may not be reduced to a single measure. We do not know whether a similar result is true for the convergence of $\{V_{E+\beta H}\}_\beta$. We nevertheless show the "projective" convergence of $\{V_{E+\beta H}\}_\beta$ in the particular case where $\Omega(H)$ can be split into disjoint irreducible components with a unique component of maximal pressure. The splitting up of $\Omega(H)$ into components uses the following notion of Peierls barrier in the sense of Mather [24, 15].

Definition 14. *Let $H : \Sigma_G^+ \to \mathbb{R}$ be a Hölder observable. We call Peierls barrier the function $h(x, y)$ defined on $\Sigma_G^+ \times \Sigma_G^+$ by*

$$h(x,y) := \lim_{\epsilon\to 0}\ \liminf_{n\to+\infty}\ S_n^\epsilon(x,y),$$

where

$$S_n^\epsilon(x,y) := \inf\Big\{\sum_{k=0}^{n-1}(H-\bar{H})\circ\sigma^k(z) \ : \ d(z,x) < \epsilon \ \text{ and } \ d(\sigma^n(z), y) < \epsilon\Big\}.$$

The Peierls barrier may be infinite. If $x \in \Omega(H)$, $h(x,y)$ is finite and Hölder with respect to $y \in \Sigma_G^+$. Notice that $\Omega(H) = \{x \in \Sigma_G^+ \ : \ h(x,x) = 0\}$. Let us recall how the minimizing non-wandering set $\Omega(H)$ can be partitioned into closed invariant sets, which uniquely characterize sub-actions.

Definition 15. *[17] We say that two points x, y of $\Omega(H)$ are equivalent, and we write $x \sim y$, whenever $h(x,y) + h(y,x) = 0$. Equivalent classes are called irreducible components. Irreducible components are σ-invariant and compact.*

We now state the main result of this section.

Theorem 16. *Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be Hölder observables. Assume that $\Omega(H) = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_r$ admits a finite decomposition into disjoint irreducible components Ω_i and*

$$\mathrm{Pres}_{\Omega(H)}(E) = \mathrm{Pres}_{\Omega_0}(E) > \mathrm{Pres}_{\Omega_1}(E) \geq \ldots \geq \mathrm{Pres}_{\Omega_r}(E).$$

Let $\Phi_{E+\beta H} = \exp(-\beta V_{E+\beta H})$ be the normalized right eigenfunction of the Ruelle transfer operator $\mathcal{L}_{E+\beta H}$. Then uniformly in $y \in \Sigma_G^+$, for any fixed $x_0 \in \Omega_0$,

$$\lim_{\beta \to +\infty} V_{E+\beta H}(y) - V_{E+\beta H}(x_0) = h(x_0, y), \quad \forall\ y \in \Sigma_G^+.$$

Notice that, in the above theorem, $\{\mu_{E+\beta H}\}_\beta$ may not converge to a unique H-minimizing measure. Indeed, any weak* limit has a support in Ω_0 which may contain many minimizing measures. Notice also that the convergence of $\{V_{E+\beta H}\}_\beta$ (as a sequence of functions) depends only on the converge of $\{V_{E+\beta H}(x_0)\}_\beta$ for any fixed $x_0 \in \Omega_0$.

3 A matrix approach to ground-state theory

We say that the interaction energy function $H : \Sigma_G^+ \to \mathbb{R}$ has finite range if it only depends on two consecutive symbols $H(x) = H(x_0, x_1)$. By allowing a larger number of vertices in another irreducible finite directed graph G', an energy function of the form $H(x_0, \ldots, x_{d-1})$ can be described by the framework we are going to develop. The main consequence of this strong assumption on the energy function is that the problem of zero-temperature phase diagram is reduced to a problem of singular perturbation of matrices of Puiseux type.

We consider a finite state space S and an irreducible directed graph $G \subset S \times S$ weighted by an energy function $\{\exp[-\beta H(x,y)]\}_{x \overset{G}{\to} y}$, where x, y are particular states in S and $x \overset{G}{\to} y$ denotes an admissible transition given by the graph G. We prefer to introduce a new parameter $\epsilon := \exp(-\beta)$, which goes to zero when β tends to $+\infty$, and a one-parameter family of transfer matrices $[M_\epsilon(x,y)]_{(x,y) \in S \times S}$, adapted to G, defined by

$$\begin{cases} M_\epsilon(x,y) := \exp[-\beta H(x,y)] = \epsilon^{H(x,y)}, & \forall\ (x,y) \in G, \\ M_\epsilon(x,y) := 0, & \forall\ (x,y) \notin G. \end{cases}$$

Notice that M_ϵ is a Perron-Frobenius matrix, that is, a matrix with nonnegative entries. Let $\lambda_\epsilon := \rho_{spec}(M_\epsilon) > 0$ be its spectral radius. Because of the irreducibility of G, λ_ϵ is an eigenvalue of multiplicity 1. Let $[L_\epsilon(x)]_{x \in S}$ and $[R_\epsilon(x)]_{x \in S}$ be the left and right eigenvector of M_ϵ for the eigenvalue λ_ϵ,

$$\sum_{x\in S} L_\epsilon(x)M_\epsilon(x,y) = \lambda_\epsilon L_\epsilon(y), \quad \forall\ y \in S,$$
$$\sum_{y\in S} M_\epsilon(x,y)R_\epsilon(y) = \lambda_\epsilon R_\epsilon(x), \quad \forall\ x \in S,$$

normalized by $\sum_{x\in S} L_\epsilon(x)R_\epsilon(x) = 1$ and $\sum_{x\in S} R_\epsilon(x) = 1$. Notice that $L_\epsilon(x) > 0$ and $R_\epsilon(x) > 0$ for all $x \in S$. Let

$$\pi_\epsilon(x) := L_\epsilon(x)R_\epsilon(x) \quad \text{and} \quad Q_\epsilon(x,y) := M_\epsilon(x,y)\frac{R_\epsilon(y)}{R_\epsilon(x)\lambda_\epsilon}, \quad \forall\ x,y \in S.$$

The Ruelle transfer operator used in the dynamical approach of section 2 is strongly related to a basic eigenvalue problem that we recall in the following remark.

Remark 17. *Assume $H(x) = H(x_0,x_1)$ has finite range. Let $\Phi_{\beta H} : \Sigma_G^+ \to \mathbb{R}$ be the right eigenfunction of $\mathcal{L}_{\beta H}$ and $\nu_{\beta H}$ be the left eigenmeasure of $\mathcal{L}_{\beta H}$. Let $\mu_{\beta H}(dx) = \Phi_{\beta H}(x)\nu_{\beta H}(dx)$ be the normalized Gibbs-equilibrium measure associated with βH. Then*

i. $\Phi_{\beta H}(x) = L_\epsilon(x_0)$, $\forall\ x = (x_0,x_1,\ldots) \in \Sigma_G^+$.

ii. $\nu_{\beta H}([x_0]) = R_\epsilon(x_0)$, $\forall\ x_0 \in S$.

iii. *$\mu_{\beta H}$ is a Markov chain on Σ_G^+ with initial law π_ϵ and transition matrix Q_ϵ. For any cylinder of size $d+1$, one has*

$$\mu_{\beta H}([x_0,x_1,\ldots,x_d]) = L_\epsilon(x_0)\big[\Pi_{i=0}^{d-1} M_\epsilon(x_i,x_{i+1})\big] R_\epsilon(x_d)/\lambda_\epsilon^d.$$

We are interested in describing the possible limits of $\{(\pi_\epsilon,Q_\epsilon)\}_{\epsilon\to 0}$ that we also call zero-temperature Gibbs measures. In an equivalent way, we want to describe all possible limits of the eigenvalue $\{\lambda_\epsilon\}_{\epsilon\to 0}$ and the projective eigenvectors $\{L_\epsilon(x)/L_\epsilon(y)\}_{\epsilon\to 0}$ and $\{R_\epsilon(x)/R_\epsilon(y)\}_{\epsilon\to 0}$. As in the dynamical system approach, the zero-temperature Gibbs measures are localized in a minimizing subgraph similar to the minimizing non-wandering set $\Omega(H)$ recalled in definition 6. We first begin by restricting the class of the one-parameter family of matrices we want to study. We introduce the notion of one-parameter family of Puiseux type in two steps.

Definition 18. *Let $G \subset S \times S$ be a (not necessarily irreducible) directed graph and $\{M_\epsilon\}_{\epsilon>0}$ be a one-parameter family of matrices indexed by S. The graph G is said to be weighted by M_ϵ if $M_\epsilon(x,y) = 0$ whenever $(x,y) \notin G$. The weighted graph (G, M_ϵ) is said to be of exact Puiseux type if there exist a nonnegative matrix $[A(x,y)]_{x,y\in S}$ and an extended real-valued matrix $[a(x,y)]_{x,y\in S}$ such that*

i. $\forall\ (x,y) \notin G$, $A(x,y) = 0$, $a(x,y) = +\infty$ *and* $M_\epsilon(x,y) = 0$.

ii. $\forall\ (x,y) \in G$, $A(x,y) > 0$, $a(x,y) \in \mathbb{R}$ *and*

$$M_\epsilon(x,y) = A(x,y)\epsilon^{a(x,y)} + o(\epsilon^{a(x,y)}).$$

We say shortly $M_\epsilon \sim A\epsilon^a$.

We call G-path of length $n \geq 1$ in S any sequence $(x_0,\ldots,x_n)$ such that $(x_k,x_{k+1}) \in G$, $\forall\ k = 0,\ldots,n-1$. The support of a G-path $(x_0,\ldots,x_n)$ is the subset $\{(x_k,x_{k+1}) : k = 0,\ldots,n-1\} \subset G$. A cycle of length $n \geq 1$ is a G-path $(x_0,\ldots,x_n)$ in S such that $x_n = x_0$. We call off-diagonal cycle any cycle $(x_0,x_1,\ldots,x_n)$ such that $x_i \neq x_{i+1}$ for all $i = 0,\ldots,n-1$. A simple cycle is a cycle $(x_0,\ldots,x_n)$ such that $x_i \neq x_j$ for all $0 \leq i \neq j < n$. A loop is a cycle (x_0,x_1) of length 1 where $(x_0,x_1) \in G$ and $x_0 = x_1$. We call mean exponent of a cycle the real number $\frac{1}{n}\sum_{i=0}^{n-1} a(x_i,x_{i+1})$.

Definition 19. *Suppose that* (G,M_ϵ) *is an irreducible weighted graph of exact Puiseux type with* $M_\epsilon \sim A\epsilon^a$.

i. We call minimizing mean exponent of (G,M_ϵ) *the real number*

$$\bar{a} := \min\Big\{\frac{1}{n}\sum_{i=0}^{n-1} a(x_i,x_{i+1}) : n \geq 1,\ (x_0,\ldots,x_n) \textit{ is a cycle } \Big\}.$$

We call minimizing cycle any cycle of mean exponent $\bar{a}$.

ii. We call minimizing subgraph the graph $G_{min} \subset S_{min} \times S_{min}$, *where* S_{min} *is the set of states belonging to some minimizing cycle and* G_{min} *is the union of supports of all minimizing cycles.*

iii. We call dominant spectral coefficient of M_ϵ *the spectral radius of* A_{min}

$$\bar{\alpha} := \sup\{|\lambda| \,:\, \lambda \in \mathrm{spec}(A_{min})\} = \rho_{spec}(A_{min}),$$

where $A_{min} = [A(x,y)\mathbb{1}_{G_{min}}(x,y)]_{x,y\in S}$. *Notice that* $\bar{\alpha} > 0$.

Notice that $\bar{a}$ may be obtained by minimizing on the finite set of simple cycles. Although we start with an irreducible graph, G_{min} may not be any more irreducible; G is nevertheless semi-irreducible as explained below.

Definition 20. *A graph* $G \subset S \times S$ *is said to be semi-irreducible if there exist a partition* $S = S_1 \cup \ldots \cup S_d$ *and irreducible subgraphs* $G_i \subset S_i \times S_i$ *such that* $G = G_1 \cup \ldots \cup G_d$. *Note that in* G *there is no transition from* $x_i \in S_i$ *to* $x_j \in S_j$ *for any* $1 \leq i \neq j \leq d$. *The subgraphs* G_i *are called the irreducible components of* G.

Lemma 21. *Let* (G,M_ϵ) *be an irreducible weighted graph of exact Puiseux type. Then the minimizing subgraph* G_{min} *is semi-irreducible.*

In the language of dynamical system, when (G,M_ϵ) is of exact Puiseux type, G_{min} describes the minimizing non-wandering set $\Omega(a)$ introduced in definition 6. More precisely:

Lemma 22. *Let G be an irreducible directed graph and $E, H : \Sigma_G^+ \to \mathbb{R}$ be finite-range observables. Let $M_\epsilon = A\epsilon^a = [\exp(E(x,y))\epsilon^{H(x,y)}\mathbb{1}_G(x,y)]_{x,y\in S}$. Then (G, M_ϵ) is of exact Puiseux type and satisfies:*

i. The minimizing mean exponent of (G, M_ϵ) is equal to the minimizing ergodic value of H, namely, $\bar{a} = \bar{H}$.

ii. The minimizing non-wandering set $\Omega(H)$ is a subshift of finite type

$$\Omega(H) = \{x \in \Sigma_G^+ : (x_k, x_{k+1}) \in G_{min},\ \forall\ k \geq 0\} = \Sigma_{G_{min}}^+.$$

iii. The splitting up of $\Omega(H)$ into irreducible components (see definition 15) corresponds to the splitting up of G_{min} into irreducible components $\{G_i\}_{i=1}^d$:

$$\Omega(H) = \Omega_1(H) \cup \ldots \cup \Omega_d(H),\ \text{where}$$
$$\Omega_i(H) := \{x \in \Sigma_G^+ : (x_k, x_{k+1}) \in G_i,\ \forall\ k \geq 0\}.$$

iv. The relative pressure of E to $\Omega(H)$ is related to the dominant spectral coefficient of M_ϵ by $\bar{\alpha} = \exp[\mathrm{Pres}_{\Omega(H)}(E)]$.

We now complete the notion of one-parameter family of Puiseux type.

Definition 23. *Let $G \subset S \times S$ be an irreducible directed graph. We call off-diagonal graph the subgraph of G defined by $G^{\mathit{off}} := G \backslash \{(x,x) : x \in S\}$. Notice that G^{off} is again irreducible. If (G, M_ϵ) is a weighted graph, we denote $M_\epsilon^{\mathit{off}}(x,y) := M_\epsilon(x,y)\mathbb{1}_{G^{\mathit{off}}}(x,y)$.*

Definition 24. *Following the definition 18, we say that an irreducible weighted graph (G, M_ϵ) is of general Puiseux type if*

i. The irreducible off-diagonal weighted graph $(G^{\mathit{off}}, M_\epsilon^{\mathit{off}})$ is of exact Puiseux type. Let $\bar{a}_{\mathit{off}}$ be the minimizing mean exponent of $(G^{\mathit{off}}, M_\epsilon^{\mathit{off}})$.

ii. For each $(x,y) \notin G$, $A(x,y) = 0$ and $a(x,y) = +\infty$ (by convention).

iii. For all $x \in S$, $(x,x) \in G$ and one of the two estimates holds

$$M_\epsilon(x,x) = o(\epsilon^{\bar{a}_{\mathit{off}}})\ \ (\text{by convention: } A(x,x) = 0,\ a(x,x) = +\infty) \quad \text{or}$$
$$M_\epsilon(x,x) = A(x,x)\epsilon^{a(x,x)} + o(\epsilon^{a(x,x)}),\ A(x,x) > 0,\ a(x,x) \leq \bar{a}_{\mathit{off}}.$$

Let $G^* := G \setminus \{(x,x) \in G : A(x,x) = 0\}$ and $M_\epsilon^*(x,y) := M_\epsilon(x,y)\mathbb{1}_{G^*}(x,y)$. Notice that G^* is an irreducible directed graph and (G^*, M_ϵ^*) becomes a weighted graph of exact Puiseux type. We call minimizing mean exponent $\bar{a}$ of (G, M_ϵ) the minimizing mean exponent of (G^*, M_ϵ^*). Let G_{min}^* be the minimizing subgraph of G^* and

$$A_{min}^* := [A(x,y)\mathbb{1}_{G_{min}^*}(x,y)]_{x,y\in S}.$$

We call dominant spectral coefficient $\bar{\alpha}$ the spectral radius of A_{min}^*. We call dominant subgraph $\bar{G}$ the subgraph of G defined by the union of all irreducible components of G_{min}^* of dominant spectral coefficient.

Notice that the only difference between the two notions of Puiseux type is that, in the weakest definition, M_ϵ may possess a diagonal term (positive or not) of the form $o(\epsilon^{\bar a_{off}})$. We will see soon that that these terms are negligible in the computation of the spectral radius of M_ϵ. Notice also that

$$\bar a = \min\{\bar a_{off}, a(x,x) : x \in S\}.$$

From lemma 21, the minimizing subgraph G^*_{min} is equal to a disjoint union of irreducible subgraphs: $G^*_{min} = G^*_1 \cup \ldots \cup G^*_d$, where $S_1 \cup \ldots \cup S_d$ is a partition of S^*_{min} and $G^*_i \subset S_i \times S_i$. By just permutating indices, we may consider that the first r subgraphs G^*_i have dominant spectral coefficient $\bar\alpha$. In order to do that, we adapt the notation and we say that $\bar G_i \subset \bar S_i \times \bar S_i$ has dominant spectral coefficient if the restricted matrix $A^{ii}_{min} = [A(x,y)\mathbb{1}_{\bar G_i}(x,y)]_{x,y\in\bar S_i}$ has spectral radius $\bar\alpha$.

Main notations 25. *Suppose (G, M_ϵ) is an irreducible weighted graph of general Puiseux type. Let $\bar G_1 \subset \bar S_1 \times \bar S_1, \ldots, \bar G_r \subset \bar S_r \times \bar S_r$, $1 \le r \le d$, be the set of irreducible components of G^*_{min} of dominant spectral coefficient $\bar\alpha$. Let $\bar G := \bar G_1 \cup \ldots \cup \bar G_r$ be the dominant subgraph, and $\bar S := \bar S_1 \cup \ldots \cup \bar S_r$ be the set of vertices of $\bar G$. Denote $G_0 = G \setminus \bar G$ and $S_0 = S \setminus \bar S$. We write M_ϵ as a $(r+1)\times(r+1)$ block matrix in the following way*

$$M_\epsilon = \begin{bmatrix} \oplus^r_{i,j=1} M^{ij}_\epsilon & \oplus^r_{i=1} M^{i0}_\epsilon \\ \oplus^r_{j=1} M^{0j}_\epsilon & M^{00}_\epsilon \end{bmatrix},$$

$$M^{00}_\epsilon = [M_\epsilon(x,y)]_{x,y\in S_0}, \quad M^{i0}_\epsilon = [M_\epsilon(x,y)]_{x\in\bar S_i, y\in S_0}, \quad M^{0j}_\epsilon = [M_\epsilon(x,y)]_{x\in S_0, y\in\bar S_j},$$
$$\text{and} \quad M^{ij}_\epsilon = [M_\epsilon(x,y)]_{x,y\in\bar S_i\times\bar S_j}, \ \forall\, 1 \le i,j \le r.$$

We call dominant matrix $\bar A$ the diagonal matrix obtained by keeping only the sub-matrices A^{ii}_{min} with dominant spectral radius

$$\bar A := [A(x,y)\mathbb{1}_{\bar G}(x,y)]_{x,y\in\bar S} = \begin{bmatrix} \bar A^{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar A^{rr} \end{bmatrix},$$
$$\bar A^{ii} = [A(x,y)\mathbb{1}_{\bar G_i}(x,y)]_{x,y\in\bar S_i} = A^{ii}_{min}, \quad \forall\, i = 1,\ldots,r.$$

By convention all matrices $\bar A^{ij}$, $1 \le i \ne j \le r$, are equal to 0. Notice that

$$\lambda_\epsilon := \sup\{|\lambda| \,:\, \lambda \in \mathrm{spec}(M_\epsilon)\} = \rho_{spec}(M_\epsilon)$$

is an eigenvalue of multiplicity 1 and unique on the circle $\{|\lambda| = \lambda_\epsilon\}$. Let L_ϵ and R_ϵ be the left and right eigenvectors of M_ϵ associated with the largest eigenvalue λ_ϵ

$$L_\epsilon = \oplus^r_{i=1} L^i_\epsilon \oplus L^0_\epsilon, \ R_\epsilon = \oplus^r_{i=1} R^i_\epsilon \oplus R^0_\epsilon,$$
$$\sum_{x\in S} L_\epsilon(x) R_\epsilon(x) = 1, \ \text{and} \ \sum_{x\in S} R_\epsilon(x) = 1,$$

where L_ϵ is a row vector and R_ϵ a column vector. Consider thus

$$\pi_\epsilon(x) := L_\epsilon(x)R_\epsilon(x),\ Q_\epsilon(x,y) := \frac{M_\epsilon(x,y)R_\epsilon(y)}{\lambda_\epsilon R_\epsilon(x)},\ \text{and}\ \mu_\epsilon(x,y) := \pi_\epsilon(x)Q_\epsilon(x,y).$$

For each $i = 1,\dots,r$, $\bar\alpha = \rho_{spec}(\bar A^{ii})$ is an eigenvalue of multiplicity 1 admitting a unique positive left row eigenvector $[\bar L^i(x)]_{x\in\bar S_i}$ and a unique right column eigenvector $[\bar R^i(x)]_{x\in\bar S_i}$ satisfying

$$\bar L^i\bar A^{ii} = \bar\alpha\ \bar L^i,\ \ \bar A^{ii}\bar R^i = \bar\alpha\ \bar R^i,$$
$$\sum_{x\in\bar S_i}\bar L^i(x)\bar R^i(x) = 1,\ \text{and} \sum_{x\in\bar S_i}\bar R^i(x) = 1.$$

Let $\bar\pi^i$, $\bar Q^{ii}$ and $\bar\mu_i$ be defined on $\bar G_i$ as follows

$$\bar\pi^i(x) := \bar L^i(x)\bar R^i(x),\ \bar Q^{ii}(x,y) := \frac{\bar A^{ii}(x,y)\bar R^i(y)}{\bar\alpha\bar R^i(x)},\ \bar\mu_i(x,y) := \bar\pi^i(x)\bar Q^{ii}(x,y).$$

We extend $\bar\mu_i$ on $G\setminus\bar G_i$ by 0.

In the language of dynamical system, the main known result in this setting is recalled in the following theorem.

Theorem 26. *[8, 23, 10] Let $E,H:\Sigma_G^+\to\mathbb{R}$ be finite-range observables defined on a transitive subshift of finite type Σ_G^+ given by an irreducible directed graph G. Let $\mu_{E+\beta H}$ be the Gibbs measure associated with $E+\beta H$. For $\epsilon = e^{-\beta}$, consider $M_\epsilon = [A(x,y)\epsilon^{a(x,y)}]_{x,y\in S}$ the transfer matrix, where*

$$\begin{cases} a(x,y) = H(x,y) & \text{and}\quad A(x,y) = e^{E(x,y)}, & \forall\ (x,y)\in G, \\ a(x,y) = +\infty & \text{and}\quad A(x,y) = 0, & \forall\ (x,y)\notin G. \end{cases}$$

We recall that $\mu_{E+\beta H}$ weights each cylinder $[x_0,\dots,x_n]\in\mathcal{C}_{n+1}(G)$ as

$$\mu_{E+\beta H}([x_0,\dots,x_n]) = L_\epsilon(x_0)\Big[\prod_{k=0}^{n-1}M_\epsilon(x_k,x_{k+1})\Big]R_\epsilon(x_n)/\lambda_\epsilon^n.$$

Let $\bar G_1,\dots,\bar G_r$ be the dominant irreducible components of G_{min}. Let $\bar\mu_i$ be the Gibbs measure associated with E restricted to $\Sigma_{\bar G_i}^+$,

$$\bar\mu_i([x_0,\dots,x_n]) = \bar L^i(x_0)\Big[\prod_{k=0}^{n-1}\bar A^{ii}(x_k,x_{k+1})\Big]\bar R^i(x_n)/\bar\alpha^n,\ \ \forall\,[x_0,\dots,x_n]\in\mathcal{C}_{n+1}(\bar G_i).$$

Then, the family $\{\mu_{E+\beta H}\}_\beta$ converges to

$$\mu_{min}^{E,H} := \lim_{\beta\to+\infty}\mu_{E+\beta H} = \sum_{i=1}^r c_i^{E,H}\bar\mu_i,$$

where $c_i^{E,H} = \mu_{min}^{E,H}(\bar G_i)\geq 0$ and $\sum_{i=1}^r c_i^{E,H} = 1$.

The existence of the limit in theorem 26 is the main point and was proved by Brémont in [8] using semi-algebraic techniques. Leplaideur in [23] gave a dynamical proof and has identified the limit as a barycenter of minimizing measure of maximal pressure. Akian, Bapat and Gaubert (see [2, 3]) using min-plus methods have obtained similar results. Chazottes, Gambaudo and Ugalde in [10] gave a more algorithmic proof. Nekhoroshev has obtained [26] the convergence to a zero-temperature Gibbs measure for generic one-dimensional spin systems with nearest-neighbors interaction. Chazottes and Hochman in [11] showed a one-dimensional counterexample for the convergence of Gibbs measures associated with an infinite-range interaction. They also showed there a tridimensional counterexample for a finite-range Hamiltonian H.

We intend to partially extend theorem 26 to the case of irreducible weighted graphs (G, M_ϵ) of general Puiseux type. We explain the first two steps of an algorithm based on Puiseux-series expansions. These two steps are enough to describe the limits $\lim_{\epsilon\to 0}\pi_\epsilon = \pi_{min}$ and $\lim_{\epsilon\to 0} Q_\epsilon = Q_{min}$ for matrices of small dimension. The main difficulty is to identify which irreducible components of G^*_{min} support μ_{min}. The first step consists in writing M_ϵ in a normal form; this step makes use of the notion of correctors (equivalent to the notion of sub-actions introduced in definition 12). The second step consists in aggregating all the states in the same irreducible component, obtaining thus a new weighted graph with a lower dimension.

Definition 27. *Suppose that (G, M_ϵ) is a weighted graph of general Puiseux type, $M_\epsilon \sim A\epsilon^a$, G^*_{min} is the minimizing subgraph of G^*, and $\bar a$ is the minimizing mean exponent of (G, M_ϵ). We call corrector any function $v : S \to \mathbb{R}$ such that*

$$a(x, y) \geq v(y) - v(x) + \bar a, \quad \forall\ (x, y) \in G^*.$$

The corrector is said to be backward or forward calibrated if

$$v(y) + \bar a = \min_{x:(x,y)\in G^*}\{v(x) + a(x, y)\}, \quad \forall\ y \in S \quad \textit{(backward)},$$
$$v(x) - \bar a = \max_{y:(x,y)\in G^*}\{v(y) - a(x, y)\}, \quad \forall\ x \in S \quad \textit{(forward)}.$$

It is said to be separating if

$$a(x, y) = v(y) - v(x) + \bar a, \quad \forall\ (x, y) \in G^*_{min},$$
$$a(x, y) > v(y) - v(x) + \bar a, \quad \forall\ (x, y) \in G^* \setminus G^*_{min}.$$

It is easy to show that separating correctors exist. We just want to make clear that this notion is a key part to understand the singular perturbations of Perron matrices.

Lemma 28. *The notations being given in definition 27, there exist (not necessarily unique) backward or forward calibrated correctors. There exist (not necessarily unique) separating correctors. The difference of two correctors is constant on each irreducible component.*

The first step of the algorithm is described below.

Algorithm 29 (I. Reduction to a normal form). *Let (G, M_ϵ) be an irreducible weighted graph of general Puiseux type, $M_\epsilon \sim A\epsilon^a$. From main notations 25, recall the partition of S into dominant and non dominant indices: $S = \cup_{i=1}^r \bar{S}_i \cup S_0$. For $v : S \to \mathbb{R}$ a separating corrector, denote $\Delta_\epsilon(v) := \mathrm{diag}[\epsilon^{v(x)} : x \in S]$ and $\tilde{a}(x,y) := a(x,y) + v(x) - v(y) - \bar{a} \geq 0$ for all $(x,y) \in G^*$. Then*

- $\tilde{M}_\epsilon := \Delta_\epsilon(v) M_\epsilon \Delta_\epsilon(v)^{-1} \epsilon^{-\bar{a}} = A^*_{min} + \tilde{N}_\epsilon$ *and* $\tilde{N}_\epsilon = o(1)$*;*
- $A^*_{\min} = \begin{bmatrix} \bar{A} & 0 \\ 0 & D \end{bmatrix}$*, where $\bar{A} := \mathrm{diag}[\bar{A}^{ii} : i = 1, \ldots, r]$ is the diagonal matrix of dominant matrices $\bar{A}^{ii}$, and D is a nonnegative matrix indexed by S_0 such that $\rho_{spec}(D) < \rho_{spec}(\bar{A}^{11}) = \ldots = \rho_{spec}(\bar{A}^{rr})$;*
- $(G^{\mathit{off}}, \tilde{M}_\epsilon^{\mathit{off}})$ *is an irreducible weighted graph of exact Puiseux type;*
- $\forall\, (x,y) \in G^{\mathit{off}}$, $\tilde{M}_\epsilon(x,y) \sim A(x,y)\epsilon^{\tilde{a}(x,y)}$, $A(x,y) > 0$, $\tilde{a}(x,y) \geq 0$.

Wa say that $(G, \tilde{M}_\epsilon)$ is a normal form of (G, M_ϵ). Let $\tilde{L}_\epsilon$ and $\tilde{R}_\epsilon$ denote the left and right eigenvectors of $\tilde{M}_\epsilon$ for $\tilde{\lambda}_\epsilon := \rho_{spec}(\tilde{M}_\epsilon)$. Then $\tilde{\lambda}_\epsilon = \lambda_\epsilon \epsilon^{-\bar{a}}$ and

$$\tilde{L}_\epsilon(x) = \epsilon^{-v(x)} L_\epsilon(x) \quad \textit{and} \quad \tilde{R}_\epsilon(x) = \epsilon^{v(x)} R_\epsilon(x), \quad \forall\, x \in S.$$

The following proposition extends proposition 8 in the sense that we admit a more general form of transfer matrix.

Proposition 30. *Let (G, M_ϵ) be an irreducible weighted graph of general Puiseux type. Then*

i. $\lambda_\epsilon \sim \bar{\alpha}\epsilon^{\bar{a}}$*;*

ii. $\mu_\epsilon(x,y) \to 0$ *for all* $(x,y) \notin \bar{G}$, $\pi_\epsilon(x) \to 0$ *for all* $x \in S_0$*;*

iii. any accumulation measure $\bar{\mu}$ of $(\mu_\epsilon)_{\epsilon>0}$ is of the form $\bar{\mu} = \sum_{i=1}^r \bar{\mu}(\bar{G}_i)\bar{\mu}_i$.

We recover the fact that, if G^*_{min} admits a unique irreducible component of dominant spectral coefficient ($r = 1$), then $\mu_\epsilon \to \bar{\mu}_1$, $\pi_\epsilon(x) \to \bar{\pi}^1(x)$ for all $x \in \bar{S}_1$ and $\pi_\epsilon(x) \to 0$ elsewhere.

The second step of the algorithm is an operation of aggregation.

Algorithm 31 (II. Reduction to an aggregated form). *Let (G, M_ϵ) be an irreducible weighted graph of general Puiseux type. Assume that $(G, \tilde{M}_\epsilon)$ is a normal form of (G, M_ϵ). We write*

$$\tilde{M}_\epsilon = \begin{bmatrix} \oplus_{i,j=1}^r \tilde{M}_\epsilon^{ij} & \oplus_{i=1}^r \tilde{M}_\epsilon^{i0} \\ \oplus_{j=1}^r \tilde{M}_\epsilon^{0j} & \tilde{M}_\epsilon^{00} \end{bmatrix} = \begin{bmatrix} \bar{A} & 0 \\ 0 & D \end{bmatrix} + \tilde{N}_\epsilon.$$

(Notice that $\bar{A}(x,y) = A(x,y)\mathbb{1}_{\tilde{a}(x,y)=0}$ for all $x, y \in \bar{S} = \bar{S}_1 \cup \ldots \cup \bar{S}_r$.) The right eigenvector $\tilde{R}_\epsilon$ is solution of the system

$$\begin{cases} \sum_{j=1}^r \tilde{M}_\epsilon^{ij} \tilde{R}_\epsilon^j + \tilde{M}_\epsilon^{i0} \tilde{R}_\epsilon^0 &= \tilde{\lambda}_\epsilon \tilde{R}_\epsilon^i, \quad \forall\, i = 1, \ldots, r, \\ \sum_{j=1}^r \tilde{M}_\epsilon^{0j} \tilde{R}_\epsilon^j + \tilde{M}_\epsilon^{00} \tilde{R}_\epsilon^0 &= \tilde{\lambda}_\epsilon \tilde{R}_\epsilon^0. \end{cases}$$

As $\rho_{spec}(\tilde M_\epsilon^{00}) \to \rho_{spec}(D) < \bar\alpha \sim \tilde\lambda_\epsilon$, $\tilde R_\epsilon^0$ can be written linearly with respect to $\tilde R_\epsilon^i$. We thus obtain

$$\sum_{j=1}^r \Big(\tilde M_\epsilon^{ij} + \tilde M_\epsilon^{i0} (\tilde\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1} \tilde M_\epsilon^{0j} \Big) \tilde R_\epsilon^j = \tilde\lambda_\epsilon \tilde R_\epsilon^i.$$

We take the scalar product of each equation by the left eigenvector $\bar L^i$. We extract the dominant term $\bar A$ and obtain a new weighted graph $(G^{(1)}, M_\epsilon^{(1)})$ indexed by $S^{(1)} := \{1,\dots,r\}$ defined in the following way. For $i \neq j$, let $\mathcal{P}(i,j)$ denote the set of G-admissible paths $\underline{x} := (x_0,\dots,x_n)$ such that $n \geq 1$, $x_0 \in \bar S_i$, $x_1,\dots,x_{n-1} \in S_0$ and $x_n \in \bar S_j$. Then

- *for all $i \neq j$, $(i,j) \in G^{(1)}$ if, and only if, $\mathcal{P}(i,j) \neq \emptyset$;*
- *for all $i = 1,\dots,r$, $(i,i) \in G^{(1)}$ (by convention);*
- $M_\epsilon^{(1)}(i,j) = \bar L^i \Big(\tilde N_\epsilon^{ij} + \tilde M_\epsilon^{i0} (\tilde\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1} \tilde M_\epsilon^{0j} \Big) \frac{\tilde R_\epsilon^j}{\bar L^j \tilde R_\epsilon^j}$.

The new eigenvalue problem is related to the previous one by

$$\sum_{j=1}^r M_\epsilon^{(1)}(i,j) R_\epsilon^{(1)}(j) = (\tilde\lambda_\epsilon - \bar\alpha) R_\epsilon^{(1)}(i), \quad R_\epsilon^{(1)}(i) = \bar L^i \tilde R_\epsilon^i, \quad \forall\ i = 1,\dots,r.$$

We say that $(G^{(1)}, M_\epsilon^{(1)})$ is an aggregated form of (G, M_ϵ). Note that $\sum_{i=1}^r R_\epsilon^{(1)}(i)$ may not be equal to 1.

Proposition 32. *Let (G, M_ϵ) be an irreducible weighted graph of general Puiseux type. Let $(G^{(1)}, M_\epsilon^{(1)})$ be its aggregated form defined by the separating corrector $v : S \to \mathbb{R}$. If $\tilde a(x,y) = a(x,y) + v(x) - v(y) - \bar a$ for all $(x,y) \in G^*$ and $\underline{x} = (x_0,\dots,x_n)$ belongs to $\mathcal{P}(i,j)$, denote $\tilde a(\underline{x}) := \sum_{i=0}^{n-1} \tilde a(x_i, x_{i+1})$. Then*

i. $(G^{(1)off}, M_\epsilon^{(1)off})$ is an irreducible weighted graph of exact Puiseux type, with $M_\epsilon^{(1)off} \sim A^{(1)} \epsilon^{a^{(1)}}$, where, for all $(i,j) \in G^{(1)off}$,

$$a^{(1)}(i,j) := \min \big\{ \tilde a(\underline{x}) \ :\ \underline{x} \in \mathcal{P}(i,j) \big\} \quad \textit{and}$$
$$A^{(1)}(i,j) := \sum_{\substack{\underline{x} = (x_0,\dots,x_n) \in \mathcal{P}(i,j) \\ \tilde a(\underline{x}) = a^{(1)}(i,j)}} \frac{\bar L^i(x_0) \Pi_{k=0}^{n-1} A(x_k, x_{k+1}) \bar R^j(x_n)}{\bar\alpha^{n(\underline{x})-1}};$$

ii. for all $i = 1,\dots,r$ and $x, y \in \bar S_i$,

$$\frac{L_\epsilon^i(x)}{L_\epsilon^i(y)} \sim \frac{\epsilon^{v(x)}}{\epsilon^{v(y)}} \frac{\bar L^i(x)}{\bar L^i(y)}, \quad \textit{and} \quad \frac{R_\epsilon^i(x)}{R_\epsilon^i(y)} \sim \frac{\epsilon^{-v(x)}}{\epsilon^{-v(y)}} \frac{\bar R^i(x)}{\bar R^i(y)};$$

iii. for all $i \neq j \in \{1,\dots,r\}$ and $x \in \bar S_i$,

$$Q_\epsilon(x,y) \to 0,\ \forall\ y \in \bar S_j \cup S_0, \qquad Q_\epsilon(x,y) \to \bar Q^{ii}(x,y),\ \forall\ y \in \bar S_i.$$

Notice that no estimate is given in the previous proposition for the quotients $R^i_\epsilon(x)/R^j_\epsilon(y)$ if $x \in \bar{S}_i$ and $y \in \bar{S}_j$.

Algorithm 33 (III. Induction). *Assume by induction one can prove*

$$\frac{R^{(1)}_\epsilon(i)}{R^{(1)}_\epsilon(j)} \sim \gamma^{(1)}(i,j)\epsilon^{c^{(1)}(i,j)}, \quad \forall\ i = 1,\dots,r,$$

for some real coefficients $\gamma^{(1)}(i,j) = \gamma^{(1)}(j,i)^{-1} > 0$ *and* $c^{(1)}(i,j) = -c^{(1)}(j,i)$. *Notice that proposition 32.ii easily implies*

$$\frac{\tilde{R}^i_\epsilon(x)}{R^{(1)}_\epsilon(i)} \sim \frac{\bar{R}^i(x)}{\bar{L}^i\bar{R}^i} = \bar{R}^i(x), \quad \forall\ i = 1,\dots,r, \quad \forall\ x \in \bar{S}_i.$$

Let G' *be the graph containing either* (x,x) *for* $x \in S_0$ *or* (x_0,x_n) *if* $(x_0,\dots,x_n)$ *is a path of* $G \cap (S_0 \times S_0)$ *such that* $D(x_k, x_k+1) > 0$. *Let* $M'_\epsilon = (\lambda_\epsilon - \tilde{M}^{00}_\epsilon)^{-1}$. *Then* (G', M'_ϵ) *is a weighted graph of exact Puiseux type (see lemma 49). It follows that*

$$\frac{\tilde{R}^0_\epsilon(x)}{R^{(1)}_\epsilon(1)} = \sum_{j=1}^{r}(\tilde{\lambda}_\epsilon - \tilde{M}^{00}_\epsilon)^{-1}\tilde{M}^{0j}_\epsilon \frac{\tilde{R}^j_\epsilon}{R^{(1)}_\epsilon(1)}(x) \sim \gamma^{(1)}(x)\epsilon^{c^{(1)}(x)}$$

for some coefficients $\gamma^{(1)}(x) > 0$ *and* $c^{(1)}(x) \in \mathbb{R}$. *One thus may obtain*

$$\frac{R_\epsilon(x)}{R_\epsilon(y)} \sim \gamma(x,y)\epsilon^{c(x,y)}, \quad \forall\ x,y \in S,$$

for some real coefficients $\gamma(x,y) = \gamma(y,x)^{-1} > 0$ *and* $c(x,y) = -c(y,x)$. *The normalization* $\sum_{x\in S} R_\epsilon(x) = 1$ *then implies*

$$R_\epsilon(x) = \frac{1}{\sum_{y\in S}\frac{R_\epsilon(y)}{R_\epsilon(x)}} \sim \rho(x)\epsilon^{r(x)}, \quad \forall\ x \in S, \quad \textit{with}$$

$$\rho(x) := \Big(\sum_{y=\arg\max c(x,y)} \gamma(y,x)\Big)^{-1} \quad \textit{and} \quad r(x) := \max_{y\in S} c(x,y).$$

Similar equivalences can be written for $L_\epsilon(x)$ *and* $Q_\epsilon(x,y)$. *In particular, the limits* $\lim_{\epsilon\to0}\pi_\epsilon(x)$ *and* $\lim_{\epsilon\to0} Q_\epsilon(x,y)$ *exist for all* $x,y \in S$.

4 Proofs of results stated in section 2

We begin by proving the results of section 2 for a transitive subshift of finite type (Σ^+_G, σ) defined by an irreducible directed graph G on a finite state space S. Let $E, H : \Sigma^+_G \to \mathbb{R}$ be two Hölder functions. Proposition 8 has been noticed many times as in [13, 23]. We nevertheless give the proof of this proposition in order to point out the following inequalities.

Lemma 34. *For any $\beta > 0$, $\mathrm{Pres}_{\Omega(H)}(E) \le \mathrm{Pres}(E + \beta H) + \beta \bar{H} \le \mathrm{Pres}(E)$. If $\mu_{E+\beta H}$ is the Gibbs-equilibrium measure of $E + \beta H$, then*

$$0 \le \beta\Big(\int H \, d\mu_{E+\beta H} - \bar{H} \Big) \le \mathrm{Pres}(E) - \mathrm{Pres}_{\Omega(H)}(E), \quad \textit{and}$$

$$\mathrm{Pres}_{\Omega(H)}(E) \le \mathrm{Ent}(\mu_{E+\beta H}) - \int E \, d\mu_{E+\beta H}.$$

Proof. On the one hand, if $\mu_{\min}$ is any H-minimizing probability with relative maximal pressure in $\Omega(H)$, then

$$\mathrm{Pres}_{\Omega(H)}(E) - \beta \bar{H} = \mathrm{Ent}(\mu_{\min}) - \int E \, d\mu - \beta \bar{H} =$$
$$= \mathrm{Ent}(\mu_{\min}) - \int (E + \beta H) \, d\mu_{\min} \le \mathrm{Pres}(E + \beta H).$$

On the other hand,

$$\begin{aligned} \mathrm{Pres}(E + \beta H) &= \mathrm{Ent}(\mu_{E+\beta H}) - \int (E + \beta H) \, d\mu_{E+\beta H}, \quad \text{either} \\ &\le \mathrm{Ent}(\mu_{E+\beta H}) - \int E \, d\mu_{E+\beta H} - \beta \bar{H}, \quad \text{or} \\ &\le \mathrm{Pres}(E) - \beta \int H \, d\mu_{E+\beta H} \le \mathrm{Pres}(E) - \beta \bar{H}. \end{aligned}$$

□

Proof of proposition 8. We first remark

$$0 \le \int H \, d\mu_{E+\beta H} - \bar{H} \le \frac{1}{\beta} [\mathrm{Pres}(E) - \mathrm{Pres}_{\Omega(H)}(E)]$$

implies that $\{\int H \, d\mu_{E+\beta H}\}_\beta$ converges to $\bar{H}$ as $\beta \to +\infty$ and that any weak* limit of $\{\mu_{E+\beta H}\}_\beta$ is actually minimizing for H. Let μ_∞ be a weak* accumulation probability. We next observe that the upper semi-continuity of the entropy map $\beta \mapsto \mathrm{Ent}(\mu_{E+\beta H})$ implies

$$\begin{aligned} \mathrm{Pres}_{\Omega(H)}(E) &\ge \mathrm{Ent}(\mu_\infty) - \int E \, d\mu_\infty \\ &\ge \limsup_{\beta \to +\infty} \Big(\mathrm{Ent}(\mu_{E+\beta H}) - \int E \, d\mu_{E+\beta H} \Big) \ge \mathrm{Pres}_{\Omega(H)}(E). \end{aligned}$$

All inequalities in the previous estimate are therefore equalities and lim sup should be understood as a limit. □

The rest of this part is now devoted to the proof of theorem 16. We first give some complements on the Peierls barrier. As usual, define the Birkhoff sum of an observable $\Psi : \Sigma_G^+ \to \mathbb{R}$ as

$$S_n \Psi(x) = \sum_{k=0}^{n-1} \Psi \circ \sigma^k(x), \quad \forall \, x \in \Sigma_G^+.$$

Lemma 35. *Let $h(x,y)$ be the Peierls barrier introduced in definition 14.*

i. The function $h : \Sigma_G^+ \times \Sigma_G^+ \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous.

ii. If $V : \Sigma_G^+ \to \mathbb{R}$ is a continuous sub-action, $V(y) - V(x) \leq h(x,y)$.

iii. For any $x \in \Omega(H)$, $h(x,\cdot) : \Sigma_G^+ \to \mathbb{R}$ is Hölder (and finite).

iv. For any $x,y,z \in \Sigma_G^+$, $h(x,z) \leq h(x,y) + h(y,z)$.

v. For any $y \in \Sigma_G^+$, $h(\cdot,y) : \Sigma_G^+ \to \mathbb{R} \cup \{+\infty\}$ is a coboundary of $H - \bar{H}$,

$$(H - \bar{H})(x) + h(\sigma(x),y) = h(x,y), \quad \forall\ x,y \in \Sigma_G^+.$$

vi. For any $x \in \Sigma_G^+$, $\sigma^n(x) \in \Omega(H) \Rightarrow h(x,\sigma^n(x)) = S_n(H - \bar{H})(x)$.

Proof. Items i, ii, iii and iv are well known and have been discussed, for instance, in [13, 16, 17].

Item v. Suppose $\epsilon \in (0,1)$. If z' is close to $\sigma(x)$, $d(z',\sigma(x)) < \epsilon$, one can find z close to x, $d(z,x) < \epsilon/2$, such that $\sigma(z) = z'$. Hence, if $\mathrm{osc}_1(H,\eta) := \sup\big\{H(x) - H(y)\ :\ d(x,y) \leq \eta\big\}$, then

$$S_{n+1}^{\epsilon/2}(x,y) \leq (H - \bar{H})(x) + S_n^{\epsilon}(\sigma(x),y) + \mathrm{osc}_1(H,\epsilon/2).$$

Conversely, if $d(z,x) < \epsilon$, then $d(\sigma(z),\sigma(x)) < 2\epsilon$. Therefore

$$S_{n+1}^{\epsilon}(x,y) \geq (H - \bar{H})(x) + S_n^{2\epsilon}(\sigma(x),y) - \mathrm{osc}_1(H,\epsilon).$$

Item v is proved by taking $\liminf_{n\to+\infty}$ first and $\lim_{\epsilon\to 0}$ afterwards.

Item vi. From the previous item, we have by induction

$$S_n(H - \bar{H})(x) + h(\sigma^n(x),y) = h(x,y).$$

If $y = \sigma^n(x) \in \Omega(H)$, then $h(y,y) = 0$ and item vi is proved. □

From now on the minimizing non-wandering set $\Omega(H)$ can be decomposed into a disjoint union of irreducible components $\Omega(H) = \Omega_0 \cup \ldots \cup \Omega_r$ (see definition 15). Each Ω_i is necessarily closed and invariant. We fixed once for all $x_i^* \in \Omega_i$. We recall that $\Omega_i = \{x \in \Sigma_G^+\ :\ h(x,x_i^*) + h(x_i^*,x) = 0\}$ and that, for any $i \neq j$, $h(x_i^*,x_j^*) + h(x_j^*,x_i^*) > 0$.

Lemma 36. *Assume $\Omega(H) = \Omega_0 \cup \ldots \cup \Omega_r$ is a disjoint union of irreducible components. Let $V : \Sigma_G^+ \to \mathbb{R}$ be any continuous sub-action. Then*

i. The quantities $\bar{h}_V(i,j) := h(x_i^*,x_j^*) - V(x_j^*) + V(x_i^*)$ are nonnegative and independent of the choice of $x_i^* \in \Omega_i$.

ii. $\bar{h}_V(i,i) = 0$ for all $i = 0,1,\ldots,r$.

iii. If $\bar{h}_V(0,j) = 0$ for all $j = 1,\ldots,r$ and V is a calibrated sub-action, then $V(y) - V(x) = h(x,y)$ for all $x \in \Omega_0$ and $y \in \Sigma_G^+$, that is, V is unique provided $V(x_0)$ is known for some $x_0 \in \Omega_0$.

Proof. Item i. Let $h_V(x,y) := h(x,y) - V(y) + V(x) \ge 0$ for all $x, y \in \Sigma_G^+$. Hence, $x \sim y$ if, and only if, $h_V(x,y) + h_V(y,x) = 0$ if, and only if, $h_V(x,y) = h_V(y,x) = 0$. Suppose $x, x', y, y' \in \Omega(H)$ satisfy $x \sim x'$ and $y \sim y'$. Because of lemma 35.*iv*,

$$h_V(x,y) \le h_V(x,x') + h_V(x',y) = h_V(x',y).$$

Equivalently $h_V(x',y) \le h_V(x,y)$ and thus $h_V(x',y) = h_V(x,y)$. For the same reason, $h_V(x',y) = h_V(x',y')$. We just have proved $h_V(x,y) = h_V(x',y')$.

Item ii. It is immediate from the definition of $\bar h_V$.

Item iii. From [5, 16], calibrated sub-actions have the following characterization $V(y) = \min\{V(x) + h(x,y) \ : \ x \in \Omega(H)\}$ for all $y \in \Sigma_G^+$. Then, for any fixed $x_0 \in \Omega_0$, on the one hand,

$$\begin{aligned} V(y) &= \min_{j=0,\ldots,r} \min_{x\in\Omega_j} [V(x) + h(x,y)] \\ &\ge \min_{j=0,\ldots,r} \min_{x\in\Omega_j} [V(x) - V(x_0) + h(x,x_0)] + V(x_0) + h(x_0,y) \\ &= V(x_0) + h(x_0,y). \end{aligned}$$

On the other hand, because V is a sub-action, $h(x_0,y) \ge V(y) - V(x_0)$. We have proved that $V(y) = V(x) + h(x,y)$ for all $x \in \Omega_0$ and $y \in \Sigma_G^+$. $\square$

Let $\Phi_{E+\beta H} = \exp(-\beta V_{E+\beta H})$ and $\nu_{E+\beta h}$ be, respectively, the eigenfunction and the eigenmeasure of the Ruelle transfer operator $\mathcal{L}_{E+\beta H}$, normalized by $\int \Phi_{E+\beta H}\, d\nu_{E+\beta H} = 1$. We know that $\{V_{E+\beta H}\}_\beta$ has uniform sup-norm and uniform Hölder norm. Let V_∞ be any accumulation point in the C^0 topology. Proposition 13 tells us that V_∞ is calibrated. We assume that $\mathrm{Pres}_{\Omega_0}(E) > \mathrm{Pres}_{\Omega_1\cup\ldots\cup\Omega_r}(E)$. We want to prove that $V_\infty(y) - V_\infty(x) = h(x,y)$ for any $x \in \Omega_0$ and $y \in \Sigma_G^+$, which will show that, for any fixed $x_0 \in \Omega_0$,

$$V_{E+\beta H}(y) - V_{E+\beta H}(x_0) \to V_\infty(y) - V_\infty(x_0), \quad \text{uniformly in } y \in \Sigma_G^+.$$

That convergence will indeed follow from lemma 36.*iii* and the next lemma.

Lemma 37. *Let $V : \Sigma_G^+ \to \mathbb{R}$ be any sub-action and $\bar h_V(i,j)$ be defined as in lemma 36. Assume, for any $j = 1,\ldots,r$, there exists $i = 0,1,\ldots,r$, $i \ne j$, such that $\bar h_V(i,j) = 0$. Then $\bar h_V(0,j) = 0$ for all $j = 1,\ldots,r$.*

Proof. Assume by contradiction that $\bar h_V(0,j_1) > 0$ for some $j_1 = 1,\ldots,r$. Define $J := \{j = 1,\ldots,r \ : \ \bar h_V(0,j) > 0\}$. Notice that if $j_1 \in J$ and $\bar h_V(j_2,j_1) = 0$ for some $j_2 = 0,1,\ldots,r$, $j_2 \ne j_1$, then necessarily $j_2 \ne 0$ and $j_2 \in J$. By hypothesis, one can therefore construct a sequence $j_1, j_2, \ldots \in J$ such that

$$\ldots = \bar h_V(j_3,j_2) = \bar h_V(j_2,j_1) = 0 \quad \text{and} \quad j_{k+1} \ne j_k.$$

Because the number of irreducible components is finite, there exist two distinct indices $s < t$ such that $\bar h_V(j_t,j_{t-1}) = \ldots = \bar h_V(j_{s+1},j_s) = 0$ and $j_s = j_t$. We obtain, for instance, $\bar h_V(j_s,j_{s+1}) = 0 = \bar h_V(j_{s+1},j_s)$, which is in contradiction with $\Omega_{j_{s+1}} \ne \Omega_{j_s}$. $\square$

In order to apply the initial assumption of lemma 37, we fix from now on $j = 1, \dots, r$, $\tilde\Omega = \Omega_j$ and $\bar\Omega = \cup_{i \neq j}\Omega_i$. Clearly, $\bar\Omega$ and $\tilde\Omega$ are disjoint closed invariant sets and $\mathrm{Pres}_{\bar\Omega}(E) > \mathrm{Pres}_{\tilde\Omega}(E)$. We want to show that

$$\min\{h(x,y) - V_\infty(y) + V_\infty(x) \ : \ x \in \bar\Omega \text{ and } y \in \tilde\Omega\} = 0.$$

We begin by introducing some notations.

Notations 38. *Let $V : \Sigma_G^+ \to \mathbb{R}$ be any Hölder sub-action. Consider the function*

$$h_V(x,y) := h(x,y) - V(y) + V(x) \geq 0, \quad \forall\ x, y \in \Sigma_G^+,$$

which is the Peierls barrier of the observable $H_V := H - \bar H - V \circ \sigma + V \geq 0$. Assume that $\Omega(H) = \bar\Omega \cup \tilde\Omega$ is a disjoint union of two closed σ-invariant sets with $\tilde\Omega$ irreducible. For $\epsilon > 0$, denote

$$K_V(\tilde\Omega, \epsilon) := \{x \in \Sigma_G^+ \ : \ \exists\ y \in \tilde\Omega \ \ s.\ t. \ \ h_V(x,y) \leq \epsilon\}.$$

We will need to approximate $\mathrm{Pres}_{\tilde\Omega}(E)$ by the pressure of E restricted to transitive subshifts of finite type $\tilde\Sigma_d \supset \tilde\Omega$ which decrease to $\tilde\Omega$. In order to introduce them, the following notion will be useful.

Definition 39. *A closed σ-invariant set $\tilde\Omega \subset \Sigma_G^+$ is said to be quasi-transitive if, for any $x, y \in \tilde\Omega$, for any $\epsilon > 0$, there exist $z \in \Sigma_G^+$ and an integer $n \geq 0$ such that*

$$d(z,x) < \epsilon, \quad d(\sigma^n(z), y) < \epsilon \quad \text{and} \quad d(\sigma^k(z), \tilde\Omega) < \epsilon, \quad \forall\ k = 0, 1, \dots, n.$$

Lemma 40. *Any isolated irreducible component $\tilde\Omega$ of $\Omega(H)$ (there exists an open set $\tilde U$ containing $\tilde\Omega$ such that $\tilde U \cap \Omega(H) = \tilde\Omega$) is quasi-transitive.*

Proof. Let V be any Hölder separating sub-action, namely, a Hölder sub-action such that $H_V^{-1}(0) = \Omega(H)$ (for details, see [17]). For $\epsilon > 0$, let U_ϵ and $\tilde U_\epsilon$ be neighborhoods of size ϵ of $\Omega(H)$ and $\tilde\Omega$, respectively. Assume ϵ is sufficiently small enough so that if $z \in \tilde U_\epsilon$ and $k \geq 1$ is the first time such that $\sigma^{k-1}(z) \in \tilde U_\epsilon$ and $\sigma^k(z) \notin \tilde U_\epsilon$, then $\sigma^k(z) \notin U_\epsilon$. Let $\eta > 0$ sufficiently small enough so that $\{z \in \Sigma_G^+ : H_V(z) < \eta\} \subset U_\epsilon$. Since $\tilde\Omega$ is irreducible, given $x, y \in \tilde\Omega$, there exist infinitely many positive integers n and points $z_n \in \Sigma_G^+$ such that

$$d(z_n, x) < \epsilon, \quad d(\sigma^n(z_n), y) < \epsilon \quad \text{and} \quad S_n H_V(z_n) < \eta.$$

Since $z_n \in \tilde U_\epsilon$ and $H_V \circ \sigma^k(z_n) < \eta$, then $\sigma^k(z_n) \in \tilde U_\epsilon$, $\forall\ k = 0, 1, \dots, n$. $\square$

Lemma 41. *Let $\tilde\Omega$ be a quasi-transitive closed σ-invariant set. Let $\tilde U_d$ be the union of all cylinders $B = [x_0, x_1, \dots, x_{d-1}]$ of length d such that $B \cap \tilde\Omega \neq \emptyset$. Consider $\tilde\Sigma_d = \{x \in \Sigma_G^+ : \sigma^n(x) \in \tilde U_d, \ \forall\ n \geq 0\} \supset \tilde\Omega$. Then*

i. $(\tilde\Sigma_d, \sigma)$ is bi-Hölder conjugate to a transitive subshift of finite type.

ii. There exists a constant $\tilde C_d > 0$ such that

$$\tilde C_d^{-1} \leq \sum_{\substack{x \in \tilde\Sigma_d \\ \sigma^n(x) = y}} \exp[-S_n(E + \mathrm{Pres}_{\tilde\Sigma_d}(E))(x)] \leq \tilde C_d, \quad \forall\ y \in \tilde\Sigma_d, \ \forall\ n \geq 0.$$

iii. $\lim_{d\to+\infty} \mathrm{Pres}_{\tilde\Sigma_d}(E) = \mathrm{Pres}_{\tilde\Omega}(E)$.

Proof. Item *i*. Let $\tilde S(d)$ be the set of cylinders $[x_0,\ldots,x_{d-1}]$ which have a non-empty intersection with $\tilde\Omega$. Let $\tilde G(d) \subset \tilde S(d) \times \tilde S(d)$ be the graph defined by the transitions

$$[x_0,\ldots,x_{d-1}] \overset{\tilde G(d)}{\longrightarrow} [x'_1,\ldots,x'_d] \;\Leftrightarrow\; (x_1,\ldots,x_{d-1}) = (x'_1,\ldots,x'_{d-1}) \text{ and } x_{d-1} \overset{G}{\to} x'_d.$$

Let $\Sigma^+_{\tilde G(d)}$ be the subshift of finite given by the graph $\tilde G(d)$. Thus $\Sigma^+_{\tilde G(d)}$ is transitive since $\tilde\Omega$ is quasi-transitive and $\Sigma^+_{\tilde G(d)}$ is bi-Hölder conjugate to $\tilde\Sigma_d$ by the conjugacy $\{[x^n_0,\ldots,x^n_{d-1}]\}_{n\ge0} \mapsto \{x^n_0\}_{n\ge0}$.

Item *ii*. This estimate is true for any transitive subshift of finite type, being invariant under topological conjugacy.

Item *iii*. Since $\tilde\Omega \subset \tilde\Sigma_d$, we have on the one hand $\mathrm{Pres}_{\tilde\Omega}(E) \le \mathrm{Pres}_{\tilde\Sigma_d}(E)$. On the other hand, if $\tilde\mu_d$ denotes the equilibrium measure associated with the observable $E : \tilde\Sigma_d \to \mathbb{R}$ and $\tilde\mu_\infty$ denotes an accumulation point of $\{\tilde\mu_d\}_{d\to+\infty}$, then $\mathrm{supp}(\tilde\mu_\infty) \subset \tilde\Omega$ and

$$\begin{aligned}\limsup_{d\to+\infty} \mathrm{Pres}_{\tilde\Sigma_d}(E) &= \limsup_{d\to+\infty} \Big(\mathrm{Ent}(\tilde\mu_d) - \int E\, d\tilde\mu_d\Big)\\ &\le \mathrm{Ent}(\tilde\mu_\infty) - \int E\, d\tilde\mu_\infty \le \mathrm{Pres}_{\tilde\Omega}(E).\end{aligned}$$

We have proved that $\mathrm{Pres}_{\tilde\Sigma_d}(E) \to \mathrm{Pres}_{\tilde\Omega}(E)$. □

Lemma 42. *Consider the decomposition $\Omega(H) = \bar\Omega \cup \tilde\Omega$ as in notations 38. For a Hölder sub-action $V : \Sigma^+_G \to \mathbb{R}$, assume $\min\{h_V(x,y) : x \in \bar\Omega$ and $y \in \tilde\Omega\} > \epsilon > 0$. Then*

i. $K_V(\tilde\Omega,\epsilon)$ is closed, invariant and disjoint from $\bar\Omega$. Moreover,

$$S_nH_V(x) \le \epsilon, \quad \forall\, x \in K_V(\tilde\Omega,\epsilon),\ \forall\, n \ge 0.$$

ii. If $\tilde U \supset \tilde\Omega$ is open and disjoint from $\bar\Omega$, then

$$\sup_{x\in K_V(\tilde\Omega,\epsilon),\ n\ge1} \mathrm{card}\{j = 0,1,\ldots,n-1 \;:\; \sigma^j(x) \notin \tilde U\} < +\infty.$$

(Every orbit of $K_V(\tilde\Omega,\epsilon)$ stays most of the time in $\tilde U$.)

iii. If $\tilde C(n) := \sup\Big\{ \sum_{x\in K_V(\tilde\Omega,\epsilon),\ \sigma^n(x)=y} \exp[-S_n(E + \mathrm{Pres}_{\tilde\Omega}(E))(x)] : y \in \tilde\Omega\Big\}$ for every $n \ge 1$, then $\limsup_{n\to+\infty} \frac1n \ln \tilde C(n) \le 0$.

Proof. For simplicity, denote $\tilde K = K_V(\tilde\Omega,\epsilon)$.

Item *i*. Since $h(x,y)$ is lower semi-continuous and $\tilde\Omega$ is compact, we deduce that $\tilde K$ is closed. From lemma 35.*v*, we have

$$h_V(\sigma(x),y) \le H_V(x) + h_V(\sigma(x),y) = h_V(x,y), \quad \forall\, x,y \in \Sigma^+_G.$$

In particular, $h_V(x,y) \le \epsilon \Rightarrow h_V(\sigma(x),y) \le \epsilon$, which shows that $\tilde{K}$ is invariant. Iterating this last formula, we also obtain

$$S_n H_V(x) \le S_n H_V(x) + h_V(\sigma^n(x),y) \le h_V(x,y), \quad \forall\, x,y \in \Sigma_G^+.$$

Hence, $S_n H_V(x)$ is uniformly bounded on $n \ge 0$ and $x \in \tilde{K}$.

Item *ii*. Suppose by contradiction there exist a sequence of points $\{x_n\}_{n\ge1}$ of $\tilde{K}$ such that

$$\mathrm{card}\{j = 0,1,\ldots,n : \sigma^j(x_n) \notin \tilde{U}\} \to +\infty.$$

Let $\eta_0 > \eta_1 > \cdots$ be a sequence of positive real numbers decreasing to 0. Let $\{B_i(\eta_0)\}_i$ be a finite cover of $\tilde{K}\backslash\tilde{U}$ by balls of radius η_0. One of these balls contains infinitely many points of $\{\sigma^j(x_n) : j = 0,1,\ldots,n,\ n \ge 1\}$. More precisely, there exist a subsequence $\{x_{k_0(n)}\}_{n\ge1}$ (with $k_0 : \mathbb{N} \to \mathbb{N}$ increasing) and a ball B_{i_0} of radius η_0 such that

$$\mathrm{card}\{j = 0,1,\ldots,k_0(n) : \sigma^j(x_{k_0(n)}) \in B_{i_0}\} \to +\infty.$$

By covering B_{i_0} by balls $\{B_i(\eta_1)\}_i$ of radius η_1, one can extract a second subsequence $\{x_{k_0\circ k_1(n)}\}_{n\ge1}$ (with $k_1 : \mathbb{N} \to \mathbb{N}$ increasing) and choose one of these balls B_{i_1} so that

$$\mathrm{card}\{j = 0,1,\ldots,k_0\circ k_1(n) : \sigma^j(x_{k_0\circ k_1(n)}) \in B_{i_1}\} \to +\infty.$$

We continue by induction. Let $k^j(n) = k_0 \circ \ldots \circ k_j(n)$ and z be an accumulation point of $\{B_{i_j}\}_{j\ge0}$. Let

$$0 = s_0^j < s_1^j < \ldots < s_{r^j(n)-1}^j < s_{r^j(n)}^j = k^j(n)$$

be the successive times $\{s_l^j\}_{l=1}^{r^j(n)-1}$ such that $\sigma^{s_l^j}(x_{k^j(n)}) \in B_{i_j}$. By construction $r^j(n) \to +\infty$. Notice that

$$\sum_{l=0}^{r^j(n)-1} S_{(s_{l+1}^j - s_l^j)} H_V \circ \sigma^{s_l^j}(x_{k^j(n)}) = S_{k^j(n)} H_V(x_{k^j(n)}) \le \epsilon.$$

Therefore, for infinitely many indices j, one can consider $z_j := \sigma^{s_l^j}(x_{k^j(n)})$ and $n_j := s_{l+1}^j - s_l^j$ for some $l = 1,\ldots,r^j(n)-1$ in such a way that $S_{n_j}H_V(z_j) \to 0$. As $z_j, \sigma^{n_j}(z_j), z \in B_{i_j}$ and $\mathrm{diam}(B_{i_j}) \to 0$, we have proved that $z \in \Omega(H) = \bar{\Omega}\cup\tilde{\Omega}$. Since $z \in \tilde{K}\setminus\tilde{U}$ and $\tilde{K}\setminus\tilde{U}$ is disjoint from $\bar{\Omega}$ and $\tilde{\Omega}$, we obtain a contradiction.

Item *iii*. Let $S(d)$ be the set of non-empty cylinders of Σ_G^+ of size d and $G(d) \subset S(d)\times S(d)$ be the graph whose transitions are given by

$$[x_0,\ldots,x_{d-1}] \overset{G(d)}{\longrightarrow} [x_1',\ldots,x_d'] \ \Leftrightarrow\ (x_1,\ldots,x_{d-1}) = (x_1',\ldots,x_{d-1}') \text{ and } x_{d-1} \overset{G}{\to} x_d'.$$

Denote the oscillation of the Birkhoff sums of E by

$$\mathrm{osc}_n(E) := \sup_{\gamma,x,y}\{S_nE(|\gamma x\rangle) - S_nE(|\gamma y\rangle) :$$
$$\gamma = v_{-n}\ldots v_{-2}v_{-1},\ v_{-1} \overset{G(d)}{\longrightarrow} x \text{ and } v_{-1} \overset{G(d)}{\longrightarrow} y\},$$

where $|\gamma x\rangle$ is the concatenation of a finite $G(d)$-admissible path $\gamma = v_{-n} \dots v_{-2}v_{-1}$ in $S(d)$ and a point x in Σ_G^+, and $v_{-1} \overset{G(d)}{\longrightarrow} x$ just denotes $v_{-1} \overset{G(d)}{\longrightarrow} [x_0, \dots, x_{d-1}]$. Hence, if $v_{-i} = [v_{-i}^0, \dots, v_{-i}^{d-1}] \in S(d)$, $i = 1, \dots, n$, then

$$|\gamma x\rangle := (v_{-n}^0, \dots, v_{-1}^0, x_0, x_1, \dots) \in \Sigma_G^+.$$

More generally, if $\gamma = v_{-n} \dots v_{-1}$ and $\gamma' = v'_{-n'} \dots v'_{-1}$ are $G(d)$-admissible paths of length n and n', we say that γ can be concatenated to γ' if $v_{-1} \overset{G(d)}{\longrightarrow} v'_{-n'}$. Write then $\gamma\gamma' = v_{-n} \dots v_{-1}v'_{-n'} \dots v'_{-1}$.

As in the proof of lemma 41.i, we also consider $\tilde{S}(d)$ the set of vertices $[x_0, \dots, x_{d-1}] \in S(d)$ such that $[x_0, \dots, x_{d-1}] \cap \tilde{\Omega} \neq \emptyset$ and the subgraph $\tilde{G}(d) = G(d) \cap \tilde{S}(d) \times \tilde{S}(d)$. We choose once for all a finite set $\tilde{\Gamma}_d$ of $\tilde{G}(d)$-admissible paths which connects all vertices of $\tilde{S}(d)$ to all vertices of $\tilde{S}(d)$. Given $y \in \tilde{\Omega}$, each inverse branch of order n of y can be written as $x = |\gamma y\rangle$, where $\gamma = v_{-n} \dots v_{-1}$ is a $G(d)$-admissible path and $v_{-1} \overset{G(d)}{\longrightarrow} v_0 := [y_0, \dots, y_{d-1}]$. We partition γ into sub-paths so that alternatively γ_{2i} is a path in $\tilde{S}(d)$ and γ_{2i+1} is a path in $S(d) \setminus \tilde{S}(d)$. More precisely, we consider $\gamma = \gamma_r \dots \gamma_1\gamma_0$ as concatenation of paths γ_i of length n_i (possibly $n_0 = 0$ if $v_{-1} \notin \tilde{S}(d)$ and γ_0 is the empty path) in such a way that

$$\begin{aligned}
&\gamma_0 = v_{-(n_0)} \dots v_{-(1)} \quad \text{is a path in } \tilde{S}(d), \\
&\gamma_1 = v_{-(n_0+n_1)} \dots v_{-(n_0+1)} \quad \text{is a path in } S(d) \setminus \tilde{S}(d), \\
&\gamma_2 = v_{-(-n_0+n_1+n_2)} \dots v_{-(n_0+n_1+1)} \quad \text{is a path in } \tilde{S}(d), \quad \textit{et cetera.}
\end{aligned}$$

We associate with each such an inverse branch γ a new path $\tilde{\gamma}$ in $\tilde{S}(d)$ of the form $\tilde{\gamma} = \tilde{\gamma}_r \dots \tilde{\gamma}_0$, given by the concatenation of paths $\tilde{\gamma}_i$ of length $\tilde{n}_i$ such that $\tilde{\gamma}_{2i} = \gamma_{2i}$ and each sub-path γ_{2i+1} outside $\tilde{S}(d)$ has been replaced by a sub-path $\tilde{\gamma}_{2i+1} = \tilde{v}_{-(\tilde{n}_0+\ldots+\tilde{n}_{2i+1})} \dots \tilde{v}_{-(\tilde{n}_0+\ldots+\tilde{n}_{2i}+1)}$ in $\tilde{S}(d)$ chosen in $\tilde{\Gamma}_d$ so that

$$\tilde{v}_{-(\tilde{n}_0+\ldots+\tilde{n}_{2i}+1)} \overset{\tilde{G}(d)}{\longrightarrow} \tilde{v}_{-(\tilde{n}_0+\ldots+\tilde{n}_{2i})} \text{ and } \tilde{v}_{-(\tilde{n}_0+\ldots+\tilde{n}_{2i+1}+1)} \overset{\tilde{G}(d)}{\longrightarrow} \tilde{v}_{-(\tilde{n}_0+\ldots+\tilde{n}_{2i+1})}.$$

Let $\tilde{n} = \tilde{n}_0 + \tilde{n}_1 \ldots + \tilde{n}_r$ be the length of the path $\tilde{\gamma}$. Denote $x_i = |\gamma_i\gamma_{i-1} \dots \gamma_0 y\rangle$ and $\tilde{x}_i = |\tilde{\gamma}_i\tilde{\gamma}_{i-1} \dots \tilde{\gamma}_0 y\rangle$. We want to compare

$$S_n E(|\gamma y\rangle) = \sum_{i=0}^{r} S_{n_i} E(x_i) \quad \text{and} \quad S_{\tilde{n}} E(|\tilde{\gamma} y\rangle) = \sum_{i=0}^{r} S_{\tilde{n}_i} E(\tilde{x}_i).$$

Either γ_i corresponds to a path outside $\tilde{S}(d)$, then

$$S_{n_i} E(x_i) \geq S_{\tilde{n}_i} E(\tilde{x}_i) - (n_i + \tilde{n}_i)\|E\|_\infty,$$

or γ_i corresponds to a path inside $\tilde{S}(d)$, then $\tilde{\gamma}_i = \gamma_i$, $\tilde{x}_i$ and x_i have the same symbols during a period $n_i = \tilde{n}_i$,

$$S_{n_i} E(x_i) \geq S_{\tilde{n}_i} E(\tilde{x}_i) - \operatorname{osc}_{n_i}(E).$$

Let $\tilde{L}_d$ be the maximal length of paths in $\tilde{\Gamma}_d$. Then

$$S_n E(|\gamma y\rangle) \geq S_{\tilde{n}} E(|\tilde{\gamma} y\rangle) - \sum_{i \text{ odd}} n_i (1 + \tilde{L}_d) \|E\|_\infty - \sum_{i \text{ even}} \sup_n \mathrm{osc}_n(E).$$

Since $\mathrm{card}\{i: \ i \text{ even}\} \leq \mathrm{card}\{i: \ i \text{ odd}\} + 1 \leq 2\sum_{i \text{ odd}} n_i$, we obtain

$$S_n E(|\gamma y\rangle) \geq S_{\tilde{n}} E(|\tilde{\gamma} y\rangle) - [(1 + \tilde{L}_d)\|E\|_\infty + 2\sup_n \mathrm{osc}_n(E)] \sum_{i \text{ odd}} n_i.$$

We assume from now on that the inverse branch $x = |\gamma y\rangle$ belongs to $\tilde{K}$. From item *ii*, we know that $\sum_{i \text{ odd}} n_i \leq \tilde{N}_d$ is bounded by a constant independent of x and n which only depends on the neighborhood of $\tilde{\Omega}$, $\tilde{U}_d = \cup\{C : C \in \tilde{S}(d)\}$ for d sufficiently large enough. Notice that

$$\sum_{i \text{ odd}} \tilde{n}_i \leq \sum_{i \text{ odd}} \tilde{L}_d \leq \sum_{i \text{ odd}} n_i \tilde{L}_d \leq \tilde{N}_d \tilde{L}_d.$$

We obtain in particular $\tilde{n} = \sum_{i=0}^r \tilde{n}_i \in [n - \tilde{N}_d, n + \tilde{N}_d \tilde{L}_d]$.

In the previous construction, we associate with an inverse branch $x = |\gamma y\rangle \in \tilde{K}$ of length n of y a new inverse branch $\tilde{x} = |\tilde{\gamma} y\rangle$ of length $\tilde{n}$ for the subshift of finite type $(\tilde{\Sigma}_d, \sigma)$ as defined in lemma 41. Since the association $x \mapsto \tilde{x}$ is not injective, we want to bound from above the cardinal of each fiber. Hence, if $\tilde{\gamma}$ has length $\tilde{n} \geq 3\tilde{N}_d$, fix a partition $\tilde{I}_r \cup \ldots \cup \tilde{I}_0$ of $\{-\tilde{n}, \ldots, -1\}$ into $r+1$ disjoint consecutive intervals, with $r \in \{1, \ldots, 3\tilde{N}_d\}$, in order to determine a decomposition $\tilde{\gamma} = \tilde{\gamma}_r \ldots \tilde{\gamma}_0$ such that $\tilde{\gamma}_i$ has length $\mathrm{card}(\tilde{I}_i)$. The possible $\gamma = \gamma_r \ldots \gamma_0$ associated with $\tilde{\gamma} = \tilde{\gamma}_r \ldots \tilde{\gamma}_0$ must have length $n \in [\tilde{n} - \tilde{N}_d \tilde{L}_d, \tilde{n} + \tilde{N}_d]$ and each γ_{2i+1} has length at most $\tilde{N}_d$. The cardinal of each fiber is thus bound from above by

$$[\tilde{N}_d(\tilde{L}_d + 1) + 1] \Big(\sum_{k=1}^{\tilde{N}_d} (\mathrm{card}(S))^k \Big)^{\tilde{N}_d} \sum_{r=1}^{3\tilde{N}_d} \binom{\tilde{n}}{r} \leq \tilde{C}'_d \ n^{3\tilde{N}_d},$$

for some constant $\tilde{C}'_d$ depending only on d. Let

$$\tilde{C}''_d := \tilde{C}'_d \exp[((1 + \tilde{L}_d)\|E\|_\infty + 2\sup_n \mathrm{osc}_n(E))\tilde{N}_d].$$

Then

$$\sum_{x \in \tilde{K}, \ \sigma^n(x) = y} \exp[-S_n E(x)] \leq \tilde{C}''_d \ n^{3\tilde{N}_d} \sum_{\tilde{n} = n - \tilde{N}_d}^{n + \tilde{N}_d \tilde{L}_d} \ \sum_{\tilde{x} \in \tilde{\Sigma}_d, \ \sigma^{\tilde{n}}(\tilde{x}) = y} \exp[-S_{\tilde{n}} E(\tilde{x})].$$

Denote $\tilde{C}'''_d := \tilde{C}''_d [\tilde{N}_d(\tilde{L}_d + 1) + 1] \tilde{C}_d \exp[\tilde{N}_d \tilde{L}_d \mathrm{Pres}_{\tilde{\Sigma}_d}(E)]$, where $\tilde{C}_d$ is the positive constant given by lemma 41.*ii*. Therefore, we get

$$\sum_{x \in \tilde{K}, \ \sigma^n(x) = y} \exp[-S_n E(x)] \leq \tilde{C}'''_d \ n^{3\tilde{N}_d} \ \exp[n \mathrm{Pres}_{\tilde{\Sigma}_d}(E)].$$

Since $\mathrm{Pres}_{\tilde{\Sigma}_d}(E) \to \mathrm{Pres}_{\tilde{\Omega}}(E)$, we finally obtain

$$\limsup_{n\to+\infty} \frac{1}{n}\ln\Big(\sup\Big\{\sum_{x\in\tilde{K},\ \sigma^n(x)=y} \exp[-S_n(E+\mathrm{Pres}_{\tilde{\Omega}}(E))(x)] : y\in\tilde{\Omega}\Big\}\Big) \le 0.$$

□

In order to prove theorem 16, we summarize in the following proposition the main technical result, which consists in relating the pressure of disjoint parts of the minimizing non-wandering set $\Omega(H)$ and the levels of the Peierls barrier $h(x,y)$ between these parts.

Proposition 43. *Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be Hölder observables. Assume $\Omega(H)$ can be written as a disjoint union $\Omega(H) = \bar{\Omega}\cup\tilde{\Omega}$ of two closed invariant sets. Assume $\tilde{\Omega}$ is irreducible. Let V_∞ be any accumulation point (in the C^0 topology) of $\{V_{E+\beta H}\}_{\beta\to+\infty}$ where $\Phi_{E+\beta H} = \exp(-V_{E+\beta H})$ is the right eigenfunction of the Ruelle operator $\mathcal{L}_{E+\beta H}$ normalized by $\int \Phi_{E+\beta H}\, d\nu_{E+\beta H} = 1$. Then*

$$\mathrm{Pres}_{\bar{\Omega}}(E) > \mathrm{Pres}_{\tilde{\Omega}}(E) \Longrightarrow \min_{x\in\bar{\Omega},\ y\in\tilde{\Omega}} h(x,y) - V_\infty(y) + V_\infty(x) = 0$$

Proof. By contradiction, we suppose that

$$\min_{x\in\bar{\Omega},\ y\in\tilde{\Omega}} h_{V_\infty}(x,y) > \epsilon > 0.$$

Let $\tilde{K} = K_V(\tilde{\Omega},\epsilon)$ as in notation 38. We consider $\Phi_{E+\beta H}$ as an eigenfunction of $\mathcal{L}^n_{E+\beta H}$ for some $n = n(\beta)$ that will be chosen later. Given $y\in\tilde{\Omega}$, we thus have

$$1 = \sum_{x\in\Sigma_G^+,\ \sigma^n(x)=y} \exp[-\beta S_n(H-\bar{H}-V_{E+\beta H}\circ\sigma + V_{E+\beta H})(x)]$$
$$\exp[-S_nE(x)]\exp[-n(\mathrm{Pres}(E+\beta H)+\beta\bar{H})].$$

We split this sum into two parts

$$I' = \sum_{x\in\Sigma_G^+\setminus\tilde{K},\ \sigma^n(x)=y} \ldots, \quad I'' = \sum_{x\in\tilde{K},\ \sigma^n(x)=y} \ldots\ .$$

We choose β large enough so that $\|V_{E+\beta H} - V_\infty\|_\infty < \frac{1}{4}\eta$, with $\eta < \epsilon$ to be determined. From lemma 35.vi, we have $S_nH_{V_\infty}(x) = h_{V_\infty}(x,y)$, which yields

$$S_n(H-\bar{H}-V_{E+\beta H}\circ\sigma + V_{E+\beta H})(x) \ge h_{V_\infty}(x,y) - 2\|V_{E+\beta H}-V_\infty\|_\infty.$$

We recall from lemma 34 the following inequalities

$$\mathrm{Pres}_{\Omega(H)}(E) \le \mathrm{Pres}(E+\beta H) + \beta\bar{H} \le \mathrm{Pres}(E).$$

We also recall how to compute the pressure using a counting argument on inverse branches ($C = \exp[2\|V_E\|_\infty]$)

$$C^{-1}\exp[n\mathrm{Pres}(E)] \le \sum_{x\in\Sigma_G^+,\ \sigma^n(x)=y} \exp[-S_nE(x)] \le C\exp[n\mathrm{Pres}(E)].$$

Therefore, the first part can be bounded from above in the following way

$$\begin{aligned} I' &\le \sum_{x \in \Sigma_G^+ \setminus \tilde{K},\ \sigma^n(x)=y} \exp[-\beta \frac{\epsilon}{2}] \exp[-S_n E(x)] \exp[-n \mathrm{Pres}_{\Omega(H)}(E)], \\ &\le C \exp[-\beta \frac{\epsilon}{2}] \exp[n(\mathrm{Pres}(E) - \mathrm{Pres}_{\Omega(H)}(E))]. \end{aligned}$$

The second part is bounded from above using the estimate of lemma 42.*iii*

$$\begin{aligned} I'' &\le \sum_{x \in \tilde{K},\ \sigma^n(x)=y} \exp[\beta \frac{\eta}{2}] \exp[-S_n E(x)] \exp[-n \mathrm{Pres}_{\Omega(H)}(E)], \\ &\le \tilde{C}(n) \exp[\beta \frac{\eta}{2}] \exp[n(\mathrm{Pres}_{\tilde{\Omega}}(E) - \mathrm{Pres}_{\bar{\Omega}}(E))]. \end{aligned}$$

We now choose η and $n = n(\beta)$ so that

$$\begin{aligned} -\beta \frac{\epsilon}{2} + n(\mathrm{Pres}(E) - \mathrm{Pres}_{\Omega(H)}(E)) &< -n \frac{\eta}{2}, \\ \beta \frac{\eta}{2} - n(\mathrm{Pres}_{\bar{\Omega}}(E) - \mathrm{Pres}_{\tilde{\Omega}}(E)) &< -n \frac{\eta}{2}, \end{aligned}$$

that is, $\eta/2 < \mathrm{Pres}_{\bar{\Omega}}(E) - \mathrm{Pres}_{\tilde{\Omega}}(E)$ and

$$\frac{\eta/2}{\mathrm{Pres}_{\bar{\Omega}}(E) - \mathrm{Pres}_{\tilde{\Omega}}(E) - \eta/2} < \frac{n}{\beta} < \frac{\epsilon/2}{\mathrm{Pres}(E) - \mathrm{Pres}_{\Omega(H)}(E) + \eta/2}.$$

We thus have obtained, for a subsequence $n \to +\infty$,

$$1 = I' + I'' \le (C + \tilde{C}(n)) \exp[-n \frac{\eta}{2}] \to 0,$$

which is clearly a contradiction. □

Proof of Theorem 16. As before, we fix an accumulation point V_∞ of the sequence $\{V_{E+\beta H}\}_{\beta \to +\infty}$. Let $\Omega(H) = \Omega_0 \cup \ldots \Omega_r$ be a disjoint union of irreducible components. By hypothesis, $\mathrm{Pres}_{\Omega_0}(E) > \mathrm{Pres}_{\Omega_1 \cup \ldots \Omega_r}(E)$. For $j = 1, \ldots, r$, denote $\bar{\Omega} = \cup_{i \ne j} \Omega_i$ and $\tilde{\Omega} = \Omega_j$. Since $\mathrm{Pres}_{\bar{\Omega}}(E) > \mathrm{Pres}_{\tilde{\Omega}}(E)$, proposition 43 implies $\bar{h}_{V_\infty}(i,j) = 0$ for some $i \ne j$. Lemma 37 shows that $\bar{h}_{V_\infty}(0,j) = 0$ for all $j = 1, \ldots, r$. Since V_∞ is calibrated, lemma 36.*iii* implies finally

$$h(x_0, y) = V_\infty(y) - V_\infty(x_0), \quad \forall\ x_0 \in \Omega_0,\ \forall\ y \in \Sigma_G^+.$$

If $x_0 \in \Omega_0$ is fixed, the sequence $\{V_{E+\beta H}(\cdot) - V_{E+\beta H}(x_0)\}_{\beta \to +\infty}$ has a unique accumulation point $h(x_0, \cdot)$ and therefore converges. □

5 Proofs of results stated in section 3

We study in this section the algorithmic aspects of singular perturbations of Perron matrices of Puiseux type. We start with a weighted irreducible graph (G, M_ϵ) of (general) Puiseux type (recall definition 24) and we write formally $M_\epsilon \sim A\epsilon^a$.

The first step of the algorithm consists in conjugating M_ϵ by a diagonal matrix $\mathrm{diag}[\epsilon^{v(x)} : x \in S]$ so that all entries in $S \times S \setminus G^*_{min}$ are negligible with respect to $\epsilon^{\bar a}$. The construction of the corrector $v(x)$ is performed in two steps: $v(x)$ is a calibrated corrector in the first step and separating in the second one. A Peierls barrier $h_a(x, y)$ between two vertices is introduced as in definition 14.

Definition 44. *Let $G \subset S \times S$ be an irreducible graph and $a : G \to \mathbb{R}$ be a weight on each edge. The Peierls barrier (associated with a) between two vertices $x, y \in S$ is defined by*

$$h_a(x,y) := \min\Big\{ \sum_{k=0}^{n-1}(a(x_k,x_{k+1}) - \bar a) : n \geq 1, \\ (x_0,\ldots,x_n) \text{ is a } G\text{-admissible path, } x_0 = x \text{ and } x_n = y \Big\}.$$

Notice that it is enough to minimize on simple path: thanks to the choice of the constant $\bar a$, each cycle $(x_0, \ldots, x_n)$ satisfies $\sum_{k=0}^{n-1}(a(x_k, x_{k+1}) - \bar a) \geq 0$ and may be eliminated from the sum.

We summarize several properties of $h_a(x, y)$. Item *vi* of the following lemma gives the definition of the irreducible components of G_{min} and proves lemma 21.

Lemma 45. *Suppose (G, M_ϵ) is an irreducible graph of exact Puiseux type, with $M_\epsilon \sim A\epsilon^a$, and $h_a(x, y)$ is the Peierls barrier associated with $a : G \to \mathbb{R}$. Then*

i. $\forall\, (x_0, \ldots, x_n)$ *G-admissible path,* $h_a(x_0, x_n) \leq \sum_{k=0}^{n-1}(a(x_k, x_{k+1}) - \bar a)$.

ii. $\forall\, x, y, z \in S$, $h_a(x, z) \leq h_a(x, y) + h_a(y, z)$.

iii. $\forall\, x \in S$, $h_a(x, x) \geq 0$.

iv. $\forall\, x \in S$, $h_a(x, x) = 0 \Leftrightarrow x \in S_{min}$.

v. A cycle has a support in G_{min} if, and only if, it is minimizing.

vi. G_{min} is semi-irreducible and its irreducible components are given by the equivalence classes of the relation

$$\forall\, x, y \in S_{min},\ x \sim_a y \Leftrightarrow h_a(x,y) + h_a(y,x) = 0 \\ \Leftrightarrow x \text{ and } y \text{ belong to the same minimizing cycle.}$$

Proof. Items *i*, *ii*, *iii* and *iv* are obvious from the definition of h_a.

Item *v*. By the definition of G_{min}, the support of all minimizing cycle is included in G_{min}. Conversely, let $(x_0, \ldots, x_n)$ be a cycle of G_{min}. Each (x_k, x_{k+1}) is the initial segment of a minimizing cycle $(z^k_0, \ldots, z^k_{p_k})$ with $p_k \geq 2$, $z^k_0 = x_k$ and $z^k_1 = x_{k+1}$. The union of the supports of these minimizing cycles can be written as a union of the supports of two (*a priori* not minimizing) cycles $(x_0, x_1, \ldots, x_n)$ and

$$(y_0,\ldots,y_{q_n}) = (z_1^{n-1},\ldots,z_{p_{n-1}}^{n-1}, z_1^{n-2},\ldots,z_{p_{n-2}}^{n-2},\ldots,z_1^0,\ldots,z_{p_0}^0)$$

of length $q_n = p_0 + \dots p_{n-1} - n$. Since

$$0 = \sum_{k=0}^{n-1} \sum_{i=0}^{p_k-1} (a(z_i^k, z_{i+1}^k) - \bar{a}) = \sum_{k=0}^{n-1} (a(x_k, x_{k+1}) - \bar{a}) + \sum_{k=0}^{q_n-1} (a(y_k, y_{k+1}) - \bar{a}),$$

both cycles $(x_0, \dots, x_n)$ and $(y_0, \dots, y_{q_n})$ are indeed minimizing.

Item *vi*. Consider the relation on S_{min}: $x \sim_a y$ if, and only if, x and y belong the support of the same minimizing cycle of length ≥ 1. Since the union of two minimizing cycles with a common point is again a minimizing cycle, the previous relation is an equivalence relation. If $x \sim_a y$, then there exists a minimizing cycle $(x_0, \dots, x_n)$ such that $x = x_0$ and $y = x_i$ for some $0 < i < n$. Therefore,

$$0 \leq h_a(x,x) \leq h_a(x,y) + h_a(y,x) \leq \sum_{k=0}^{i-1} (a(x_k, x_{k+1}) - \bar{a}) + \sum_{k=i}^{n-1} (a(x_k, x_{k+1}) - \bar{a}) = 0,$$

and $h_a(x,y) + h_a(y,x) = 0$. Conversely, suppose $h_a(x,y) + h_a(y,x) = 0$. So each minimum $h_a(x,y)$ or $h_a(y,x)$ is reached by a G-admissible path $(x_0, \dots, x_i)$ or $(x_i, \dots, x_n)$, with $x_0 = x$, $x_i = y$ and $x_n = y$. Then $(x_0, \dots, x_n)$ is a minimizing cycle containing both x and y. $\square$

In the framework of a dynamical system where the weighted graph (G, M_ϵ) is given by $M_\epsilon(x,y) = \exp(E(x,y))\epsilon^{H(x,y)} \mathbb{1}_G(x,y)$ for two finite-range potentials $E, H : \Sigma_G^+ \to \mathbb{R}$, we show that the two notions of minimizing non-wandering set $\Omega(H)$ and minimizing subgraphs coincide. Let $a(x,y) = H(x,y)$ if $(x,y) \in G$ and $a(x,y) = +\infty$ otherwise.

Proof of Lemma 22. Item i. Let $x = (x_0, x_1, \dots) \in \Sigma_G^+$. Since G is irreducible, there is a G-admissible path joining x_n to x_0, $(x_0^n, x_1^n, \dots, x_{p_n}^n)$ of length p_n at most the cardinal of S. Then $(y_0, \dots, y_{n+p_n}) = (x_0, \dots, x_{n-1}, x_0^n, \dots, x_{p_n}^n)$ is a cycle and

$$\bar{H} = \inf_{x \in \Sigma_G^+} \liminf_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} H \circ \sigma^k(x) = \inf_{x \in \Sigma_G^+} \liminf_{n \to +\infty} \frac{1}{n + p_n} \sum_{k=0}^{n+p_n-1} a(y_k, y_{k+1}) \geq \bar{a}.$$

The converse $\bar{H} \leq \bar{a}$ is obtained by taking a periodic point $x = (x_0, \dots, x_n)^\infty$ with $(x_0, \dots, x_n)$ a minimizing cycle.

Item ii. Let $h(x,y) = \lim_{\epsilon \to 0} \liminf_{n \to +\infty} S_n^\epsilon(x,y)$ be the Peierls barrier introduced in definition 14. We first show that $h(x,y) \geq h_a(x_0, y_0)$ for any $x, y \in \Sigma_G^+$. Indeed, for ϵ sufficiently small, for any $z = (z_0, z_1, \dots) \in \Sigma_G^+$ satisfying $d(x,z) < \epsilon$ and $d(\sigma^n(z), y) < \epsilon$, we have $z_0 = x$ and $z_n = y_0$ and therefore $S_n^\epsilon(x,y) \geq h_a(x,y)$. Let $x = (x_0, x_1, \dots) \in \Omega(H)$. Since $0 = h(x,x) \geq h_a(x_0, x_0) \geq 0$, $x_0 \in S_{min}$. Hence $\sigma^n(x) \in \Omega(H)$ implies $x_n \in S_{min}$ for any n. Moreover,

$$\begin{aligned} 0 = h(x,x) &= (H - \bar{H})(x) + h(\sigma(x), x) \\ &\geq (a(x_0, x_1) - \bar{a}) + h_a(x_1, x_0) \geq h_a(x_0, x_1) + h_a(x_1, x_0) \geq 0. \end{aligned}$$

In particular, $(a(x_0,x_1)-\bar a)+h_a(x_1,x_0)=0$. By choosing a path $(y_1,\ldots,y_n)$ joining x_1 to x_0 which realizes the minimum in $h_a(x_1,x_0)$, we obtain a minimizing cycle $(x_0,x_1,y_2,\ldots,y_n)$. We have just proved $(x_0,x_1)\in G_{min}$ and more generally $(x_k,x_{k+1})\in G_{min}$. Thus, $\Omega(H)\subset\Sigma^+_{G_{min}}$. Conversely, suppose $x\in\Sigma^+_{G_{min}}$. Let $n\geq 1$ and $k=0,\ldots,n-1$. Then any (x_k,x_{k+1}) is the beginning of a minimizing cycle $(x_0^k,x_1^k,\ldots,x_{p_k}^k)$ with $p_k\geq 2$. Consider z_n the periodic point of period $q_n=p_0+\ldots+p_{n-1}+n$ given by

$$z_n=(x_0,\ldots,x_{n-1},x_1^{n-1},\ldots,x_{p_{n-1}-1}^{n-1},x_1^{n-2},\ldots,x_{p_{n-2}-1}^{n-2},\ldots,x_1^0,\ldots,x_{p_0}^0)^\infty.$$

Then $d(z_n,x)\to 0$ when $n\to+\infty$ and $\sum_{k=0}^{q_n-1}(a(z_k,z_{k+1})-\bar a)=0$. We have proved that $x\in\Omega(H)$.

Item *iii*. We first show that, if $x=(x_0,x_1,\ldots),y=(y_0,y_1,\ldots)\in\Omega(H)$, then $x\sim y$ if, and only if, $x_0\sim_a y_0$. Indeed, on the one hand,

$$x\sim y\Leftrightarrow h(x,y)+h(y,x)=0\Rightarrow h_a(x_0,y_0)+h_a(y_0,x_0)=0\Leftrightarrow x_0\sim_a y_0.$$

On the other hand, suppose $x_0\sim_a y_0$. Since $(x_k,x_{k+1}),(y_k,y_{k+1})\in G_{min}$ for all $k=0,\ldots,p-1$, by transitivity we have that $x_p\sim_a y_0$ and $y_p\sim_a x_0$. For infinitely many m and n, one can find a G_{min}-cycle of length $q=2p+m+n$ containing both $(x_0,\ldots,x_{p-1})$ and $(y_0,\ldots,y_{p-1})$ of the following form

$$(x_0,\ldots,x_{p-1},z_p,\ldots,z_{p+m-1},y_0,\ldots,y_{p-1},z_{2p+m},\ldots,z_{2p+m+n}).$$

Let $z\in\Sigma^+_{G_{min}}$ be the corresponding periodic point. For any $\epsilon>0$, if p is large enough, for infinitely many m and n, one has

$$d(z,x)<\epsilon,\quad d(\sigma^{p+m}(z),y)<\epsilon,\quad d(\sigma^{2p+m+n}(z),x)<\epsilon,$$
$$S^\epsilon_{p+m}(x,y)+S^\epsilon_{p+n}(y,x)\leq\sum_{k=0}^{2p+m+n-1}(H-\bar H)\circ\sigma^k(z)=0.$$

By taking lim inf when $m\to\infty$ and $n\to\infty$ first and lim when $\epsilon\to 0$, one obtains $h(x,y)+h(y,x)=0$, that is, $x\sim y$. Since G_{min} is equal to the disjoint union of irreducible components $G_i\subset S_i\times S_i$ with no transition from S_i to S_j when $i\neq j$, $\Omega(H)=\Sigma^+_{G_{min}}$ is equal to the disjoint union of $\Omega_i(H)=\Sigma^+_{G_i}$. The equivalence between $x\sim y$ and $x_0\sim_a y_0$ shows that $\Omega_1(H),\ldots,\Omega_d(H)$ are the irreducible components of $\Omega(H)$.

Item *iv*. The pressure of E restricted to $\Omega(H)$ is equal to the maximum of the pressure of E restricted on each $\Omega_i(H)$. It is well known (see, for instance, [27]) that the two notions of spectral radius α_i of the matrix $A^{ii}_{\min}=[e^{E(x,y)}\mathbb{1}_{G_i}(x,y)]_{x,y\in S_i}$ and the pressure of E restricted to $\Sigma^+_{G_i}$ coincide: $\alpha_i=\exp[\mathrm{Pres}_{\Omega_i(H)}(E)]$ and $\bar\alpha=\max_{1\leq i\leq d}\alpha_i=\exp[\mathrm{Pres}_{\Omega(H)}(E)]$. □

The first step of the algorithm consists in finding a normal form for M_ϵ. This step is done using a diagonal matrix $\mathrm{diag}[\epsilon^{v(x)}:x\in S]$ where $v:S\to\mathbb{R}$ is a separating corrector. We prove the existence of such a corrector.

Proof of Lemma 28. Given $z^* \in S_{min}$, consider

$$u(x) := h_a(z^*, x), \quad \forall\, x \in S,$$

where h_a is the Peierls barrier associated with a introduced in definition 44. Items i and ii of lemma 45 and the fact that the Peierls barrier between two vertices is realized by a G-admissible path easily show that u is a backward calibrated corrector. Let $G_1 \subset S_1 \times S_1, \ldots, G_d \subset S_d \times S_d$ be the irreducible components of the minimizing subgraph $G_{min} \subset S_{min} \times S_{min}$. Denote $S_0 = S \setminus (S_1 \cup \ldots \cup S_d)$. We consider then

$$\tilde{a}(x,y) := a(x,y) - u(y) + u(x) - \bar{a} \geq 0, \quad \forall\, x, y \in S.$$

Notice that the mean of $\tilde{a}$ on any minimizing cycle is zero and therefore $\tilde{a}(x,y) = 0$ whenever $(x,y) \in G_{min}$. We introduce a new directed graph. The set of vertices $\tilde{S}$ is made of classes of two kinds: a class $[x]$ reduced to one point for all $x \in S_0$ and d classes $[G_1] \ldots [G_d]$ where all vertices in each G_i are identified into one vertex. For any $x \in S$, we note by $[x]$ the class containing x. Let $\tilde{G} \subset \tilde{S} \times \tilde{S}$ be the graph whose transitions are defined as follows

$$[x] \overset{\tilde{G}}{\to} [y] \iff [x] \neq [y] \quad \text{and} \quad \min\{\tilde{a}(x', y') \,:\, x' \in [x],\, y' \in [y]\} = 0.$$

The main observation is that there is no cycle in $\tilde{G}$ and we can define a decreasing "height" function $\eta : S \to [0, \epsilon]$ as small as we want so that η is constant on each class $[x]$ and

$$[x] \overset{\tilde{G}}{\to} [y] \iff \eta(x) > \eta(y), \quad \forall\, x, y \in S.$$

We claim that, for ϵ small enough,

$$v(x) := u(x) + \eta(x), \quad \forall\, x \in S$$

is a separating corrector for $a(x,y)$ or equivalently $\eta(x)$ is a separating corrector for $\tilde{a}(x,y)$. Indeed, on the one hand, if $(x,y) \in G_{min}$, x and y belong to the same irreducible component of G, $\eta(x) = \eta(y)$ and $\tilde{a}(x,y) = 0 = \eta(y) - \eta(x)$. On the other hand, if $(x,y) \in G \setminus G_{min}$, we discuss two cases. In the first case, $([x],[y])$ is not an edge of $\tilde{G}$. This implies $\tilde{a}(x,y) > 0$ since $(x,y) \notin G_{min}$. We choose then $\epsilon > 0$ such that $\tilde{a}(x,y) > \eta(y) - \eta(x)$. In the second case, $([x],[y])$ is an edge of $\tilde{G}$. Since η is decreasing along the edges, $\tilde{a}(x,y) \geq 0 > \eta(y) - \eta(x)$ independently of ϵ. As S is finite, the number of constraints on ϵ is finite. $\square$

In order to prove proposition 30, we recall some notions of entropy and pressure for graphs weighted by Perron matrices.

Definition 46. *Let $G \subset S \times S$ be a directed graph weighted by a Perron matrix $[M(x,y)]_{x,y \in S}$. We call transshipment any a probability measure $\mu(x,y)$ on G such that $\pi(y) := \sum_{x \in S} \mu(x,y) = \sum_{x \in S} \mu(y,x)$, for all $y \in S$. The entropy of a transshipment μ is given by*

$$\mathrm{Ent}(\mu) := \sum_{(x,y) \in G} -\mu(x,y) \ln \frac{\mu(x,y)}{\pi(x)}.$$

We say the transshipment μ is supported by M if $M(x,y)=0$ implies $\mu(x,y)=0$. In this case, the pressure of M with respect to μ is given by

$$\mathrm{Pres}(M,\mu) := \mathrm{Ent}(\mu) + \sum_{(x,y)\in G} \mu(x,y)\ln M(x,y).$$

We recall that, if G is irreducible and $\lambda = \rho_{spec}(M)$, then $\mathrm{Pres}(M,\mu) \le \ln\lambda$ for any transshipment μ supported by M, with equality if, and only if, $\mu(x,y) = L(x)M(x,y)R(y)/\lambda$, where $[L(x)]_{x\in S}$ and $[R(x)]_{x\in S}$ are the left and right eigenvectors of M for the eigenvalue λ.

We shall also use a known result on the perturbation of the spectrum of matrices. See Kato's monograph [21] for more elaborate statements.

Lemma 47. *For any matrix $M \in \mathrm{Mat}(n,\mathbb{C})$, for any $\epsilon > 0$, there exists $\eta > 0$ such that, if $H \in \mathrm{Mat}(n,\mathbb{C})$ and $\|H\| < \eta$, then $\mathrm{spec}(M+H) \subset \mathrm{spec}(M) + B_\epsilon$, where B_ϵ denotes the disk of radius ϵ centered at 0. In particular, $M \mapsto \rho_{spec}(M)$ is continuous on $\mathrm{Mat}(n,\mathbb{C})$.*

Proof of proposition 30. Notice that it is enough to assume M_ϵ is written in a normal form

$$M_\epsilon = \hat M + N_\epsilon,\ \hat M = \begin{bmatrix} \bar A & 0 \\ 0 & D \end{bmatrix},\ \bar A = \mathrm{diag}[\bar A^{11},\ldots,\bar A^{rr}],\ \bar\alpha = \rho_{\mathrm{spec}}(\bar A^{ii}),$$

where $\bar A^{ii}$ is nonnegative irreducible, D is nonnegative with $\rho_{\mathrm{spec}}(D) < \bar\alpha$, and $N_\epsilon = o(1)$. We also assume M_ϵ is nonnegative by changing if necessary M_ϵ to $M_\epsilon - \eta_\epsilon \mathrm{Id}$ where $\eta_\epsilon = 0 \wedge \min\{M_\epsilon(x,x) : x \in S\}$. Notice that L_ϵ and R_ϵ do not change and that $\eta_\epsilon = o(1)$.

Let thus $\hat G$ be the subgraph of G defined by $(x,y) \in \hat G \Leftrightarrow \bar A(x,y) > 0$ or $D(x,y) > 0$. Let $\hat M_\epsilon(x,y) = M_\epsilon(x,y)$ if $(x,y) \in \hat G$, $\hat M_\epsilon(x,y) = M_\epsilon^{1/2}(x,y)$ if $(x,y) \in G \setminus \hat G$. On the one hand, we remark that

$$\begin{aligned}\ln\lambda_\epsilon = \mathrm{Pres}(M_\epsilon,\mu_\epsilon) = \mathrm{Pres}(\hat M_\epsilon,\mu_\epsilon) + \sum_{(x,y)\in G\setminus\hat G} \mu_\epsilon(x,y)\ln M_\epsilon^{1/2}(x,y) \le \\ \le \ln\rho_{\mathrm{spec}}(\hat M_\epsilon) + \sum_{(x,y)\in G\setminus\hat G} \mu_\epsilon(x,y)\ln M_\epsilon^{1/2}(x,y) \le \ln\rho_{\mathrm{spec}}(\hat M_\epsilon).\end{aligned}$$

Consider now $\bar G_1$ (an irreducible component of $G^*_{\min}$ of dominant spectral coefficient $\bar\alpha$) weighted by $\hat M^{11}_\epsilon(x,y) = M_\epsilon(x,y)\mathbb{1}_{\bar G_1}(x,y)$. Let $\hat\mu^1_\epsilon$ be the transshipment defined on $\bar G_1$ by

$$\hat\mu^1_\epsilon(x,y) = \hat L^1_\epsilon(x)\hat M^{11}_\epsilon(x,y)\hat R^1_\epsilon(y)/\rho_{\mathrm{spec}}(\hat M^{11}_\epsilon),$$

and extended by 0 on $G \setminus \bar G_1$. Then, on the other hand, one has

$$\ln\lambda_\epsilon \ge \mathrm{Pres}(\hat M^{11}_\epsilon, \hat\mu^1_\epsilon) = \ln\rho_{\mathrm{spec}}(\hat M^{11}_\epsilon).$$

Lemma 47 tells us that $\rho_{\text{spec}}(\hat{M}_\epsilon) \sim \rho_{\text{spec}}(\hat{M}_\epsilon^{11}) \sim \bar{\alpha}$. Hence, the two previous inequalities show that $\lambda_\epsilon \sim \bar{\alpha}$ (item *i*), as well as $\mu_\epsilon(x,y) \to 0$ whenever $(x,y) \notin \hat{G}$. They also show that any accumulation point $\bar{\mu}$ of $(\mu_\epsilon)_{\epsilon>0}$ has maximal pressure

$$\ln \bar{\alpha} = \lim_{\epsilon\to 0} \ln \lambda_\epsilon \leq \lim_{\epsilon\to 0} \Big[\text{Ent}(\mu_\epsilon) + \sum_{(x,y)\in\hat{G}} \mu_\epsilon(x,y) \ln M_\epsilon(x,y)\Big] = \text{Pres}(\hat{M},\bar{\mu}) \leq \ln \bar{\alpha}.$$

(The first inequality comes from the fact that $\ln M_\epsilon(x,y) < 0$ if $(x,y) \in G \setminus \hat{G}$. Notice also that $\bar{\mu}$ has support on $\hat{G}$.) For $\bar{G}$ the dominant subgraph, let $\bar{\mu}_{\bar{G}}$ and $\bar{\mu}_{\hat{G}\setminus\bar{G}}$ be the induced transshipments on $\bar{G}$ and $\hat{G}\setminus\bar{G}$, respectively. Since

$$\ln \bar{\alpha} = \text{Pres}(\hat{M},\bar{\mu}) = \bar{\mu}(\bar{G})\text{Pres}(\bar{A},\bar{\mu}_{\bar{G}}) + \bar{\mu}(\hat{G}\setminus\bar{G})\text{Pres}(D,\bar{\mu}_{\hat{G}\setminus\bar{G}}),$$

we obtain $\bar{\mu}(\hat{G}\setminus\bar{G}) = 0$, that is, $\mu_\epsilon(x,y) \to 0$ whenever $(x,y) \notin \bar{G}$ (item *ii*).

Consider $\bar{\pi}^i(x) = \sum_{y\in\bar{S}_i} \bar{\mu}(x,y)/\bar{\mu}(\bar{G}_i)$ for any $x \in \bar{S}_i$. Let $\bar{\mu}_i$ be the induced transshipment on $\bar{G}_i$, $\bar{\mu}_i(x,y) = \bar{\mu}(x,y)/\bar{\mu}(\bar{G}_i)$ whenever $\bar{\mu}(\bar{G}_i) \neq 0$. The main remark is the following coboundary property

$$\sum_{x\in\bar{S}_i} \bar{\mu}_i(x,y) = \sum_{x\in\bar{S}_i} \bar{\mu}_i(y,x),\ \forall\ y \in \bar{S}_i\ \Rightarrow \sum_{(x,y)\in\bar{S}_i\times\bar{S}_i} \bar{\mu}_i(x,y) \ln\Big(\frac{\bar{R}^i(y)}{\bar{R}^i(x)}\Big) = 0.$$

Then $\ln\bar{\alpha} = \sum_{i=1}^r \bar{\mu}(\bar{G}_i)\text{Pres}(\bar{A}^{ii},\bar{\mu}_i)$ and

$$\text{Pres}(\bar{A}^{ii},\bar{\mu}_i) = \sum_{\substack{x\in\bar{S}_i\\ \bar{\pi}^i(x)\neq 0}} \bar{\pi}^i(x) \sum_{y\in\bar{S}_i} \frac{\bar{\mu}_i(x,y)}{\bar{\pi}^i(x)} \ln\Big(\frac{\bar{A}^{ii}(x,y)\bar{R}^i(y)/\bar{R}^i(x)}{\bar{\mu}_i(x,y)/\bar{\pi}_i(x)}\Big).$$

Each sum over $y \in \bar{S}_i$ is bounded from above by

$$\ln\Big(\sum_{y\in\bar{S}_i} \bar{A}^{ii}(x,y)\bar{R}^i(y)/\bar{R}^i(x)\Big) = \ln\bar{\alpha},$$

with equality if, and only if, $\bar{\mu}_i(x,y)/\bar{\pi}^i(x) = \bar{A}^{ii}(x,y)\bar{R}^i(y)/(\bar{\alpha}\bar{R}^i(x))$, $\forall\, y \in \bar{S}_i$. We thus have proved (whether or not $\bar{\pi}_i(x) = 0$)

$$\frac{\bar{\pi}^i(x)}{\bar{R}^i(x)}\bar{A}^{ii}(x,y) = \bar{\alpha}\frac{\bar{\mu}_i(x,y)}{\bar{R}^i(y)}, \quad \forall\ x,y \in \bar{S}_i.$$

By summing over x, using the fact that $\bar{\mu}_i$ is a transshipment, we obtain that $[\bar{\pi}^i(x)/\bar{R}^i(x)]_{x\in\bar{S}_i}$ is a left eigenvector of $\bar{A}^{ii}$ for the eigenvalue $\bar{\alpha}$. In particular, if $\bar{\pi}^i(x) \neq 0$ for some $x \in \bar{S}_i$, $\bar{\pi}^i(y) \neq 0$ for all $y \in \bar{S}_i$ and

$$\bar{\pi}^i(y) = \bar{L}^i(y)\bar{R}^i(y), \quad \bar{\mu}_i(x,y) = \bar{L}^i(x)\bar{A}^{ii}(x,y)\bar{R}^i(y)/\bar{\alpha}.$$

(Item *iii* is proved.) □

Before proving proposition 32, we give some complements to the theory of series of equivalences.

Lemma 48. *Let $(A_n)_{n\geq 0}$ be a sequence of positive numbers and $(A_n(\epsilon))_{n\geq 0}$ be a sequence of functions. We assume that $A_n = O(\delta^n)$ for some $\delta \in (0,1)$ and $\big(A_n(\epsilon)/A_n\big)^{1/n} \to 1$ as $\epsilon \to 0$ uniformly in $n \geq 0$. Then*

$$\sum_{n\geq 0} A_n(\epsilon) \sim \sum_{n\geq 0} A_n.$$

Proof. Denote $h_n(\epsilon) := \big(A_n(\epsilon)/A_n\big)^{1/n} - 1$. Let $\eta \in (0,1)$ be small enough so that $\delta(1+\eta) < 1$. Fix a constant $C > 0$ such that $A_n \leq C\delta^n$, for all $n \geq 0$. Choose a positive integer N large enough so that

$$(1-\eta)\sum_{n\geq N} A_n < \eta \sum_{n\geq 0} A_n \quad \text{and} \quad C\sum_{n\geq N} \delta^n(1+\eta)^n < \eta \sum_{n\geq 0} A_n.$$

For ϵ small enough, one has $(1-\eta)\sum_{n=0}^{N-1} A_n \leq \sum_{n=0}^{N-1} A_n(\epsilon) \leq (1+\eta)\sum_{n=0}^{N-1} A_n$, as well as $h_n(\epsilon) < \eta$ uniformly in n, which in particular yields

$$\sum_{n\geq N} A_n(\epsilon) < \sum_{n\geq N} A_n(1+\eta)^n \leq C\sum_{n\geq N} \delta^n(1+\eta)^n.$$

Considering all these inequalities, for all ϵ small enough, we obtain that

$$(1-2\eta)\sum_{n\geq 0} A_n < \sum_{n\geq 0} A_n(\epsilon) < (1+2\eta)\sum_{n\geq 0} A_n.$$

□

In the following lemma, we extend the notion of weighted graph (G, M_ϵ) of general Puiseux type to the case in which G is not irreducible and we show that the resolvent is of exact Puiseux type.

Lemma 49. *Let (G, M_ϵ) be a (not necessarily irreducible) weighted graph. Assume $M_\epsilon = D + N_\epsilon$, where D is nonnegative, $\rho_{spec}(D) < 1$, $N_\epsilon = o(1)$. Suppose (G, M_ϵ) is of general Puiseux type in the following sense:*

$$M_\epsilon(x,y) = \begin{cases} 0 & \text{if } (x,y) \notin G, \\ A_\epsilon(x,y)\epsilon^{a(x,y)} & \text{if } (x,y) \in G \text{ and } x \neq y, \\ A_\epsilon(x,y) & \text{if } (x,x) \in G,\ x = y \text{ and } D(x,x) > 0, \\ o(1) & \text{if } (x,x) \in G,\ x = y \text{ and } D(x,x) = 0, \end{cases}$$

where $A_\epsilon(x,y) \sim A(x,y) > 0$ and $a(x,y) \geq 0$ in the second and third cases, and by convention $A(x,y) = 0$ and $a(x,y) = +\infty$ in the other cases. Let $\mathcal{P}(x,y)$ be the set of G-admissible paths $\underline{x} = (x_0, \ldots, x_n)$ of length $n \geq 1$ such that $x_0 = x$ and $x_n = y$. Consider the directed graph

$$G' = \big\{(x,x) \,:\, x \in S\big\} \cup \big\{(x,y) \in S \times S \,:\, \mathcal{P}(x,y) \neq \emptyset\big\}$$

and define $M'_\epsilon := (Id - M_\epsilon)^{-1}$. Then (G', M'_ϵ) is a weighted graph of exact Puiseux type. More precisely,

$$M'_\epsilon(x,y) = 0 \Leftrightarrow (x,y) \notin G' \quad \text{and} \quad M'_\epsilon(x,y) \sim A'(x,y)\epsilon^{a'(x,y)} \Leftrightarrow (x,y) \in G',$$

with $a'(x,y) = \begin{cases} 0 & \textit{if } x = y \\ \min\{a(\underline{x}) \,:\, \underline{x} \in \mathcal{P}(x,y)\} & \textit{if } x \neq y \end{cases}, \quad \forall\ (x,y) \in G',$
and $A'(x,y) = \mathbb{1}_{(x=y)} + \sum_{\underline{x}\in\mathcal{P}(x,y)\,:\,a(\underline{x})=a'(x,y)} \Pi_{i=0}^{n(\underline{x})-1} A(x_i,x_{i+1}),\ \forall\ (x,y) \in G',$
where $n(\underline{x})$ *is the length of the path* $\underline{x} \in \mathcal{P}(x,y)$ *and* $a(\underline{x}) := \sum_{i=0}^{n(\underline{x})-1} a(x_i,x_{i+1})$. *(By convention* $A'(x,y) = 0$ *and* $a'(x,y) = +\infty$ *for all* $(x,y) \notin G'$.*)*

Proof. Part 1. We first assume that (G, M_ϵ) is of exact Puiseux type,

$$M_\epsilon(x,y) = \begin{cases} 0 & \forall\ (x,y) \notin G, \\ A_\epsilon(x,y)\epsilon^{a(x,y)} & \forall\ (x,y) \in G, \end{cases}$$

where $A_\epsilon(x,y) \sim A(x,y) > 0$ and $a(x,y) \geq 0$ if $(x,y) \in G$, $A(x,y) = 0$ and $a(x,y) = +\infty$ if $(x,y) \notin G$. Note that $D(x,y) > 0$ if, and only if, $a(x,y) = 0$. Since $\rho_{\text{spec}}(M_\epsilon)$ converges to $\rho_{\text{spec}}(D) < 1$, $(\text{Id} - M_\epsilon)$ is invertible and

$$M'_\epsilon(x,y) = \sum_{n\geq 0} M_\epsilon^n(x,y) = \mathbb{1}_{(x=y)} + \sum_{\underline{x}\in\mathcal{P}(x,y)} \Pi_{i=0}^{n(\underline{x})-1} M_\epsilon(x_i,x_{i+1}).$$

Since M_ϵ is a nonnegative matrix, M'_ϵ is nonnegative too. Moreover,

$$M'_\epsilon(x,y) = 0 \iff x \neq y \ \text{ and } \ \mathcal{P}(x,y) = \emptyset \iff (x,y) \notin G'.$$

For $(x,y) \in G'$, let $\mathcal{P}(x,y,k)$ be the subset of paths $\underline{x} \in \mathcal{P}(x,y)$ such that

$$k = \text{card}\{i = 0, \ldots, n(\underline{x}) - 1 \,:\, a(x_i,x_{i+1}) > 0\}.$$

If $\underline{x} \in \mathcal{P}(x,y,k)$ and $k \geq 1$, then $a(\underline{x})$ takes a finite number of distinct values $a_{k,l}$,

$$0 < ka_{min} \leq a_{k,1} < a_{k,2} < \cdots < a_{k,p_k} \leq ka_{max},$$

with $a_{min} := \min\{a(x,y) : a(x,y) > 0\}$ and $a_{max} := \max\{a(x,y) : a(x,y) < +\infty\}$. Notice that the set of exponents $\{a_{k,l} \,:\, k \geq 1,\ 1 \leq l \leq p_k\}$ is finite on each bounded interval. Let $\mathcal{P}(x,y,k,l)$ be the subset of paths $\underline{x} \in \mathcal{P}(x,y,k)$ such that $a(\underline{x}) = a_{k,l}$. By developing all products M_ϵ^n, one obtains

$$\begin{aligned} M'_\epsilon(x,y) = \mathbb{1}_{(x=y)} &+ \sum_{\underline{x}\in\mathcal{P}(x,y,0)} \Pi_{i=0}^{n(\underline{x})-1} A_\epsilon(x_i,x_{i+1}) \\ &+ \sum_{k\geq 1}\sum_{l=1}^{p_k} \Big(\sum_{\underline{x}\in\mathcal{P}(x,y,k,l)} \Pi_{i=0}^{n(\underline{x})-1} A_\epsilon(x_i,x_{i+1}) \Big) \epsilon^{a_{k,l}}. \end{aligned}$$

Let $\mathcal{P}(x,y,0,0) := \mathcal{P}(x,y,0)$ by convention and $\mathcal{P}_n(x,y,k,l)$ be the set of paths $\underline{x} \in \mathcal{P}(x,y,k,l)$ of length $n(\underline{x}) = n$. Denote

$$A_{n,k,l}(\epsilon) := \sum_{\underline{x}\in\mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} A_\epsilon(x_i,x_{i+1}), \quad A_{n,k,l} := \sum_{\underline{x}\in\mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} A(x_i,x_{i+1}).$$

We use lemma 48 to show that $\sum_{n\geq1} A_{n,k,l}(\epsilon) \sim \sum_{n\geq1} A_{n,k,l}$ (one only considers terms (n,k,l) such that $\mathcal{P}_n(x,y,k,l) \neq \emptyset$). Since $\rho_{\text{spec}}(D) < 1$, there exists a positive matrix $[\tilde D(x,y)]_{x,y\in S}$ such that

$$\rho_{\text{spec}}(\tilde D) < 1 \quad \text{and} \quad \tilde D(x,y) > D(x,y), \quad \forall\, x,y \in S.$$

Since $A(x,y) = D(x,y)$ whenever $D(x,y) > 0$, one obtains

$$A_{n,k,l} \leq \sum_{\underline{x}\in\mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} \tilde D(x_i,x_{i+1}) \Big(\frac{\max A}{\min \tilde D}\Big)^k \leq \tilde D^n(x,y) \Big(\frac{\max A}{\min \tilde D}\Big)^k.$$

Choose $\tilde\delta$ such that $\rho_{\text{spec}}(\tilde D) < \tilde\delta < 1$. Then $\tilde D^n(x,y) = O(\tilde\delta^n)$, and in particular $A_{n,k,l} = O(\tilde\delta^n)$. Given $\eta \in (0,1)$, for ϵ small enough,

$$(1-\eta)A(x,y) < A_\epsilon(x,y) < (1+\eta)A(x,y), \quad \forall\,(x,y) \in G.$$

For all non empty set $\mathcal{P}_n(x,y,k,l)$,

$$(1-\eta)^n < \frac{\sum_{\underline{x}\in\mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} A_\epsilon(x_i,x_{i+1})}{\sum_{\underline{x}\in\mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} A(x_i,x_{i+1})} < (1+\eta)^n.$$

We have thus obtained $\big(A_{n,k,l}(\epsilon)/A_{n,k,l}\big)^{1/n} \to 1$ uniformly in n.

We now show that the rest of the series

$$R_K(\epsilon) := \sum_{k\geq K} \sum_{l=1}^{p_k} \Big(\sum_{n\geq1} A_{n,k,l}(\epsilon)\Big)\epsilon^{a_{k,l}}$$

is negligible with respect to the first non zero term $(\sum_{n\geq1} A_{n,k,l})\epsilon^{a_{k,l}}$. More precisely, we show that, for any $\mathsf{a} > 0$, there exists $K \geq 1$ such that $R_K(\epsilon) = o(\epsilon^{\mathsf{a}})$ as $\epsilon \to 0$. Indeed, let d be the dimension of the matrix M_ϵ, then $p_k \leq d^{2k}$ and

$$R_K(\epsilon) \leq \sum_{k\geq K} \Big(\sum_{n\geq1} \|\tilde D^n\|\Big)\Big(d^2 \frac{\max A}{\min \tilde D}\epsilon^{a_{min}}\Big)^k \leq C_K \epsilon^{K a_{min}} = o(\epsilon^{\mathsf{a}})$$

as soon as $\mathsf{a} < K a_{min}$.

Therefore, $M'_\epsilon(x,y) \sim A'(x,y)\epsilon^{a'(x,y)}$ for all $(x,y) \in G'$.

Part 2. We now assume that (G, M_ϵ) is of general Puiseux type as described in the statement. We first notice that M'_ϵ admits a different expression

$$M'_\epsilon = \frac{1}{2}\Big(\text{Id} - \frac{\text{Id}+M_\epsilon}{2}\Big)^{-1} \quad \text{where} \quad \frac{\text{Id}+M_\epsilon}{2} = \frac{\text{Id}+D}{2} + \frac{N_\epsilon}{2},$$

with $\rho_{\text{spec}}(\frac12(\text{Id}+D)) < 1$ and $\frac12 N_\epsilon = o(1)$. Since $(G, \frac12(\text{Id}+M_\epsilon))$ is of exact Puiseux type, one obtains from part 1 that (G', M'_ϵ) is of exact Puiseux type.

We now want to determine a' and A' in this case. Let Δ_ϵ be the diagonal matrix built from the principal diagonal of N_ϵ. Hence,

$$N_\epsilon = \Delta_\epsilon + \tilde N_\epsilon, \quad \tilde N_\epsilon(x,x) = 0,\ \forall\, x \in S.$$

Let $\tilde M_\epsilon := D + \tilde N_\epsilon$, $\tilde G := G \setminus \{(x,x) \,:\, D(x,x) = 0\}$. Then $(\tilde G, \tilde M_\epsilon)$ is of exact Puiseux type. Moreover,

$$M'_\epsilon = (\mathrm{Id} - \tilde M_\epsilon - \Delta_\epsilon)^{-1} = (\mathrm{Id} - \tilde M'_\epsilon\Delta_\epsilon)^{-1}\tilde M'_\epsilon = \sum_{n\geq 0}(\tilde M'_\epsilon\Delta_\epsilon)^n\tilde M'_\epsilon,$$

where $\tilde M'_\epsilon := (\mathrm{Id} - \tilde M_\epsilon)^{-1}$. From part 1, we know that $(\tilde G', \tilde M'_\epsilon)$ is of exact Puiseux type. Let a' and A' be defined as in part 1 by using $(\tilde G, \tilde M_\epsilon)$. Then

$$\tilde M'_\epsilon(x,y) = A'_\epsilon(x,y)\epsilon^{a'(x,y)}, \quad \forall\, x,y \in S,$$

with $A'_\epsilon(x,y) \sim A'(x,y) > 0$ if $(x,y) \in \tilde G'$ and $A'_\epsilon(x,y) = 0$ if $(x,y) \notin \tilde G'$. Since G and $\tilde G$ have the same off-diagonal entries, $G' = \tilde G'$. We show by induction there exist matrices $(B_{n,\epsilon})_{n\geq 1}$ such that

$$\begin{cases} (\tilde M'_\epsilon\Delta_\epsilon)^n\tilde M'_\epsilon(x,y) = B_{n,\epsilon}(x,y)\epsilon^{a'(x,y)} & \forall\, (x,y) \in G' \\ (\tilde M'_\epsilon\Delta_\epsilon)^n\tilde M'_\epsilon(x,y) = 0 = B_{n,\epsilon}(x,y) & \forall\, (x,y) \notin G' \end{cases} \quad \text{and}$$

$$\lim_{\epsilon\to 0}\big(B_{n,\epsilon}(x,y)\big)^{1/n} = 0, \quad \text{uniformly in } n \geq 1.$$

Since $(\tilde M'_\epsilon\Delta_\epsilon)^{n+1}\tilde M'_\epsilon = (\tilde M'_\epsilon\Delta_\epsilon)^n\tilde M'_\epsilon\Delta_\epsilon\tilde M'_\epsilon$, for all $x,y \in S$ one has

$$\begin{aligned}(\tilde M'_\epsilon\Delta_\epsilon)^{n+1}\tilde M'_\epsilon(x,y) &= \sum_{z\in S}(\tilde M'_\epsilon\Delta_\epsilon)^n\tilde M'_\epsilon(x,z)\Delta_\epsilon(z,z)\tilde M'_\epsilon(z,y) \\ &= \sum_{z\in S} B_{n,\epsilon}(x,z)\Delta_\epsilon(z,z)A'_\epsilon(z,y)\epsilon^{a'(x,z)+a'(z,y)}.\end{aligned}$$

If $(x,y) \notin G'$, then $(x,z) \notin G'$ or $(z,y) \notin G'$ and the above sum is null. Thus by convention $B_{n+1,\epsilon}(x,y) = 0$. If $(x,y) \in G'$ and $z \in S$ is such that $(x,z) \in G'$ and $(z,y) \in G'$, then $a'(x,y) \leq a'(x,z) + a'(z,y)$. Let

$$B_{n+1,\epsilon}(x,y) := \sum_{z\in S} B_{n,\epsilon}(x,z)\Delta_\epsilon(z,z)A'_\epsilon(z,y)\epsilon^{a'(x,z)+a'(z,y)-a'(x,y)}.$$

By taking the supremum in $x,y \in S$, we obtain

$$\sup_{x,y}\Big(B_{n+1,\epsilon}(x,y)\Big) \leq \sup_{x,y}\Big(B_{n,\epsilon}(x,y)\Big)\sup_{x,y}\Big(d\Delta_\epsilon(x,y)A'(x,y)\Big).$$

As $\Delta_\epsilon = o(1)$, we have proved that $(B_{n,\epsilon}(x,y))^{1/n} \to 0$ uniformly in n. Besides,

$$M'_\epsilon(x,y) = A'_\epsilon(x,y)\epsilon^{a'(x,y)}\Big[1 + \sum_{n\geq 1}\frac{B_{n,\epsilon}(x,y)}{A'_\epsilon(x,y)}\Big] \sim A'(x,y)\epsilon^{a'(x,y)} \quad \text{for all } (x,y) \in G',$$

and $M'_\epsilon(x,y) = 0$ for all $(x,y) \notin G'$. □

Proof of proposition 32. Notice that it is enough to assume (G, M_ϵ) is reduced to a normal form and $M_\epsilon = \tilde M_\epsilon$ is nonnegative (by possibly subtracting $\eta_\epsilon\mathrm{Id}$, where $\eta_\epsilon := 0 \wedge \min\{M_\epsilon(x,x) \,:\, x \in S\}$ is negligible with respect to λ_ϵ). We prove item i at the end.

Item *ii.* We only prove the equivalence $\tilde R^i_\epsilon(x)/\tilde R^i_\epsilon(y) \sim \bar R^i(x)/\bar R^i(y)$. We consider the vector space indexed by $\bar S_i$. The vectors are supposed to be column vectors. Let us consider the projector onto $\bar R^i$ defined by

$$V \mapsto (\bar L^i V)\bar R^i, \quad (V \text{ is a column vector}),$$

or as a (square) matrix $\bar R^i \bar L^i$. Notice that the kernel $\{V : \bar L^i V = 0\}$ is invariant by $\bar A^{ii}$. The complementary projector is denoted $\bar P^{ii} := \mathrm{Id} - \bar R^i \bar L^i$. We then obtain a decomposition of $\bar A^{ii}$

$$\bar A^{ii} = \bar\alpha \bar R^i \bar L^i + \bar D^{ii} \quad \text{or} \quad \bar D^{ii} = \bar P^{ii} \bar A^{ii} = \bar A^{ii} \bar P^{ii}.$$

Since $\bar A^{ii}$ is irreducible, $\bar\alpha$ has multiplicity 1 and $\rho_{\mathrm{spec}}(\bar D^{ii}) < \bar\alpha$. By multiplying by $\bar P^{ii}$ the equation

$$\sum_{j=1}^r \Big(\tilde M^{ij}_\epsilon + \tilde M^{i0}_\epsilon(\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}\tilde M^{0j}_\epsilon\Big)\tilde R^j_\epsilon = \tilde\lambda_\epsilon \tilde R^i_\epsilon,$$

one obtains

$$\sum_{j=1}^r (\tilde\lambda_\epsilon - \bar D^{ii})^{-1}\bar P^{ii}\Big(\tilde N^{ij}_\epsilon + \tilde M^{i0}_\epsilon(\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}\tilde M^{0j}_\epsilon\Big)\tilde R^j_\epsilon = \bar P^{ii}\tilde R^i_\epsilon.$$

(We use the fact that $\tilde N^{ij}_\epsilon = \tilde M^{ij}_\epsilon$ when $i \neq j$ and that $\bar A^{ii}\bar P^{ii} = \bar D^{ii}\bar P^{ii}$.) We first claim that $\tilde R^i_\epsilon/\bar L^i \tilde R^i_\epsilon$ is bounded, or equivalently that $\tilde R^i_\epsilon(x)/\tilde R^i_\epsilon(y)$ is bounded for all $x, y \in \bar S_i$. Notice that all following terms are nonnegative

$$\tilde M^{ij}_\epsilon(x,y) \geq 0 \quad \text{or} \quad \tilde M^{i0}_\epsilon(\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}\tilde M^{0j}_\epsilon(x,y) \geq 0.$$

(The second inequality follows from lemma 49.) By the irreducibility of $\bar A^{ii}$, if $(x_0, \ldots, x_n)$ is a path joining x to y such that $\bar A^{ii}(x_k, x_{k+1}) > 0$, then

$$\frac{\tilde R^i_\epsilon(x_0)}{\tilde R^i_\epsilon(x_n)} \geq \frac{\Pi_{k=0}^{n-1}\tilde M^{ii}_\epsilon(x_k, x_{k+1})}{\tilde\lambda^n_\epsilon} \sim \frac{\Pi_{k=0}^{n-1}\bar A^{ii}(x_k, x_{k+1})}{\bar\alpha^n} > 0.$$

By reversing x and y, we prove the claim. We now claim that all following terms are negligible

$$\frac{\tilde N^{ij}_\epsilon \tilde R^j_\epsilon}{\bar L^i \tilde R^i_\epsilon} = o(1) \quad \text{or} \quad \frac{\tilde M^{i0}_\epsilon(\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}\tilde M^{0j}_\epsilon \tilde R^j_\epsilon}{\bar L^i \tilde R^i_\epsilon} = o(1).$$

Notice that these terms are nonnegative, except perhaps $\tilde\mu^i_\epsilon := \tilde N^{ii}_\epsilon \tilde R^i_\epsilon/\bar L^i \tilde R^i_\epsilon$ which is negligible because of the first claim. We conclude by observing that all terms on the left hand side of the following equality are nonnegative and that the right hand side is negligible

$$\sum_{j=1}^r \frac{\bar L^i\big(\tilde N^{ij}_\epsilon \delta_{(i\neq j)} + \tilde M^{i0}_\epsilon(\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}\tilde M^{0j}_\epsilon\big)\tilde R^j_\epsilon}{\bar L^i \tilde R^i_\epsilon} = \tilde\lambda_\epsilon - \bar\alpha - \tilde\mu^i_\epsilon = o(1).$$

Therefeore, we have proved that $\dfrac{\tilde R_\epsilon^i}{\bar L^i \tilde R_\epsilon^i} - \bar R^i = \dfrac{\bar P^{ii}\tilde R_\epsilon^i}{\bar L^i \tilde R_\epsilon^i} = o(1)$.

Item *iii*. Let $i,j \in \{1,\dots,r\}$, $x \in \bar S_i$ and $y \in S$. We have already proved in the first part that

$$\frac{\tilde M_\epsilon^{ii}(x,y)\tilde R_\epsilon^i(y)}{\tilde\lambda_\epsilon \tilde R_\epsilon^i(x)} \sim \frac{\bar A^{ii}(x,y)\bar R^i(y)}{\bar\alpha \bar R^i(x)} = \bar Q^{ii}(x,y), \quad \forall\ x,y \in \bar S_i,$$

$$\frac{\tilde M_\epsilon^{ij}(x,y)\tilde R_\epsilon^j(y)}{\tilde\lambda_\epsilon \tilde R_\epsilon^i(x)} = o(1), \quad \forall\ x \in \bar S_i,\ \forall\ y \in \bar S_j,\ i \neq j,$$

$$\frac{\tilde M_\epsilon^{i0}(x,y)\tilde R_\epsilon^0(y)}{\tilde\lambda_\epsilon \tilde R_\epsilon^i(x)} = \frac{\tilde M_\epsilon^{i0}(x,y)\big(\sum_{j=1}^r(\tilde\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1}\tilde M_\epsilon^{0j}\tilde R_\epsilon^j\big)(y)}{\tilde\lambda_\epsilon \tilde R_\epsilon^i(x)} = o(1).$$

(In the two last estimates, we use the fact that the sum over y in each case is negligible.) We then obtain

$$Q_\epsilon(x,y) = \frac{\tilde M_\epsilon(x,y)\tilde R_\epsilon(y)}{\tilde\lambda_\epsilon \tilde R_\epsilon(x)} \to \begin{cases} \bar Q^{ii}(x,y), & \forall\ x,y \in \bar S_i, \\ 0, & \forall\ x \in \bar S_i,\ \forall\ y \in \bar S_j \cup S_0,\ i \neq j. \end{cases}$$

Item *i*. Let $i \neq j$, then $M_\epsilon^{(1)}(i,j) = \bar L^i\Big(\tilde M_\epsilon^{ij} + \tilde M_\epsilon^{i0}(\tilde\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1}\tilde M_\epsilon^{0j}\Big)\frac{\tilde R_\epsilon^j}{\bar L^j \tilde R_\epsilon^j}$. We want to show that

$$M_\epsilon^{(1)}(i,j) = \begin{cases} 0 & \forall\ (i,j) \notin G^{(1)}, \\ A_\epsilon^{(1)}(i,j)\epsilon^{a^{(1)}(i,j)} & \forall\ (i,j) \in G^{(1)},\ i \neq j, \end{cases}$$

where $A_\epsilon^{(1)}(i,j) = 0$ in the first case and $A^{(1)}(i,j) \sim A^{(1)}(i,j) > 0$ in the second one. From item *ii*, we know that $\tilde R_\epsilon^j/\bar L^j \tilde R_\epsilon^j \sim \bar R^j$. Since $\bar L^i$ and $\bar R^j$ have positive coefficients, it is enough to determine equivalences to the terms $\tilde M_\epsilon^{ij}(x,y)$ and $\tilde M_\epsilon^{i0}(\tilde\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1}\tilde M_\epsilon^{0j}(x,y)$ when $x \in \bar S_i$ and $y \in \bar S_j$. From lemma 49, we know that the matrix $(\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1}$ is of exact Puiseux type on the graph containing either $\{(x_1,x_1)\ :\ x_1 \in S_0\}$ or $\{(x_1,x_{n-1})\ :\ (x_1,\dots,x_{n-1})$ is a path of $\tilde G \cap S_0 \times S_0\}$, where $\tilde G$ is obtained from G by subtracting all loops (x,x) such that $D(x,x) = 0$. We write $(\tilde\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1}(x,y) \sim \bar\alpha^{-1}A'(x,y)\epsilon^{a'(x,y)}$. Therefore, for $x \in \bar S_i$ and $y \in \bar S_j$, one has $\tilde M_\epsilon^{ij}(x,y) \sim A(x,y)\epsilon^{a(x,y)}$ and

$$\tilde M_\epsilon^{i0}(\tilde\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1}\tilde M_\epsilon^{0j}(x,y) \sim \sum_{z,w \in S_0} A(x,z)\frac{A'(z,w)}{\bar\alpha}A(w,y)\epsilon^{a(x,z)+a'(z,w)+a(w,y)}.$$

One can see the previous estimate as a sum over paths $\underline{x}$ of two kinds. Either there exists a G-admissible path $\underline{x} = (x,z,y)$ (for $z = w$), or there exists a G-admissible path $\underline{x} = (x_0,\dots,x_n)$ of length $n \geq 3$, with $x_0 = x$, $x_1 = z$, $x_{n-1} = w$, $x_n = y$, such that the intermediate path $(x_1,\dots,x_{n-1})$ is $(\tilde G \cap S_0 \times S_0)$-admissible and realizes the minimum in the definition of $a'(z,w)$. Each one of these terms is of the form

$$\Big[\Pi_{k=0}^{n-1}A(x_k,x_{k+1})/\bar\alpha^{n-1}\Big]\epsilon^{\sum_{k=0}^{n-1}a(x_k,x_{k+1})}.$$

The dominant term is obtained by minimizing $a(\underline{x})$ over $\underline{x}$. $\square$

6 Complete classification for 3-states spin systems

We consider in this section a full weighted graph of exact Puiseux type on 3 states. More precisely, for $S = \{1,2,3\}$, we consider $G = S \times S$ weighted by

$$M_\epsilon(x,y) = \exp[-\beta H(x,y)] = \epsilon^{H(x,y)}, \quad \epsilon = e^{-\beta}, \quad \forall\, x,y \in S.$$

We assume (by subtracting $\bar{H}$) that H has been normalized: $\bar{H} = 0$. We are interested in describing the unique zero-temperature Gibbs measure μ^H_{min} (notations of section 3) obtained as a limit of

$$\big(\pi_\epsilon(x), Q_\epsilon(x,y)\big) = \Big(L_\epsilon(x)R_\epsilon(x), \frac{M_\epsilon(x,y)R_\epsilon(y)}{\lambda_\epsilon R_\epsilon(x)}\Big)$$

as $\epsilon \to 0$. As it will be clear from the computation, the limit depends from the possibility to expand each quotient $R_\epsilon(x)/R_\epsilon(y)$ and $L_\epsilon(x)/L_\epsilon(y)$ into a Puiseux series of an *a priori* arbitrarily large precision. The algorithm is based on the dimension of the matrix M_ϵ. We will obtain a finite set of possible μ^H_{min} and for each of them we describe the space of parameters $\{H(x,y) \,:\, x,y \in S\}$ which exhibit that zero-temperature Gibbs measure. The dimension of this space of parameters is *a priori* 9; we will reduce it to 2 in the following discussion. We describe each domain according to the number of irreducible components of the minimizing subgraph. We use algorithm 29 to conjugate M_ϵ to a simpler matrix $M'_\epsilon = \Delta_\epsilon M_\epsilon \Delta_\epsilon^{-1}$, which (by possibly permuting $\{1,2,3\}$) takes one of the following form.

i. A unique dominant irreducible component.

   – When the dominant spectral radius $\bar{\alpha}$ is equal to 1, $G_{min} = \bar{G}$ is irreducible and there are three possibilities corresponding respectively with $\bar{S} = \{1,2,3\}$, $\bar{S} = \{1,2\}$ or $\bar{S} = \{1\}$,

$$M'_\epsilon = \begin{bmatrix} \epsilon^a & 1 & \epsilon^{b'} \\ \epsilon^{c'} & \epsilon^b & 1 \\ 1 & \epsilon^{a'} & \epsilon^c \end{bmatrix}, \; M''_\epsilon = \begin{bmatrix} \epsilon^a & 1 & \epsilon^d \\ 1 & \epsilon^b & \epsilon^e \\ \epsilon^{d'} & \epsilon^{e'} & \epsilon^c \end{bmatrix}, \; M'''_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^c \\ \epsilon^{a'} & \epsilon^b & \epsilon^d \\ \epsilon^{c'} & \epsilon^{d'} & \epsilon^e \end{bmatrix}.$$

   (Notice that all coefficients $a, a', b, \ldots$ are positive.)

   – When $\bar{\alpha} > 1$, $\bar{G} = G_{min}$ is obtained by replacing in the previous M'_ϵ any (but at least one) $a, a', b, \ldots$ by 0, and in M''_ϵ one of the two coefficients a and/or b by 0 and leaving $c, c', d, \ldots$ positive. When $\bar{\alpha} > 1$, $\bar{G} \subset G_{min}$ with two irreducible components is obtained by replacing a and/or b in M''_ϵ by 0 and c by 0. Notice that we obtain a finite list of possible $\bar{\alpha}$.

ii. Two irreducible components with equal dominant spectral radius:

$$\bar{\alpha} = 1, \quad M'_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & \epsilon^c & 1 \\ \epsilon^{b'} & 1 & \epsilon^d \end{bmatrix}, \quad \text{or} \quad M''_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & \epsilon^d \end{bmatrix}.$$

iii. Three irreducible components with dominant spectral radius 1:

$$\bar{\alpha} = 1, \quad M'_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & 1 \end{bmatrix}.$$

In order to simplify notations, we introduce the following convention

$$a\#b = 1 \quad \text{if} \quad a \neq b, \quad a\#b = 2 \quad \text{if} \quad a = b.$$

In the case of one irreducible component with dominant spectral coefficient ($r = 1$), $\pi_\epsilon(x) \to 0$ for all $x \in S \setminus \bar{S}$ and $\pi_\epsilon(x) \to \bar{\pi}^1(x)$ for all $x \in \bar{S}$. For instance, for M'_ϵ, M''_ϵ and M'''_ϵ, respectively, π_ϵ converges to $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$, $[\frac{1}{2}, \frac{1}{2}, 0]$ and $[1, 0, 0]$. We now treat in detail the two remaining cases *ii* and *iii*.

6.1 Two irreducible components. Part I

We first consider the matrix

$$M_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & \epsilon^c & 1 \\ \epsilon^{b'} & 1 & \epsilon^d \end{bmatrix}, \quad a, a', b, b', c, d > 0.$$

We already know that $\lambda_\epsilon \sim 1$, $R_\epsilon(2) \sim R_\epsilon(3)$ and $L_\epsilon(2) \sim L_\epsilon(3)$. We collapse the two components 2 and 3 and obtain for the right eigenvector

$$M_\epsilon^{(1)} = \begin{bmatrix} 0 & (\epsilon^a R_2 + \epsilon^b R_3)/(R_2 + R_3) \\ \epsilon^{a'} + \epsilon^{b'} & (\epsilon^c R_2 + \epsilon^d R_3)/(R_2 + R_3) \end{bmatrix} \sim \begin{bmatrix} 0 & \frac{a\#b}{2}\epsilon^{a\wedge b} \\ a'\#b'\epsilon^{a'\wedge b'} & \frac{c\#d}{2}\epsilon^{c\wedge d} \end{bmatrix}.$$

Note that $M_\epsilon^{(1)}$ is of exact Puiseux type. Let r and ρ be the minimizing mean exponent and the dominant spectral radius of $M_\epsilon^{(1)}$. Then $\lambda_\epsilon^{(1)} = \lambda_\epsilon - 1 \sim \rho\epsilon^r$,

$$r = \min\Big(c \wedge d, \frac{a \wedge b + a' \wedge b'}{2}\Big), \ \frac{R_1}{R_3} \sim \frac{a\#b}{\rho}\epsilon^{a\wedge b - r}, \ \frac{L_1}{L_3} \sim \frac{a'\#b'}{\rho}\epsilon^{a'\wedge b' - r}.$$

We thus obtain a complete formula for the transition matrix

$$Q_\epsilon \sim \begin{bmatrix} 1 & \frac{\rho}{a\#b}\epsilon^{a - a\wedge b + r} & \frac{\rho}{a\#b}\epsilon^{b - a\wedge b + r} \\ \frac{a\#b}{\rho}\epsilon^{a' + a\wedge b - r} & \epsilon^c & 1 \\ \frac{a\#b}{\rho}\epsilon^{b' + a\wedge b - r} & 1 & \epsilon^d \end{bmatrix} \to Q_{min}^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and for the zero-temperature Gibbs measure

$$\frac{\pi_\epsilon(2)}{\pi_\epsilon(3)} \sim 1 \quad \text{and} \quad \frac{\pi_\epsilon(1)}{\pi_\epsilon(3)} \sim \frac{(a\#b)(a'\#b')}{\rho^2}\epsilon^{a\wedge b + a'\wedge b' - 2r}.$$

We are left to discuss the value of ρ according to the choice of the exponents contributing in the definition of r. We recall that ρ is the largest eigenvalue of the dominant matrix $A_{min}^{(1)}$.

6.1.1 Case $c \wedge d < (a \wedge b + a' \wedge b')/2$:

In this case, $r = c \wedge d$,

$$A^{(1)}_{min} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{c\#d}{2} \end{bmatrix}, \ \rho = \frac{c \wedge d}{2}, \ \lambda_\epsilon = 1 + \frac{c \wedge d}{2}\epsilon^{c\wedge d} + \ldots, \ \mu^H_{min} = \left[0, \tfrac{1}{2}, \tfrac{1}{2}\right].$$

6.1.2 Case $c \wedge d > (a \wedge b + a' \wedge b')/2$:

In this case, $r = \frac{1}{2}(a \wedge b + a' \wedge b')$,

$$A^{(1)}_{min} = \begin{bmatrix} 0 & \frac{a\#b}{2} \\ a'\#b' & 0 \end{bmatrix}, \ \rho = \sqrt{\frac{(a\#b)(a'\#b')}{2}}, \ \mu^H_{min} = \left[\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}\right].$$

6.1.3 Case $c \wedge d = (a \wedge b + a' \wedge b')/2$:

In this case, $r = c \wedge d$,

$$A^H_{min} = \begin{bmatrix} 0 & \frac{a\wedge b}{2} \\ a' \wedge b' & \frac{c\wedge d}{2} \end{bmatrix}, \quad \rho = \frac{c\#d}{4}\left[1 + \sqrt{1 + 8\frac{(a\#b)(a'\#b')}{(c\#d)^2}}\ \right]$$

and the zero-temperature Gibbs measure is proportional to

$$\mu^H_{min} \propto \begin{bmatrix} 16(a\#b)(a'\#b')/(c\#d)^2 \\ \left[1 + \sqrt{1 + 8(a\#b)(a'\#b')/(c\#d)^2}\right]^2 \\ \left[1 + \sqrt{1 + 8(a\#b)(a'\#b')/(c\#d)^2}\right]^2 \end{bmatrix}$$

or $\mu^H_{min}(1) = \dfrac{4(a\#b)(a'\#b')/(c\#d)^2}{1 + 8(a\#b)(a'\#b')/(c\#d)^2 + \sqrt{1 + 8(a\#b)(a'\#b')/(c\#d)^2}}.$

We summarize the discussion in figure 5.

6.2 Two irreducible components. Part II

We consider now the matrix

$$M_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & \epsilon^d \end{bmatrix}, \quad a, a', b, b', c, c', d > 0.$$

Let $[L_\epsilon(x)]_{x=1,2,3}$ and $[R_\epsilon(x)]_{x=1,2,3}$ be the left and right eigenvector for the largest eigenvalue λ_ϵ. We eliminate the negligible variable $x = 3$ by substituting $L_\epsilon(3)$ or $R_\epsilon(3)$ in the first two equations. We subtract the dominant term 1 of λ_ϵ and obtain

$$L^{(1)}_\epsilon M^{(1)}_\epsilon = \lambda^{(1)}_\epsilon L^{(1)}_\epsilon, \quad M^{(1)}_\epsilon R^{(1)}_\epsilon = \lambda^{(1)}_\epsilon R^{(1)}_\epsilon.$$

We summarize the discussion in figure 6.

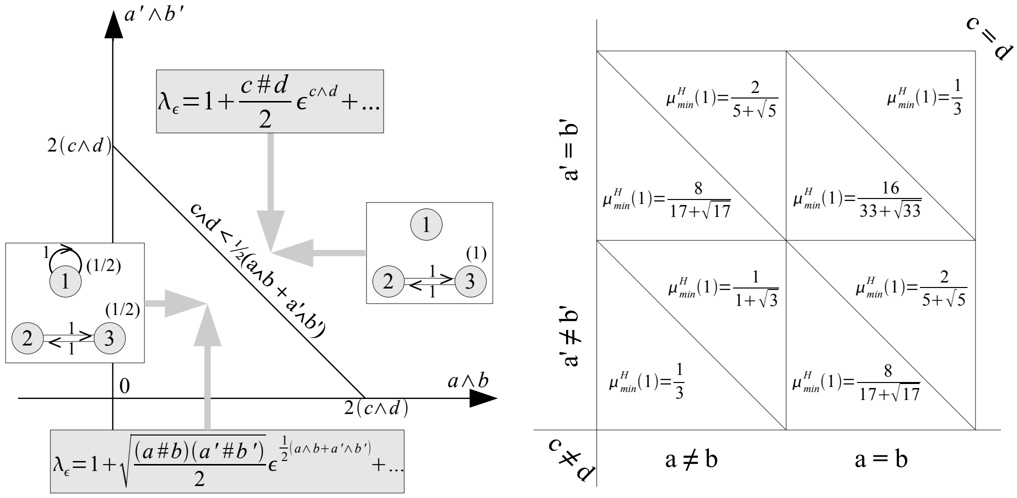


Figure 5: Phase diagram for a 3×3 matrix with two irreducible components: part I. In the left diagram, numbers in parentheses indicate the weight of each irreducible components. In the right diagram, the value of $\mu^H_{min}(1)$ is shown for the case $c \wedge d = \frac{1}{2}(a \wedge b + a' \wedge b')$.

6.3 Three irreducible components

We consider the matrix

$$M_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & 1 \end{bmatrix}, \quad a, a', b, b', c, c' > 0.$$

We know from propositions 30 and 32 that $\lambda_\epsilon \sim 1$ and $Q_\epsilon \to \mathrm{Id}$. We want to show that $[\pi_\epsilon(x)]_{x=1,2,3} = [L_\epsilon(x)R_\epsilon(x)]_{x=1,2,3}$ converges to some raw vector $[\mu^H_{min}(x)]_{x=1,2,3}$ identified to the zero-temperature Gibbs measure as a barycenter of 3 Dirac masses:

$$\mu^H_{min} = \mu^H_{min}(1)\delta_{<1^\infty>} + \mu^H_{min}(2)\delta_{<2^\infty>} + \mu^H_{min}(3)\delta_{<3^\infty>}.$$

Thanks to the special form of the matrix, the steps of algorithm 31 are immediate: $M^{(1)}_\epsilon = M_\epsilon - \mathrm{Id}$, $\lambda^{(1)}_\epsilon = \lambda_\epsilon - 1$, $L^{(1)}_\epsilon = L_\epsilon$ and $R^{(1)}_\epsilon = R_\epsilon$. We want to apply again algorithm 31 by reducing $M^{(1)}_\epsilon$ to a normal form as in algorithm 29. We call $\bar{a}^{(1)}$ the minimizing mean exponent of $M^{(1)}_\epsilon$ and $A^{(1)}_{min}$ the matrix associated with the graph of minimizing cycles. Notice that $A^{(1)}_{min}$ admits a unique irreducible component. Let $v : S \to \mathbb{R}$ be a separating corrector and $\tilde{M}_\epsilon := \Delta_\epsilon(v)M^{(1)}_\epsilon\Delta_\epsilon(v)^{-1}\epsilon^{-\bar{a}^{(1)}} = A^{(1)}_{min} + \tilde{N}_\epsilon$. Denote $\tilde{L}_\epsilon(x) = \epsilon^{-v(x)}L^{(1)}_\epsilon(x)$ and $\tilde{R}_\epsilon(x) = \epsilon^{v(x)}R^{(1)}_\epsilon(x)$. Proposition 32 tells us that

$$\frac{\tilde{L}_\epsilon(x)}{\tilde{L}_\epsilon(y)} \sim \frac{\bar{L}(x)}{\bar{L}(y)}, \quad \frac{\tilde{R}_\epsilon(x)}{\tilde{R}_\epsilon(y)} \sim \frac{\bar{R}(x)}{\bar{R}(y)}, \ \forall\, x, y \in \bar{S}, \quad \text{and} \quad \tilde{L}_\epsilon(x)\tilde{R}_\epsilon(x) \to 0, \ \forall\, x \in S_0,$$

where $\bar{L}$ and $\bar{R}$ are the left and right eigenvectors of the dominant matrix $\bar{A}$.

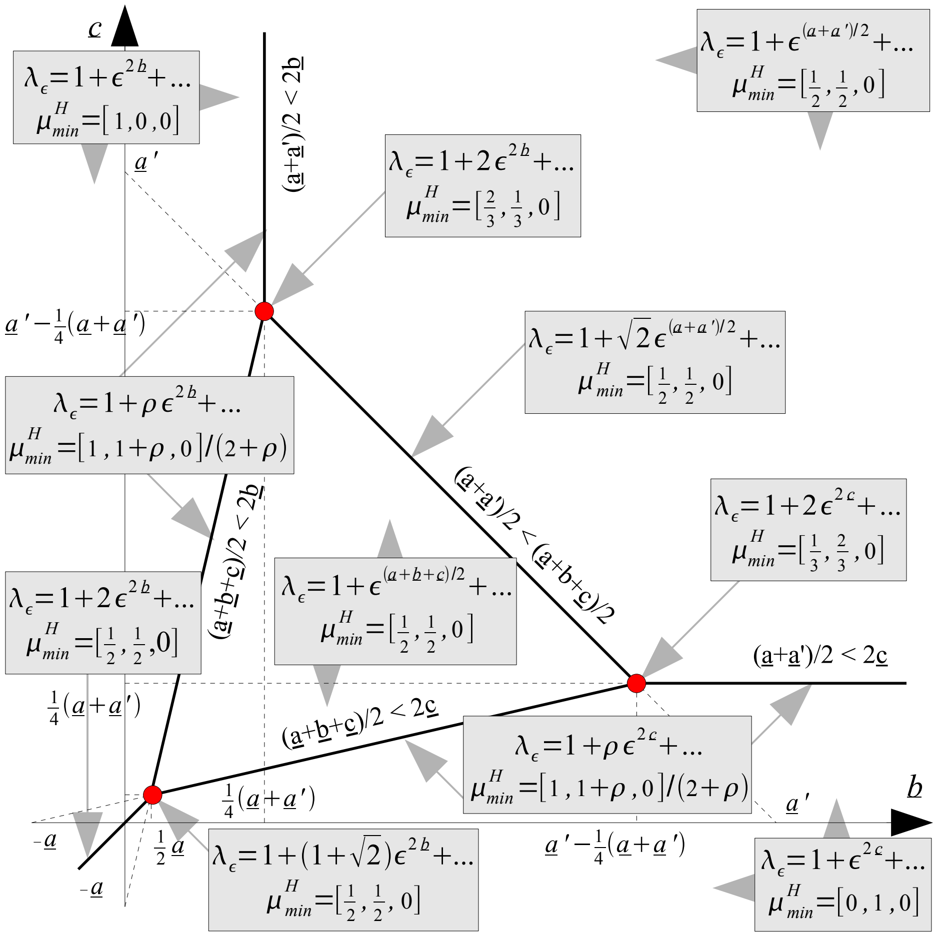


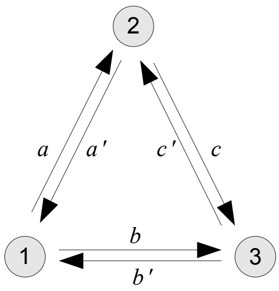
Figure 6: Phase diagram for a 3×3 matrix with two irreducible components: part II. We assume $\underline{a} < \underline{a}'$. The zero-temperature Gibbs measure is a barycenter of the periodic measures δ_1 and δ_2.

In order to simplify the phase transition diagram, we change the coefficients:

$$\underline{a} := \frac{1}{2}(a+a'), \quad \underline{b} := \frac{1}{2}(b+b'),$$
$$\underline{c} := c + \frac{1}{2}(b'-b) + \frac{1}{2}(a-a'), \quad \underline{c}' := c' + \frac{1}{2}(b-b') + \frac{1}{2}(a'-a).$$

Then $\dfrac{c+c'}{2} = \dfrac{\underline{c}+\underline{c}'}{2}$, $\dfrac{a+b'+c}{3} = \dfrac{\underline{a}+\underline{b}+\underline{c}}{3}$, $\dfrac{a'+b+c'}{3} = \dfrac{\underline{a}+\underline{b}+\underline{c}'}{3}$.

We now discuss the different phases according to the coincidence set of multiple order of minimizing cycles. We discuss only the case $\underline{c} < \underline{c}'$. The purely symmetric case $a=a'$, $b=b'$, $c=c'$ is done in section 7. We show in figure 7 the location of all possible minimizing cycles.



Mean along the cycles of order 2 and 3:

cycles of order 2	cycles of order 3
$\frac{1}{2}(a+a')$	$\frac{1}{3}(a+b'+c)$
$\frac{1}{2}(b+b')$	$\frac{1}{3}(a'+b+c')$
$\frac{1}{2}(c+c')$	

Figure 7: Graph of interactions and minimizing cycles of $M_\epsilon - \mathrm{Id}$.

6.3.1 Case $\underline{a} < \min\{\underline{b}, \frac{1}{2}(\underline{c}+\underline{c}'), \frac{1}{3}(\underline{a}+\underline{b}+\underline{c})\}$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \epsilon^{\underline{a}}, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{2}, \frac{1}{2}, 0].$$

6.3.2 Case $\underline{b} < \min\{\underline{a}, \frac{1}{2}(\underline{c}+\underline{c}'), \frac{1}{3}(\underline{a}+\underline{b}+\underline{c})\}$:

$$\bar{a}^{(1)} = \underline{b}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \epsilon^{\underline{b}}, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{2}, 0, \frac{1}{2}].$$

6.3.3 Case $\frac{1}{2}(\underline{c}+\underline{c}') < \min\{\underline{a}, \underline{b}, \frac{1}{3}(\underline{a}+\underline{b}+\underline{c})\}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c}+\underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \epsilon^{(\underline{c}+\underline{c}')/2}, \quad \text{and} \quad \mu^H_{min} = [0, \frac{1}{2}, \frac{1}{2}].$$

6.3.4 Case $\frac{1}{3}(\underline{a}+\underline{b}+\underline{c}) < \min\{\underline{a}, \underline{b}, \frac{1}{2}(\underline{c}+\underline{c}')\}$:

$$\bar{a}^{(1)} = \frac{1}{3}(\underline{a}+\underline{b}+\underline{c}), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \epsilon^{(\underline{a}+\underline{b}+\underline{c})/3},$$

$$\tilde{L}_\epsilon \propto [1,1,1], \quad \tilde{R}_\epsilon \propto [1,1,1]^T, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}].$$

Notice that the reverse cycle $1 \to 3 \to 2 \to 1$ is negligible against the cycle $1 \to 2 \to 3 \to 1$ since its exponent is higher.

6.3.5 Case $\underline{a} = \frac{1}{3}(\underline{a}+\underline{b}+\underline{c}) < \min\{\underline{b}, \frac{1}{2}(\underline{c}+\underline{c}')\}$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \kappa\epsilon^{\underline{a}},$$

$$\tilde{L}_\epsilon \propto [\kappa^2, \kappa, 1], \quad \tilde{R}_\epsilon \propto [\kappa, \kappa^2, 1]^T, \quad \text{and} \quad \mu^H_{min} = [1+\kappa, 1+\kappa, 1]/(3+2\kappa),$$

where κ is the largest eigenvalue of $A^{(1)}_{min}$ and satisfies $\kappa^3 - \kappa - 1 = 0$.

6.3.6 Case $\underline{b} = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \min\{\underline{a}, \frac{1}{2}(\underline{c} + \underline{c}')\}$:

$$\bar{a}^{(1)} = \underline{b}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \kappa \epsilon^{\underline{b}},$$

$$\tilde{L}_\epsilon \propto [\kappa, 1, \kappa^2], \quad \tilde{R}_\epsilon \propto [\kappa^2, 1, \kappa]^T, \quad \text{and} \quad \mu^H_{min} = [1+\kappa, 1, 1+\kappa]/(3+2\kappa).$$

($A^{(1)}_{min}$ admits the same characteristic polynomial as before.)

6.3.7 Case $\frac{1}{2}(\underline{c} + \underline{c}') = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \min\{\underline{a}, \underline{b}\}$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \kappa \epsilon^{(\underline{c}+\underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [1, \kappa^2, \kappa], \quad \tilde{R}_\epsilon \propto [1, \kappa, \kappa^2]^T, \quad \text{and} \quad \mu^H_{min} = [1, 1+\kappa, 1+\kappa]/(3+2\kappa).$$

6.3.8 Case $\underline{a} = \underline{b} < \min\{\frac{1}{2}(\underline{c} + \underline{c}'), \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})\}$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \sqrt{2} \epsilon^{\underline{a}},$$

$$\tilde{L}_\epsilon \propto [\sqrt{2}, 1, 1], \quad \tilde{R}_\epsilon \propto [\sqrt{2}, 1, 1]^T, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{2}, \frac{1}{4}, \frac{1}{4}].$$

6.3.9 Case $\underline{a} = \frac{1}{2}(\underline{c} + \underline{c}') < \min\{\underline{b}, \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})\}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c} + \underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \sqrt{2} \epsilon^{(\underline{c}+\underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [1, \sqrt{2}, 1], \quad \tilde{R}_\epsilon \propto [1, \sqrt{2}, 1]^T, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}].$$

6.3.10 Case $\underline{b} = \frac{1}{2}(\underline{c} + \underline{c}') < \min\{\underline{a}, \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})\}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c} + \underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \sqrt{2} \epsilon^{(\underline{c}+\underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [1, 1, \sqrt{2}], \quad \tilde{R}_\epsilon \propto [1, 1, \sqrt{2}]^T, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{2}].$$

6.3.11 Case $\underline{a} = \underline{b} = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \frac{1}{2}(\underline{c} + \underline{c}')$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \rho\epsilon^{\underline{a}},$$

$$\tilde{L}_\epsilon \propto [\rho, 1, \rho], \quad \tilde{R}_\epsilon \propto [\rho, \rho, 1]^T, \quad \text{and} \quad \mu^H_{min} = [\rho, 1, 1]/(2+\rho),$$

where ρ is the positive root of $\rho^3 - 2\rho - 1 = (\rho+1)(\rho^2 - \rho - 1) = 0$.

6.3.12 Case $\underline{a} = \frac{1}{2}(\underline{c} + \underline{c}') = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \underline{b}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c} + \underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \rho\epsilon^{(\underline{c}+\underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [\rho, \rho, 1], \quad \tilde{R}_\epsilon \propto [1, \rho, \rho]^T, \quad \text{and} \quad \mu^H_{min} = [1, \rho, 1]/(2+\rho).$$

6.3.13 Case $\underline{b} = \frac{1}{2}(\underline{c} + \underline{c}') = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \underline{a}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c} + \underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \rho\epsilon^{(\underline{c}+\underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [1, \rho, \rho], \quad \tilde{R}_\epsilon \propto [\rho, 1, \rho]^T, \quad \text{and} \quad \mu^H_{min} = [1, 1, \rho]/(2+\rho).$$

We summarize the preceding discussion in the figure 8.

7 Zero-temperature phase diagram for BEG model

We give in this section a complete description of the zero-temperature phase diagram for the Blume-Emery-Griffiths model. We apply the algorithm proposed in section 3 to $S = \{-, 0, +\}$, $G = S \times S$ and $M_\epsilon(x, y) = \epsilon^{H_0(x,y)}$ for all $x, y \in S$, where

$$H_0 = \begin{bmatrix} -J - K + \Delta & \frac{1}{2}\Delta & J - K + \Delta \\ \frac{1}{2}\Delta & 0 & \frac{1}{2}\Delta \\ J - K + \Delta & \frac{1}{2}\Delta & -J - K + \Delta \end{bmatrix}.$$

We discuss the different cases according to the choice of the parameters which contribute to the minimizing mean exponent $\bar{a}$. In all cases, we have

$$M_\epsilon = \begin{bmatrix} \epsilon^a & \epsilon^b & \epsilon^c \\ \epsilon^b & 1 & \epsilon^b \\ \epsilon^c & \epsilon^b & \epsilon^a \end{bmatrix}, \quad \pi_\epsilon = \begin{bmatrix} L_\epsilon(-)R_\epsilon(-) \\ L_\epsilon(0)R_\epsilon(0) \\ L_\epsilon(+)R_\epsilon(+) \end{bmatrix}, \quad Q_\epsilon = \begin{bmatrix} \frac{\epsilon^a R_\epsilon(-)}{\lambda_\epsilon R_\epsilon(-)} & \frac{\epsilon^b R_\epsilon(0)}{\lambda_\epsilon R_\epsilon(-)} & \frac{\epsilon^c R_\epsilon(+)}{\lambda_\epsilon R_\epsilon(-)} \\ \frac{\epsilon^b R_\epsilon(-)}{\lambda_\epsilon R_\epsilon(0)} & \frac{R_\epsilon(0)}{\lambda_\epsilon R_\epsilon(0)} & \frac{\epsilon^b R_\epsilon(+)}{\lambda_\epsilon R_\epsilon(0)} \\ \frac{\epsilon^c R_\epsilon(-)}{\lambda_\epsilon R_\epsilon(+)} & \frac{\epsilon^b R_\epsilon(0)}{\lambda_\epsilon R_\epsilon(+)} & \frac{\epsilon^a R_\epsilon(+)}{\lambda_\epsilon R_\epsilon(+)} \end{bmatrix},$$

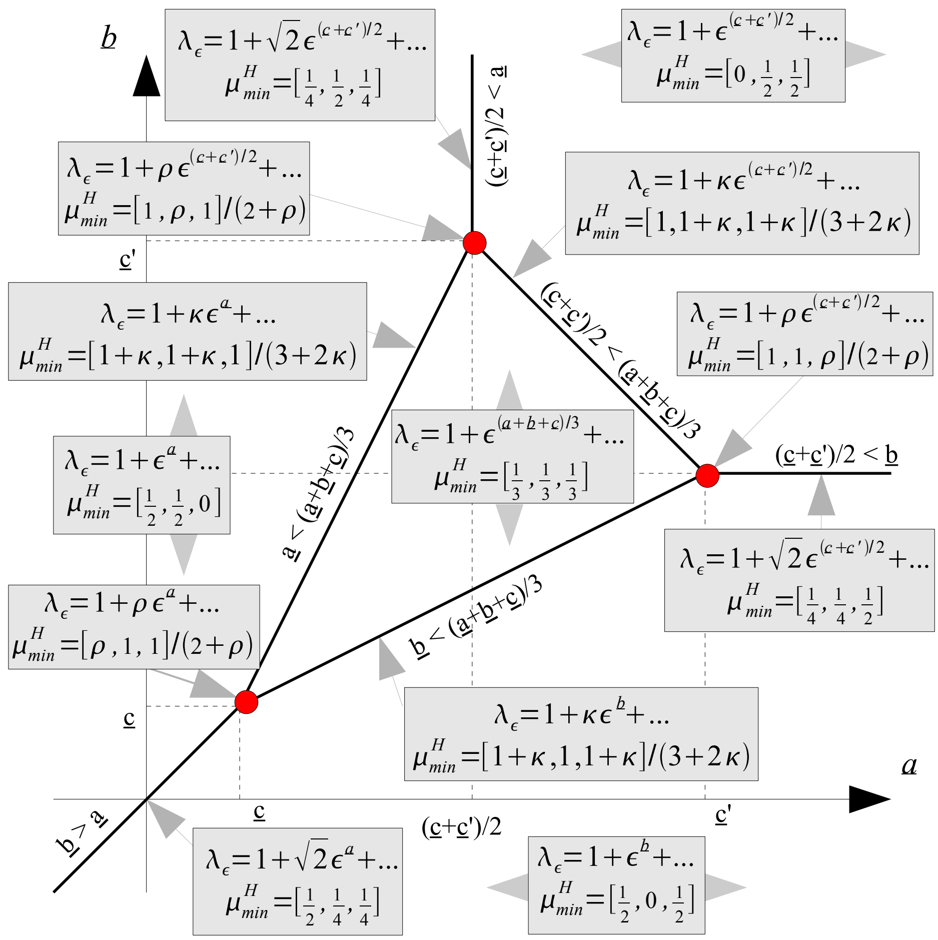


Figure 8: Phase diagram for a 3×3 matrix with three irreducible components. We assume $\underline{c} < \underline{c}'$. The zero-temperature Gibbs measure is a barycenter of the three periodic measures $\delta_{<1\infty>}$, $\delta_{<2\infty>}$ and $\delta_{<3\infty>}$. The constants ρ and κ are solutions of $\rho^2 - \rho - 1 = 0$ and $\kappa^3 - \kappa - 1 = 0$. The exact values of these constants are $\rho = \frac{1}{2}(1+\sqrt{5})$ and $\kappa = \sqrt[3]{\frac{1}{2}(1-\sqrt{23/27})} + \sqrt[3]{\frac{1}{2}(1+\sqrt{23/27})}$.

normalized by $\sum_{x\in S} L_\epsilon(x)R_\epsilon(x) = 1$ and $\sum_{x\in S} R_\epsilon(x) = 1$. Because of the symmetry of M_ϵ, $L_\epsilon = R_\epsilon$ and $\pi_\epsilon(x) = R_\epsilon^2(x)/\sum_x R_\epsilon^2(x)$. We also simplify the computation by noticing that $R_\epsilon(-) = R_\epsilon(+)$. We recall that G_{min} is the minimizing subgraph and $\bar{\alpha}$ is the dominant spectral coefficient. We only present the details of the computations for $\Delta > 0$, the other situations being analogous.

7.1 Case $J - K + \Delta < 0$, $J < 0$:

Case: $c < \min(0, a, b)$. We know that

$$\bar{a} = c, \quad A_{min} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \bar{\alpha} = 1, \quad \lambda_\epsilon \sim \epsilon^c$$

and G_{min} has one irreducible component $(-) \rightleftharpoons (+)$. We aggregate the two components $(\pm)$ by adding $R_\epsilon(\pm) := R_\epsilon(-) + R_\epsilon(+)$ and eliminate the negligible term $R_\epsilon(0)$. The new singular eigenvalue problem obtained in algorithm 31, $M_\epsilon^{(1)} R_\epsilon^{(1)} = \lambda_\epsilon^{(1)} R_\epsilon^{(1)}$, is actually reduced to a unique equation with unique unknown $R_\epsilon^{(1)} := R_\epsilon(\pm)$. More precisely,

$$\begin{cases} (\epsilon^a + \epsilon^c) R_\epsilon(\pm) + 2\epsilon^b R_\epsilon(0) &= \lambda_\epsilon R_\epsilon(\pm), \\ \epsilon^b R_\epsilon(\pm) + R_\epsilon(0) &= \lambda_\epsilon R_\epsilon(0), \end{cases}$$

$$R_\epsilon(0) = \frac{\epsilon^b}{\lambda_\epsilon - 1} R_\epsilon(\pm) \ll R_\epsilon(\pm), \quad \lambda_\epsilon^{(1)} := \lambda_\epsilon - \epsilon^c = \epsilon^a + \frac{2\epsilon^{2b}}{\lambda_\epsilon - 1},$$

which yields

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-c} \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \epsilon^{-J+K-\Delta/2} \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2(-J+K-\Delta/2)} \\ 1/2 \end{bmatrix},$$

$$Q_\epsilon \sim \begin{bmatrix} \epsilon^{a-c} & 2\epsilon^{2(b-c)} & 1 \\ 1/2 & \epsilon^{-c} & 1/2 \\ 1 & 2\epsilon^{2(b-c)} & \epsilon^{a-c} \end{bmatrix} = \begin{bmatrix} \epsilon^{-2J} & 2\epsilon^{2(-J+K-\Delta/2)} & 1 \\ 1/2 & \epsilon^{-J+K-\Delta} & 1/2 \\ 1 & 2\epsilon^{2(-J+K-\Delta/2)} & \epsilon^{-2J} \end{bmatrix}.$$

7.2 Case $-J - K + \Delta < 0$, $J > 0$:

Case: $a < \min(0, b, c)$. G_{min} has two irreducible components with identical spectral coefficient, $(-) \leftrightarrow (-)$ and $(+) \leftrightarrow (+)$, and as before $R_\epsilon(0) \ll R_\epsilon(-) = R_\epsilon(+)$. We thus obtain

$$\bar{a} = a, \quad A_{min} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{\alpha} = 1, \quad \lambda_\epsilon \sim \epsilon^a,$$

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-a} \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2(b-a)} \\ 1/2 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} 1 & 2\epsilon^{2(b-a)} & \epsilon^{c-a} \\ 1/2 & \epsilon^{-a} & 1/2 \\ \epsilon^{c-a} & 2\epsilon^{2(b-a)} & 1 \end{bmatrix}.$$

7.3 Case $-J - K + \Delta > 0$, $J - K + \Delta > 0$:

Case: $0 < \min(a, b, c)$. G_{min} has one irreducible component $(0) \leftrightarrow (0)$, $\bar{\alpha} = 1$ and $R_\epsilon(-) = R_\epsilon(+) \ll R_\epsilon(0)$. We obtain

$$\bar{a} = 0, \quad A_{min} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\alpha} = 1, \quad \lambda_\epsilon \sim 1,$$

$$R_\epsilon \sim \begin{bmatrix} \epsilon^b \\ 1 \\ \epsilon^b \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} \epsilon^{2b} \\ 1 \\ \epsilon^{2b} \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} \epsilon^a & 1 & \epsilon^c \\ \epsilon^{2b} & 1 & \epsilon^{2b} \\ \epsilon^c & 1 & \epsilon^a \end{bmatrix}.$$

As in case 7.1, we eliminate the negligible term $R_\epsilon(\pm)$ and get a new graph $G^{(1)}$ reduced to a singleton

$$R_\epsilon(\pm) = \frac{2\epsilon^b}{\lambda_\epsilon - (\epsilon^a + \epsilon^c)} R_\epsilon(0) \quad \text{and} \quad \lambda_\epsilon - 1 = \frac{2\epsilon^{2b}}{\lambda_\epsilon - (\epsilon^a + \epsilon^c)} \sim 2\epsilon^{2b}.$$

7.4 Case $J - K + \Delta = 0$, $J < 0$:

Case: $c = 0 < \min(a, b)$. We know that

$$\bar{a} = 0, \quad A_{min} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \bar{\alpha} = 1, \quad \lambda_\epsilon \sim 1,$$

and G_{min} has two irreducible components $(-) \rightleftharpoons (+)$ and $(0) \leftrightarrow (0)$. No state $x \in S$ is *a priori* negligible. We then aggregate the states $(\pm)$ by adding $R_\epsilon(\pm) := R_\epsilon(+) + R_\epsilon(-)$ and obtain a new eigenvalue problem $M_\epsilon^{(1)} R_\epsilon^{(1)} = \lambda_\epsilon^{(1)} R_\epsilon^{(1)}$, where

$$M_\epsilon^{(1)} := \begin{bmatrix} \epsilon^a & 2\epsilon^b \\ \epsilon^b & 0 \end{bmatrix}, \quad R_\epsilon^{(1)} := \begin{bmatrix} R_\epsilon(\pm) \\ R_\epsilon(0) \end{bmatrix} \quad \text{and} \quad \lambda_\epsilon^{(1)} := \lambda_\epsilon - 1.$$

We then have to discuss three subcases.

7.4.1 Subcase $J < -\frac{1}{4}\Delta < 0$:

Subcase: $b < a$. The minimizing subgraph $G_{min}^{(1)}$ has one irreducible component $(\pm) \rightleftharpoons (0)$ with minimizing mean exponent $\bar{a}^{(1)} = b$ and dominant spectral coefficient $\bar{\alpha}^{(1)} = \sqrt{2}$. We obtain

$$\lambda_\epsilon^{(1)} \sim \sqrt{2}\epsilon^b, \quad R_\epsilon^{(1)} \propto \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, \quad R_\epsilon \propto \begin{bmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and}$$

$$\lambda_\epsilon = 1 + \sqrt{2}\epsilon^b + \dots, \quad \pi_\epsilon \sim \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} \epsilon^a & \sqrt{2}\epsilon^b & 1 \\ \epsilon^b/\sqrt{2} & 1 & \epsilon^b/\sqrt{2} \\ 1 & \sqrt{2}\epsilon^b & \epsilon^a \end{bmatrix}.$$

7.4.2 Subcase $J = -\frac{1}{4}\Delta < 0$:

Subcase: $a = b$. $G_{min}^{(1)}$ has one irreducible component $(\pm) \leftrightarrow (\pm) \rightleftharpoons (0)$ with dominant spectral coefficient $\bar{\alpha}^{(1)} = 2$ (the spectral radius of $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$), and the right eigenvector $R_\epsilon^{(1)}$ is proportional to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. We obtain $\lambda_\epsilon^{(1)} \sim 2\epsilon^b$ and

$$\lambda_\epsilon = 1 + 2\epsilon^b + \dots, \quad R_\epsilon \sim \pi_\epsilon \sim \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} \epsilon^b & \epsilon^b & 1 \\ \epsilon^b & 1 & \epsilon^b \\ 1 & \epsilon^b & \epsilon^b \end{bmatrix}.$$

7.4.3 Subcase $-\frac{1}{4}\Delta < J < 0$:

Subcase: $a < b$. The minimizing subgraph $G_{min}^{(1)}$ has one irreducible component $(\pm) \leftrightarrow (\pm)$ with dominant spectral coefficient $\bar{\alpha}^{(1)} = 1$. We obtain therefore

$\lambda_\epsilon^{(1)} \sim \epsilon^a$, $R_\epsilon(0) = \epsilon^{b-a} R_\epsilon(\pm) \ll R_\epsilon(\pm)$, $\lambda_\epsilon = 1 + \epsilon^a + \dots$ and

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-a} \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2(b-a)} \\ 1/2 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} \epsilon^a & 2\epsilon^{2b-a} & 1 \\ \epsilon^a/2 & 1 & \epsilon^a/2 \\ 1 & 2\epsilon^{2b-a} & \epsilon^a \end{bmatrix}.$$

7.5 Case $-J - K + \Delta = 0$, $J > 0$:

Case: $a = 0 < \min(b, c)$. One then has

$$\bar{a} = 0, \quad A_{min} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{\alpha} = 1, \quad \lambda_\epsilon \sim 1.$$

The minimizing subgraph G_{min} has three irreducible components $(-) \leftrightarrow (-)$, $(0) \leftrightarrow (0)$ and $(+) \leftrightarrow (+)$. Once again we simplify the proof by noticing that $R_\epsilon(-) = R_\epsilon(+)$, but it is so far not clear which state dominates. The reduction to an aggregated form consists in simply eliminating the first term of λ_ϵ in the Puiseux series:

$$M_\epsilon^{(1)} = M_\epsilon - \mathrm{Id}, \quad M_\epsilon^{(1)} R_\epsilon^{(1)} = \lambda_\epsilon^{(1)} R_\epsilon^{(2)}, \quad R_\epsilon^{(1)} = R_\epsilon, \quad \lambda_\epsilon^{(1)} = \lambda_\epsilon - 1.$$

The new graph $G^{(1)}$ has possible minimizing mean exponents $\bar{a}^{(1)} = b$ or c. Let $\bar{\alpha}^{(1)}$ be the associated dominant spectral coefficient. We discuss three subcases.

7.5.1 Subcase $0 < \frac{1}{4}\Delta < J$:

Subcase: $b < c$. $G_{min}^{(1)}$ has one irreducible component $(-) \rightleftharpoons (0) \rightleftharpoons (+)$ with $\bar{a}^{(1)} = b$. Moreover,

$$A_{min}^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{\alpha}^{(1)} = \sqrt{2}, \quad R_\epsilon^{(1)} \propto \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

Then $\lambda_\epsilon = 1 + \sqrt{2}\epsilon^b + \dots$ and

$$R_\epsilon \propto \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} 1 & \sqrt{2}\epsilon^b & \epsilon^c \\ \epsilon^b/\sqrt{2} & 1 & \epsilon^b/\sqrt{2} \\ \epsilon^c & \sqrt{2}\epsilon^b & 1 \end{bmatrix}.$$

7.5.2 Subcase $0 < \frac{1}{4}\Delta = J$:

Subcase: $c = b$. The subgraph $G_{min}^{(1)}$ has one irreducible component $(-) \rightleftharpoons (0) \rightleftharpoons (+) \rightleftharpoons (-)$ and

$$A_{min}^{(1)} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \bar{\alpha}^{(1)} = 2, \quad R_\epsilon^{(1)} \propto \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We thus obtain

$$\lambda_\epsilon = 1 + 2\epsilon^b + \ldots, \quad R_\epsilon \sim \pi_\epsilon \sim \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} 1 & \epsilon^b & \epsilon^b \\ \epsilon^b & 1 & \epsilon^b \\ \epsilon^b & \epsilon^b & 1 \end{bmatrix}.$$

7.5.3 Subcase $0 < J < \frac{1}{4}\Delta$:

Subcase: $c < b$. $G^{(1)}_{min}$ has one irreducible component $(-) \rightleftharpoons (+)$ with minimizing mean exponent $\bar{a}^{(1)} = c$ and $\bar{\alpha}^{(1)} = 1$. We aggregate the states $(\pm)$, $R^{(1)}_\epsilon(\pm) := R^{(1)}_\epsilon(-) + R^{(1)}_\epsilon(+)$, and eliminate $R^{(1)}_\epsilon(0) \ll R^{(1)}_\epsilon(\pm)$ to obtain a third graph (reduced to a singleton)

$$\begin{cases} \epsilon^c R^{(1)}_\epsilon(\pm) + 2\epsilon^b R^{(1)}_\epsilon(0) & = \lambda^{(1)}_\epsilon R^{(1)}_\epsilon(\pm), \\ \epsilon^b R^{(1)}_\epsilon(\pm) & = \lambda^{(1)}_\epsilon R^{(1)}_\epsilon(0), \end{cases} \qquad \epsilon^c + \frac{2\epsilon^{2b}}{\lambda^{(1)}_\epsilon} = \lambda^{(1)}_\epsilon.$$

We get $\lambda_\epsilon = 1 + \epsilon^c + \ldots$ and

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-c} \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2(b-c)} \\ 1/2 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} 1 & 2\epsilon^{2b-c} & \epsilon^c \\ \epsilon^c/2 & 1 & \epsilon^c/2 \\ \epsilon^c & 2\epsilon^{2b-a} & 1 \end{bmatrix}.$$

7.6 Case $J = 0 < \Delta < K$:

Case: $a = c < \min(0, b)$. One has

$$\bar{a} = a, \quad A_{min} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \bar{\alpha} = 2, \quad \lambda_\epsilon \sim 2\epsilon^a.$$

G_{min} has one irreducible component $(-) \leftrightarrow (-) \rightleftharpoons (+) \leftrightarrow (+)$ with minimizing mean exponent $\bar{a} = a$ and dominant spectral coefficient $\bar{\alpha} = 2$. We again aggregate the states $(\pm)$, $R_\epsilon(\pm) := R_\epsilon(-) + R_\epsilon(+)$, and eliminate $R_\epsilon(0) \ll R_\epsilon(\pm)$ in order to introduce a new singular eigenvalue problem

$$\begin{cases} 2\epsilon^a R_\epsilon(\pm) + 2\epsilon^R_\epsilon(0) & = \lambda_\epsilon R_\epsilon(\pm), \\ \epsilon^b R_\epsilon(\pm) + R_\epsilon(0) & = \lambda_\epsilon R_\epsilon(0), \end{cases} \qquad 2\epsilon^a + \frac{2\epsilon^{2b}}{\lambda_\epsilon - 1} = \lambda_\epsilon.$$

We thus obtain

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-a}/2 \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{2(b-a)}/2 \\ 1/2 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} 1/2 & \epsilon^{2(b-a)}/2 & 1/2 \\ 1/2 & \epsilon^{-a}/2 & 1/2 \\ 1/2 & \epsilon^{2(b-a)} & 1/2 \end{bmatrix}.$$

7.7 Case $J = 0 < \Delta = K$:

Case: $a = c = 0 < b$. We have

$$\bar{a} = 0, \quad A_{min} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \bar{\alpha} = 2, \quad \lambda_\epsilon \sim 2.$$

G_{min} has two irreducible components with spectral coefficients equal to 1 and 2, whose graphs are $(0) \leftrightarrow (0)$ and $(-) \leftrightarrow (-) \rightleftharpoons (+) \leftrightarrow (+)$, respectively. We aggregate $(\pm)$ into a unique state $R_\epsilon(\pm) := R_\epsilon(-) + R_\epsilon(+)$ and obtain

$$\begin{cases} 2R_\epsilon(\pm) + 2\epsilon^b R_\epsilon(0) &= \lambda_\epsilon R_\epsilon(\pm), \\ \epsilon^b R_\epsilon(\pm) + R_\epsilon(0) &= \lambda_\epsilon R_\epsilon(0), \end{cases} \qquad 2 + \frac{2\epsilon^{2b}}{\lambda_\epsilon - 1} = \lambda_\epsilon.$$

We thus get $\lambda_\epsilon = 2 + 2\epsilon^{2b} + \dots$ and

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^b \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2b} \\ 1/2 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} 1/2 & \epsilon^{2b} & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & \epsilon^{2b} & 1/2 \end{bmatrix}.$$

We recall that the previous discussion is summarized in figures 3 and 4.

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