Dense periodic optimization for countable Markov shift via Aubry points

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Abstract.
For transitive Markov subshifts over countable alphabets, this note ensures that a dense subclass of locally Hölder continuous potentials admits at most a single periodic probability as a maximizing measure with compact support. We resort to concepts analogous to those introduced by Mather and Mañé in the study of globally minimizing curves in Lagrangian dynamics. In particular, given a summable variation potential, we show the existence of a continuous sub-action in the presence of an Aubry point.

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1. Introduction and Main Results

Ergodic optimization in the context of countable Markov shifts registers works mainly focused on ensuring existence and describing properties of optimizing measures [11, 3, 2]. By a perturbative approach, we show here the denseness into a class of locally Hölder continuous potentials of the subset formed by those that admit at most one periodic probability as maximizing measure with compact support. To the best of our knowledge, this is an inaugural result of typicality for optimal measures in this scenario in which the compactness of the phase space is not assumed. Our strategy follows the one adopted for the case of finite alphabets [5, 10, 4]. In addition to the main result, other contribution of these notes to ergodic optimization over non-compact phase spaces is the adaptation to a symbolic dynamic setting of the viewpoint developed by Mañé for Lagrangian systems [6]. Concepts discussed here had not yet been considered with the depth that we were led to take into account. In particular, we had to return to the very concept of the ergodic maximizing constant via Mañé's critical value, an attitude that, to our knowledge, had not yet been used in ergodic optimization.

Let Σ denote a one-sided Markov shift on a countable alphabet and $\sigma : \Sigma \to \Sigma$ the left shift map. To be concrete, we assume from this point forward that the alphabet is the set of non-negative integers $\mathbb{Z}_+$. As usual $\mathbb{Z}_+$ is provided with the discrete topology and Σ with the product topology, being metrizable when, for a fixed $\lambda \in (0, 1)$, the distance between two sequences $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$ is $d(x, y) = \lambda^\ell$, where ℓ indicates the first position of disagreement. We assume throughout the text that the dynamics (Σ, σ) is topologically transitive.

We call a potential any continuous function $A : \Sigma \to \mathbb{R}$ bounded from above. The main objective is to study (when they exist) the σ-invariant probabilities that maximize the integral of A over Σ. The regularity of the potential plays an important role in this analysis. In fact, much of the subtlety of results in ergodic optimization lies in this specific aspect. For subshifts over finite alphabets, for instance, one can either generically have a single maximizing probability supported in a periodic orbit by focusing on Hölder continuous potentials [5], or observe in a dense way the phenomenon of the existence of uncountably many ergodic maximizing measures with full support and positive entropy when taking into account continuous potentials in general [14].

We first consider potentials with summable variation. For a function $A : \Sigma \to \mathbb{R}$ and a non-negative integer $\ell \geq 0$, the ℓ-th variation is defined as

$$\mathrm{Var}_\ell(A) = \sup\left\{A(x) - A(y) \,:\, d(x, y) \leq \lambda^\ell\right\}.$$

For all $0 \leq n < m \leq \infty$, we denote $\mathrm{Var}_n^m(A) := \sum_{\ell=n}^m \mathrm{Var}_\ell(A)$. We say that A is of summable variation when

$$\mathrm{Var}_1^\infty(A) := \sum_{\ell=1}^\infty \mathrm{Var}_\ell(A) < +\infty.$$

(Note that no restriction is imposed on the zeroth variation of A.) The space of real-valued functions of summable variation is a disjoint collection of affine spaces, each of which is a metric space with respect to the distance

$$\|A - B\|_{\mathrm{sv}} := \mathrm{Var}_1^\infty(A - B) + \|A - B\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the supremum norm. A specific class will receive our attention: a potential A is said to be locally Hölder continuous when there exists a positive constant $\mathrm{Lip}_{\mathrm{loc}}(A)$ such that $\mathrm{Var}_\ell(A) \leq \mathrm{Lip}_{\mathrm{loc}}(A)\, \lambda^\ell$ for all $\ell \geq 1$.

Given a potential A, we introduce the ergodic maximizing constant as

$$\beta_A = \sup\left\{\int_\Sigma A \,\mathrm{d}\mu \,:\, \mu \text{ is } \sigma\text{-invariant probability}\right\}.$$

Obviously $\beta_A \in (-\infty, \sup A]$. We say that an invariant probability measure μ is maximizing whenever $\int A \,\mathrm{d}\mu = \beta_A$. The existence of maximizing measures is far from obvious in the general context of Markov shifts. When these exist, however, they are expected to be supported in the Aubry set, the habitat of maximizing trajectories (for details, see Chapter 4 of [8]).

Definition 1.1 (Aubry set)**.** We say that $x \in \Sigma$ is an Aubry point for the potential A when, for any $\varepsilon > 0$, there are a point $w \in \Sigma$ and an integer $n > 0$ such that $d(x, w) < \varepsilon$, $d(\sigma^n(w), x) < \varepsilon$ and

$$-\varepsilon < S_n(\beta_A - A)(w) < \varepsilon,$$

where S_n indicates the n-th Birkhoff sum. The set of Aubry points is denoted by $\Omega(A)$.

Our main result can be stated as follows.

Main Theorem. *Let (Σ, σ) be a topologically transitive Markov shift on a countable alphabet. Let A be a locally Hölder continuous potential. Suppose that a non-empty compact invariant subset of its Aubry set is contained in a subshift over a finite alphabet. Then, for any $\varepsilon > 0$, there is a locally Hölder continuous potential B such that $\|A - B\|_{\text{sv}} < \varepsilon$ and B has at most a single periodic probability as maximizing measure with compact support.*

Note that, thanks to Atkinson's theorem [1], the hypothesis on the Aubry set is equivalent to the existence of a maximizing probability with compact support. Examples of potentials that fulfill this last requirement are provided by the coercive ones [2, Theorem 1]. Remember that a potential A is said to be coercive when $\lim_{i \to +\infty} \sup A|_{[i]} = -\infty$, where $[i]$ is the cylinder set $\{x = (x_0, x_1, \ldots) \in \Sigma : x_0 = i\}$. Thus, a particular consequence of our main result is the following one.

Corollary for Coercive Potentials. *Let (Σ, σ) be a topologically transitive Markov shift on a countable alphabet. Let A be a coercive locally Hölder continuous potential. Then, given $\varepsilon > 0$, there exists a coercive locally Hölder continuous potential B such that $\|A - B\|_{\text{sv}} < \varepsilon$ and B admits a single periodic probability as maximizing measure.*

For primitive subshifts, the so-called oscillation condition (for details, see [11, Definition 5.1]) actually allows to consider more general class of examples [11, Theorem 6.1]. As a matter of fact, the central results in [11, 3, 2] ensure that, in the cases analyzed, any maximizing probability is indeed supported in a subshift over a finite subalphabet. Our main theorem takes into account also other situations. For instance, if $\Sigma_0, \Sigma_1 \subset \Sigma$ are disjoint subshifts, the first one over a finite subalphabet and the second one over a countable subalphabet, the result applies to the potential $A = -d(\Sigma_0, \cdot) d(\Sigma_1, \cdot)$, whose maximizing probabilities are clearly all those invariant ones supported either in Σ_0 or in Σ_1.

A key ingredient for obtaining the above theorem is the notion of a sub-action, namely, a continuous function $u : \Sigma \to \mathbb{R}$ such that

$$A + u \circ \sigma - u \leq \beta_A \qquad \text{everywhere on} \quad \Sigma.$$

Previously the notion of normal form [11, Definition 2.2] consisted of an interesting proposal to extend the role played by sub-actions in characterizing maximizing probabilities. An ancillary result of this work is the guarantee of existence of a sub-action via the Mañé potential ϕ_A or the Peierls barrier h_A, both defined on $\Sigma \times \Sigma$, objects hitherto unexplored in the general context of potentials of summable variation on transitive countable Markov shifts. Their precise definition is postponed until the next session. In the Lagrangian setting, the pioneer notions were introduced in [13, 12].

Collateral Theorem. *Let (Σ, σ) be a topologically transitive Markov shift on a countable alphabet. Let A be a potential of summable variation. Suppose there exists an Aubry point for A. Then, for any $x \in \Omega(A)$ fixed, the potential admits a sub-action defined as*

$$u_x(\,\cdot\,) = \phi_A(x,\,\cdot\,) = h_A(x,\,\cdot\,),$$

which has ℓ-th variation bounded from above by $\mathrm{Var}_\ell^\infty(A)$. If $\sum_{\ell=1}^\infty \mathrm{Var}_\ell^\infty(A) < +\infty$, the sub-action u_x has summable variation. In particular, a locally Hölder continuous potential admits a locally Hölder continuous sub-action.

The paper is organized as follows. In section 2, we introduce both the Mañé potential and the Peierls barrier and highlight their main properties. The third section is devoted to the Aubry set, with special attention to its interactions with these action potentials. In particular, Collateral Theorem corresponds to Theorem 3.9. The proof of Main Theorem is presented in the final section of this note.

2. Mañé Potential and Peierls Barrier

2.1. Fundamental Facts

Both the Mañé potential and the Peierls barrier are action potentials between points, the first considers trajectories of any size, the second focuses on arbitrarily long trajectories. We can introduce them through the following auxiliary function.

Definition 2.1. Let $A : \Sigma \to \mathbb{R}$ be a potential and $\gamma \in \mathbb{R}$ be a constant. Given integers $k \geq 0$ and $l \geq 0$, we define for $x, y \in \Sigma$,

$$^{\gamma}_{A}\mathfrak{S}_l^k(x,y) := \inf_{n \geq l} \inf_{\substack{d(x,w) \leq \lambda^k \\ d(\sigma^n(w),y) \leq \lambda^k}} S_n(\gamma - A)(w).$$

To avoid cumbersome notation, when it is clear the potential taken into account, we will simply denote $^{\gamma}\mathfrak{S}_l^k(x,y)$. Likewise, when $\gamma = \beta_A$, we will just use $\mathfrak{S}_l^k(x,y)$.

Concerning its basic properties, this function clearly fulfills, for all $x, y \in \Sigma$,

$$0 \leq k, l \implies {}^{\gamma}\mathfrak{S}_l^k(x,y) < +\infty; \tag{1}$$

$$0 \leq l_1 \leq l_2 \implies {}^{\gamma}\mathfrak{S}_{l_1}^k(x,y) \leq {}^{\gamma}\mathfrak{S}_{l_2}^k(x,y); \tag{2}$$

$$0 \leq k_1 \leq k_2 \implies {}^{\gamma}\mathfrak{S}_l^{k_1}(x,y) \leq {}^{\gamma}\mathfrak{S}_l^{k_2}(x,y). \tag{3}$$

Besides, the auxiliary function is locally constant, i. e.,

$${}^{\gamma}\mathfrak{S}_l^k(x,y) = {}^{\gamma}\mathfrak{S}_l^k(x',y') \quad \text{whenever} \quad d(x,x') \leq \lambda^k \text{ and } d(y,y') \leq \lambda^k. \tag{4}$$

Note that even if the infimum in the definition of auxiliary function is not $+\infty$, the above result does not prevent the supremum with respect to k of ${}^{\gamma}\mathfrak{S}_l^k$ to be $+\infty$. This fact will lead us to pay close attention to situations in which $\pm\infty$ values can be present.

A fundamental inequality involving the auxiliary function is the following one.

Lemma 2.2. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation and $\gamma \in \mathbb{R}$ be a constant. For all integers $k \geq \bar{k} > 0$ and $l \geq k - \bar{k}$, $m \geq 0$ and for any points $x, y, z \in \Sigma$, we have*

$$^{\gamma}\mathfrak{S}^{k}_{l+m}(x, z) \leq {}^{\gamma}\mathfrak{S}^{k}_{l}(x, y) + {}^{\gamma}\mathfrak{S}^{\bar{k}}_{m}(y, z) + \mathrm{Var}^{\infty}_{\bar{k}}(A).$$

Proof. This result corresponds to a version of Lemma 5.1 of [8] for countable alphabets and potentials of summable variation, the proof of that result being easily adjustable. □

We initially consider versions of Mañé potential and Peierls barrier at any level γ.

Definition 2.3. Let $A : \Sigma \to \mathbb{R}$ be a potential and $\gamma \in \mathbb{R}$ be a constant.

(i) We define the Mañé potential as the function $\phi^{\gamma}_{A} : \Sigma \times \Sigma \to \mathbb{R} \cup \{\pm\infty\}$ given as

$$\phi^{\gamma}_{A}(x, y) := \lim_{k\to\infty} {}^{\gamma}\mathfrak{S}^{k}_{1}(x, y) = \sup_{k\geq 0} \inf_{n\geq 1} \inf_{\substack{d(x, w) \leq \lambda^k \\ d(\sigma^n(w), y) \leq \lambda^k}} S_n(\gamma - A)(w)$$

for all x, $y \in \Sigma$.

(ii) The Peierls barrier is the function $h^{\gamma}_{A} : \Sigma \times \Sigma \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$\begin{aligned} h^{\gamma}_{A}(x, y) &:= \sup_{k\geq 0} \sup_{l\geq 1} {}^{\gamma}\mathfrak{S}^{k}_{l}(x, y) \\ &= \lim_{k\to\infty} \liminf_{n\to\infty} \inf_{\substack{d(x, w) \leq \lambda^k \\ d(\sigma^n(w), y) \leq \lambda^k}} S_n(\gamma - A)(w) \end{aligned}$$

for every x, $y \in \Sigma$.

It is immediate from these definitions that

$$\phi^{\gamma}_{A}(x, y) \leq h^{\gamma}_{A}(x, y) \tag{5}$$

for all $x, y \in \Sigma$. It is also easy to see that ${}^{\gamma}\mathfrak{S}^{k}_{l}(x, \sigma^n(x)) \leq S_n(\gamma - A)(x)$, for all $k \geq 0$ and $n \geq l$, from which we obtain a fundamental inequality over an orbit

$$\phi^{\gamma}_{A}(x, \sigma^n(x)) \leq S_n(\gamma - A)(x), \tag{6}$$

for every $x \in \Sigma$ and for all $n \geq 1$.

We can present basic "triangle inequalities" involving the Mañé potential and the Peierls barrier.

Proposition 2.4. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation and $\gamma \in \mathbb{R}$ be a constant. Then, for every point $x, y, z \in \Sigma$, the following inequalities hold*

$$\phi^{\gamma}_{A}(x, z) \leq \phi^{\gamma}_{A}(x, y) + \phi^{\gamma}_{A}(y, z), \tag{7}$$

$$h^{\gamma}_{A}(x, z) \leq \phi^{\gamma}_{A}(x, y) + h^{\gamma}_{A}(y, z), \tag{8}$$

$$h^{\gamma}_{A}(x, z) \leq h^{\gamma}_{A}(x, y) + \phi^{\gamma}_{A}(y, z), \tag{9}$$

$$h^{\gamma}_{A}(x, z) \leq h^{\gamma}_{A}(x, y) + h^{\gamma}_{A}(y, z). \tag{10}$$

Proof. From Lemma 2.2, the proof follows the same lines as the proof of item ii of Proposition 5.2 of [8]. □

2.2. Minus Infinity Dichotomy

We analyze a central dichotomy of the Mañé potential (and the Peierls barrier) with respect to the value $-\infty$, which is intrinsically related with the ergodic maximizing constant.

Lemma 2.5. *Let $A : \Sigma \to \mathbb{R}$ be a potential. Then*

$$\beta_A = \sup\left\{\kappa \in \mathbb{R} \ : \ \begin{array}{c} \text{there exists a periodic point } x = \sigma^p(x) \\ \text{with } S_p(\kappa - A)(x) < 0 \end{array}\right\}$$
$$= \min\left\{\kappa \in \mathbb{R} \ : \ \begin{array}{c} S_p(\kappa - A)(x) \geq 0 \\ \text{for all periodic point } x = \sigma^p(x) \end{array}\right\}.$$

This corresponds to the Mañé critical value characterization for β_A (see [6]).

Proof. Since $S_p(\kappa - A)(x) < 0 \Leftrightarrow p\,\kappa < S_pA(x)$ and $S_p(\kappa - A)(x) \geq 0 \Leftrightarrow S_pA(x) \leq p\,\kappa$, the sets

$$\mathcal{I} := \{\kappa \in \mathbb{R} \ : \ \text{there is a periodic point } x = \sigma^p(x) \text{ with } p\,\kappa < S_pA(x)\}\,,$$
$$\mathcal{J} := \{\kappa \in \mathbb{R} \ : \ S_pA(x) \leq p\,\kappa \text{ for all periodic point } x = \sigma^p(x)\}$$

are complementary intervals with infinite endpoints such that $\sup \mathcal{I} = \inf \mathcal{I}^{\complement} = \inf \mathcal{J}$. As $\mathcal{J}$ is a closed set, the infimum is in fact a minimum.

Note that $\beta_A \geq \min \mathcal{J}$. In fact, given a periodic point $x = \sigma^p(x)$, for the associated σ-invariant probability $\mu_x := \frac{1}{p}\sum_{i=0}^{p-1}\delta_{\sigma^i(x)}$, we obtain $S_pA(x) = p\int A\,\mathrm{d}\mu_x \leq p\,\beta_A$.

From topological transitivity, periodic probabilities are dense among invariant measures, by Theorem 4.2 and Section 6 of [7], so that

$$\beta_A = \sup\left\{\tfrac{1}{p}S_pA(x) \ : \ x = \sigma^p(x) \text{ is a } p\text{-periodic point of } \Sigma\right\}.$$

It is easy to see that every $\kappa \in \mathcal{J}$ is greater than or equal to β_A, thus $\beta_A \leq \min \mathcal{J}$. □

Now we can precisely state the fundamental dichotomy.

Proposition 2.6. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation and $\gamma \in \mathbb{R}$ be a constant. The following assertions are equivalent:*

(i) $\gamma \geq \beta_A$;

(ii) $\phi_A^\gamma(x, x) > -\infty$ *for every* $x \in \Sigma$.

From the above proposition and the corresponding triangle inequality, the Mañé potential (and the Peierls barrier) assumes the value $-\infty$ everywhere or nowhere.

Proof. Let us prove the contrapositive statements. Suppose that $\phi_A^\gamma(x, x) = -\infty$ for some $x \in \Sigma$. It is immediate that ${}^\gamma\mathfrak{S}_1^k(x, x) = -\infty$ for any k. Since ${}^\gamma\mathfrak{S}_1^1(x, x)$ is an infimum, there exist $w \in \Sigma$ and integer $n \geq 1$ fulfilling

$$d(x, w) \leq \lambda, \quad d(\sigma^n(w), x) \leq \lambda \quad \text{and} \quad S_n(\gamma - A)(w) < -\mathrm{Var}_1^\infty(A).$$

In particular, $w_0 = x_0 = w_n$, so we can consider the periodic point

$$z := (x_0, w_1, \ldots, w_{n-1},\ \ x_0, w_1, \ldots, w_{n-1},\ \ \ldots) = \sigma^n(z) \in \Sigma.$$

Note that $S_n(\gamma - A)(z) \le S_n(\gamma - A)(w) + \mathrm{Var}_1^\infty(A) < 0$. From Lemma 2.5, $\gamma < \beta_A$.

Assume now $\gamma < \beta_A$. Again Lemma 2.5 ensures that there is a periodic point $x = \sigma^p(x)$ such that $-M := S_p(\gamma - A)(x) < 0$. By the periodicity of x, from inequality (6) we conclude that

$$\begin{aligned}\phi_{A,\gamma}(x,x) &= \lim_{k\to\infty} \phi_{A,\gamma}\left(x, \sigma^{k\,p}(x)\right) \\ &\le \lim_{k\to\infty} S_{k\,p}(\gamma - A)(x) = \lim_{k\to\infty} -k\,M = -\infty.\end{aligned}$$ □

Corollary 2.7. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation. Then, for every $x, y \in \Sigma$,*

$$-\infty < \phi_A\,(x,y) \qquad \text{and} \qquad -\infty < h_A\,(x,y)\,.$$

Proof. By Proposition 2.6, we have $\phi_A\,(z,z) > -\infty$. Apply twice inequality (7) in order to obtain $-\infty < \phi_A\,(z,z) \le \phi_A\,(z,x) + \phi_A\,(x,y) + \phi_A\,(y,z)$ for any x, $y \in \Sigma$. Thus, inequality (5) provides $-\infty < \phi_A\,(x,y) \le h_A\,(x,y)$. □

3. Aubry Set

The Aubry set was already introduced in Definition 1.1. We provide below a list of the main properties of this set that remain unchanged regardless the non-compact scenario. Proofs may be found in Chapter 4 of [8].

Proposition 3.1. *Let $A : \Sigma \to \mathbb{R}$ be a potential. The following properties hold.*

(i) *For every continuous function $f : \Sigma \to \mathbb{R}$ and any constant $c \in \mathbb{R}$, we have $\Omega(A) = \Omega\,(A + f\circ\sigma - f - c)$.*

(ii) *$\Omega(A)$ is an invariant set, i. e., $\sigma\,(\Omega(A)) \subset \Omega(A)$.*

(iii) *$\Omega(A)$ is a closed set.*

(iv) *If μ is an A-maximizing measure, then $\operatorname{supp}\mu \subset \Omega(A)$. In particular, the existence of a maximizing probability implies the Aubry set is non-empty.*

The behavior of the Mañé potential and the Peierls barrier on the diagonal and the Aubry set are intimately related. The following alternative characterization of Aubry points, given by Corollary 4.5 of [8], allows us to be more precise.

Lemma 3.2. *Let $A : \Sigma \to \mathbb{R}$ be a potential. Then, $x \in \Omega(A)$ if and only if for any $\varepsilon > 0$ and for all integer $L \ge 1$, there are a point $w \in \Sigma$ and an integer $n \ge L$ such that $d(x,w) < \varepsilon$, $d(\sigma^n(w), x) < \varepsilon$ and*

$$-\varepsilon \le S_n(\beta_A - A)(w) \le \varepsilon.$$

Proposition 3.3. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation. Then*

$$x \in \Omega(A) \qquad \Longleftrightarrow \qquad \phi_A(x,x) = h_A(x,x) = 0.$$

Proof. Note first that $0 \leq \phi_A(x,x) \leq h_A(x,x) \leq +\infty$ for any $x \in \Sigma$. As a matter of fact, $\phi_A(x,\sigma(x)) \in \mathbb{R}$ by inequality (6) and Corollary 2.7. From inequalities (7) and (5), it follows that $0 = \phi_A(x,\sigma(x)) - \phi_A(x,\sigma(x)) \leq \phi_A(x,x) \leq h_A(x,x)$ for all $x \in \Sigma$.

Let $x \in \Omega(A)$. From Lemma 3.2, for every $k \geq 0$ and for all integer $L \geq 1$, there are $w \in \Sigma$ and $n \geq L$ such that $d(x,w) < \lambda^k$, $d(\sigma^n(w),x) < \lambda^k$ and

$$\mathfrak{S}_L^k(x,x) \leq S_n(\beta_A - A)(w) \leq \lambda^k.$$

By taking the supremum with respect to $L \geq 1$ and passing to the limit as $k \to \infty$, we obtain $h_A(x,x) \leq 0$.

Reciprocally, suppose $h_A(x,x) = 0$. In particular, $\phi_A(x,x) = \sup_{k\geq 0} \mathfrak{S}_1^k(x,x) = 0$. Thus, given $\varepsilon > 0$, there is $K \geq 0$ such that $-\varepsilon < \mathfrak{S}_1^k(x,x) \leq 0$ for any $k \geq K$. We may assume that $\lambda^K \leq \varepsilon$. For a fixed $k \geq K$, since $\mathfrak{S}_1^k(x,x)$ is an infimum, there are a point $w \in \Sigma$ and an integer $n \geq 1$ fulfilling $d(x,w) \leq \lambda^k \leq \varepsilon$, $\ d(\sigma^n(w),x) \leq \lambda^k \leq \varepsilon$ and

$$-\varepsilon < \mathfrak{S}_1^k(x,x) \leq S_n(\beta_A - A)(w) < \mathfrak{S}_1^k(x,x) + \varepsilon \leq \varepsilon.$$

Therefore, x is an Aubry point. □

3.1. Sub-action

We will show that the existence of an Aubry point x ensures that there is always a continuous sub-action, precisely the function

$$y \in \Sigma \ \longmapsto \ u_x(y) := \phi_A(x,y) = h_A(x,y) \in \mathbb{R}.$$

The first step is to observe that we are dealing with a real-valued function.

Proposition 3.4. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation. If $h_A(x,z) \in \mathbb{R}$ for some $x, z \in \Sigma$, then*

$$\phi_A(x,y) \in \mathbb{R} \qquad \textit{and} \qquad h_A(x,y) \in \mathbb{R}, \qquad \forall\, y \in \Sigma.$$

Proof. Property (2) and Lemma 2.2 (with $m = 0$) provide for any $x, y, z \in \Sigma$,

$$\sup_{l\geq 1} \mathfrak{S}_l^k(x,y) \leq \sup_{l\geq k-\bar{k}} \mathfrak{S}_l^k(x,y) \leq \sup_{l\geq 1} \mathfrak{S}_l^k(x,z) + \mathfrak{S}_0^{\bar{k}}(z,y) + \mathrm{Var}_1^\infty(A),$$

where $k \geq \bar{k} > 0$. By passing to the limit as $k \to \infty$, and recalling inequality (5), we obtain for any $\bar{k} > 0$,

$$\phi_A(x,y) \leq h_A(x,y) \leq h_A(x,z) + \mathfrak{S}_0^{\bar{k}}(z,y) + \mathrm{Var}_1^\infty(A).$$

We conclude the result applying Corollary 2.7, the hypothesis and (1). □

Corollary 3.5. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation. If $x \in \Omega(A)$, then*

$$\phi_A(x,y) = h_A(x,y) \in \mathbb{R}, \qquad \forall\, y \in \Sigma.$$

Proof. The equality between the Mañé potential and the Peierls barrier in this case follows from inequalities (5) and (9), and from Proposition 3.3, since

$$\phi_A(x,y) \le h_A(x,y) \le h_A(x,x) + \phi_A(x,y) = \phi_A(x,y).$$

Besides, $h_A(x,y) \in \mathbb{R}$ thanks to Propositions 3.3 and 3.4. □

The Peierls barrier is continuous with respect to the second variable.

Proposition 3.6. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation. If $h_A(\bar{x}, z) \in \mathbb{R}$ for some $\bar{x}, z \in \Sigma$, then the map*

$$y \in \Sigma \;\longmapsto\; h_A(\bar{x}, y) \in \mathbb{R}$$

is continuous with ℓ-th variation bounded from above by $\mathrm{Var}_\ell^\infty(A)$. In particular, if $\sum_{\ell=1}^\infty \mathrm{Var}_\ell^\infty(A) < +\infty$, then $h_A(\bar{x}, \cdot)$ is of summable variation.

Proof. Proposition 3.4 guarantee that $h_A(\bar{x}, \cdot)$ is a real-valued function. Consider points $x, y \in \Sigma$ such that $d(x,y) \le \lambda^{\bar{k}}$, with $\bar{k} > 0$. From property (2) and Lemma 2.2 (with $m = 0$), we obtain

$$\sup_{l \ge 1} \mathfrak{S}_l^k(\bar{x}, x) \le \sup_{l \ge k - \bar{k}} \mathfrak{S}_l^k(\bar{x}, x) \le \sup_{l \ge 1} \mathfrak{S}_l^k(\bar{x}, y) + \mathfrak{S}_0^{\bar{k}}(y, x) + \mathrm{Var}_{\bar{k}}^\infty(A),$$

where $k \ge \bar{k} > 0$. Note that $\mathfrak{S}_0^{\bar{k}}(y,x) \le S_0(\beta_A - A)(x) = 0$. As $k \to \infty$, it follows that $h_A(\bar{x}, x) - h_A(\bar{x}, y) \le \mathrm{Var}_{\bar{k}}^\infty(A)$, for any $x, y \in \Sigma$ with $d(x,y) \le \lambda^{\bar{k}}$. In other terms,

$$\mathrm{Var}_\ell\left(h_A(\bar{x}, \cdot)\right) \le \mathrm{Var}_\ell^\infty(A).$$ □

Remark 3.7. There is a slight loss of regularity between the potential A and the associated Peierls barrier, which seems to be natural on non-compact scenarios, see [9].

Regularity on the first coordinate can be verified for Aubry points.

Proposition 3.8. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation. Then, for any $\bar{y} \in \Sigma$ fixed, the map*

$$x \in \Omega(A) \;\longmapsto\; \phi_A(x, \bar{y}) = h_A(x, \bar{y}) \in \mathbb{R}$$

is continuous with ℓ-th variation bounded from above by $\mathrm{Var}_\ell^\infty(A)$. In particular, if $\sum_{\ell=1}^\infty \mathrm{Var}_\ell^\infty(A) < +\infty$, then $h_A(\cdot, \bar{y})|_{\Omega(A)}$ is of summable variation.

Proof. Thanks to Corollary 3.5, $\phi_A(x, \cdot) = h_A(x, \cdot) \in \mathbb{R}$, for every $x \in \Omega(A)$. By inequality (10) and Proposition 3.3, we obtain

$$h_A(x, \bar{y}) - h_A(y, \bar{y}) \le h_A(x, y) = h_A(x, y) - h_A(x, x),$$

for all $x, y \in \Omega(A)$. Hence, if $d(x,y) \le \lambda^\ell$ with $\ell \ge 1$, Proposition 3.6 ensures that

$$\mathrm{Var}_\ell\left(h_A(\cdot, \bar{y})|_{\Omega(A)}\right) \le \mathrm{Var}_\ell^\infty(A).$$ □

We can now ensure the existence of a basic sub-action, obtained from the Mañé potential and the Peierls barrier (also known as Collateral Theorem).

Theorem 3.9. *Let $A : \Sigma \to \mathbb{R}$ be a potential of summable variation. Then, for any $x \in \Omega(A)$ fixed, the map*

$$\begin{array}{rcl} u_x(\,\cdot\,) = \phi_A(x,\,\cdot\,) = h_A(x,\,\cdot\,) \,:\, \Sigma & \longrightarrow & \mathbb{R} \\ y & \longmapsto & u_x(y) = \phi_A(x,y) = h_A(x,y) \end{array}$$

is a continuous sub-action with ℓ-th variation bounded from above by $\mathrm{Var}_\ell^\infty(A)$. In particular, if $\sum_{\ell=1}^\infty \mathrm{Var}_\ell^\infty(A) < +\infty$, then u_x is a sub-action of summable variation. More specifically, if $A : \Sigma \to \mathbb{R}$ is a locally Hölder continuous potential, then u_x is a locally Hölder continuous sub-action.

Proof. First, note that u_x fulfills the inequality in the definition of a sub-action. Indeed, by applying inequalities (7) and (6), for every $y \in \Sigma$, we get

$$\begin{aligned} u_x \circ \sigma(y) = \phi_A\left(x, \sigma(y)\right) &\le \phi_A\left(x, y\right) + \phi_A\left(y, \sigma(y)\right) \\ &\le u_x(y) + (\beta_A - A)(y) = u_x(y) - A(y) + \beta_A. \end{aligned}$$

Next, the statements about the regularity of the function u_x are direct consequences of Corollary 3.5 and Proposition 3.6. □

When the potential A admits a sub-action u, we introduce its contact locus as

$$\mathbb{M}_A\left(u\right) := \left(A + u \circ \sigma - u\right)^{-1}\left(\beta_A\right).$$

We summarize the main properties of this set including its relations with the Aubry set and the maximizing measures. Proofs may be found in [8] and remain unchanged regardless the non-compact scenario.

Proposition 3.10. *Let $A : \Sigma \to \mathbb{R}$ be a potential and $u : \Sigma \to \mathbb{R}$ be any sub-action of A. The following properties hold.*

(i) $\mathbb{M}_A\left(u\right)$ *is a closed set.*

(ii) $\Omega(A) \subset \mathbb{M}_A\left(u\right)$.

(iii) If μ is an A-maximizing measure, then $\mathrm{supp}\,\mu \subset \mathbb{M}_A\left(u\right)$. *In particular, $\mathbb{M}_A\left(u\right)$ is a non-empty set whenever there exists a maximizing probability.*

4. Densely Periodic Optimization

This section is dedicated to prove the Main Theorem. The argument is inspired by the proof of Contreras theorem for shifts over finite alphabets [5, 10, 4]. In particular, we will make use of the following result.

Lemma 4.1 (Huang, Lian, Ma, Xu, Zhang)**.** *Let Ω be a compact invariant subset of a Markov subshift over a finite alphabet. Then, for any $\tau > 0$, there exists a periodic orbit $\mathcal{O}$ such that*

$$\sum_{z \in \mathcal{O}} d(\Omega, z) < \tau \Delta(\mathcal{O}),$$

where $\Delta(\mathcal{O})$ denotes the half-gap of the orbit: $\Delta(\mathcal{O}) = \frac{1}{2} \min \left\{\lambda, \min_{\substack{y, z \in \mathcal{O} \\ y \neq z}} d(y,z)\right\}$‡.

‡ It is conventional that $\min \emptyset = \infty$, so that $\Delta(\mathcal{O}) = \lambda/2$ if $\mathcal{O}$ consists of a single fixed point.

For a proof of this lemma, see [10, Proposition 3.1] or [4, Lemma 2.3].

Proof of the Main Theorem. Let Ω denote the non-empty compact invariant subset of the Aubry set of the potential A that is contained in a subshift over a finite alphabet. For $x \in \Omega$, we consider the associated locally Hölder continuous sub-action $u := \phi_A(x, \cdot) = h_A(x, \cdot)$, given by Theorem 3.9, and we introduce

$$\hat{A} := \frac{1}{1 + \|A + u \circ \sigma - u - \beta_A\|_{\mathrm{sv}}} (A + u \circ \sigma - u - \beta_A).$$

Note that $\hat{A} \leq 0$ is a locally Hölder continuous potential of norm less than 1, which admits among its maximizing probabilities all A-maximizing measures with compact support.

As maximizing measures are preserved when multiplying a potential by a positive constant, it suffices to show that, given $\varepsilon > 0$, there exists a periodic point $y = y(\varepsilon) \in \Sigma$ such that, if the perturbed potential

$$\hat{A} - \varepsilon d(\mathrm{orb}(y), \cdot)$$

has a maximizing probability with compact support, then this is necessarily the periodic measure

$$\mu_y := \frac{1}{\#\mathrm{orb}(y)} \sum_{z \in \mathrm{orb}(y)} \delta_z.$$

By taking $\tau \in (0, \varepsilon)$, which will be specified later, we apply Lemma 4.1 to determine a periodic point y fulfilling

$$\sum_{z \in \mathrm{orb}(y)} d(\Omega, z) < \tau \Delta,$$

where we abbreviate $\Delta = \Delta(\mathrm{orb}(y))$. Hence, denote

$$\beta(\varepsilon) := \beta_{\hat{A} - \varepsilon d(\mathrm{orb}(y), \cdot)} \leq 0$$

and define

$$\hat{B} := \hat{A} - \varepsilon d(\mathrm{orb}(y), \cdot) - \beta(\varepsilon).$$

It is now sufficient to show that if there exists a $\hat{B}$-maximizing probability with compact support, then it must necessarily be μ_y.

We have $\int \hat{A} \, d\mu_y = \int [\hat{A} - \varepsilon d(\mathrm{orb}(y), \cdot)] \, d\mu_y \leq \beta(\varepsilon)$. In particular, note that

$$\int \hat{B} \, d\mu_y \leq 0. \tag{11}$$

Besides, as $\|\hat{A}\|_{\mathrm{sv}} \leq 1$ and $\hat{A}|_\Omega \equiv 0$ (by item ii of Proposition 3.10),

$$\int \hat{A} \, d\mu_y = \frac{1}{\#\mathrm{orb}(y)} \sum_{z \in \mathrm{orb}(y)} \hat{A}(z) \geq -\frac{1}{\#\mathrm{orb}(y)} \sum_{z \in \mathrm{orb}(y)} d(\Omega, z),$$

which yields

$$\int \hat{A}\, d\mu_y \geq -\frac{1}{\#\mathrm{orb}(y)}\tau\Delta.$$

We then obtain that

$$\hat{B} \leq -\beta(\varepsilon) \leq \frac{1}{\#\mathrm{orb}(y)}\tau\Delta. \tag{12}$$

Let us denote $r := -\beta(\varepsilon)/\varepsilon$. We point out that

$$d(\mathrm{orb}(y), z) > r \qquad \Rightarrow \qquad \hat{B}(z) \leq 0. \tag{13}$$

Indeed, in this case,

$$\hat{B}(z) = \hat{A}(z) - \varepsilon d(\mathrm{orb}(y), z) - \beta(\varepsilon) \leq \beta(\varepsilon) - \beta(\varepsilon) = 0.$$

Suppose that $\hat{B}$ admits a maximizing probability with compact support, say, S. Recall that $\mathrm{S} \subset \Omega(\hat{B})$ by item iv of Proposition 3.1. Let us fix $w \in \mathrm{S}$ and consider $v := \phi_{\hat{B}}(w, \cdot) = h_{\hat{B}}(w, \cdot)$ sub-action for $\hat{B}$. It is essential to remember that, thanks to item iii of Proposition 3.10,

$$\hat{B} \circ \sigma^{\ell}(w) + v \circ \sigma^{\ell+1}(w) - v \circ \sigma^{\ell}(w) = 0 \qquad \forall\, \ell \geq 0,$$

which brings us in particular

$$\sum_{p \leq j \leq q} \hat{B}(\sigma^j(w)) \geq -2\, \|v|_{\mathrm{S}}\|_\infty \qquad \forall\, 0 \leq p \leq q < \infty.$$

Because of the last inequality, we will show that $\sigma^{\ell}(w)$ can be far from the periodic orbit of y for at most a finite number of indices $\ell \geq 0$. By the expansiveness, for ℓ sufficiently large, this will automatically lead to $\sigma^{\ell}(w) \in \mathrm{orb}(y) = \mathrm{supp}(\mu_y)$. Consequently, the unique possible $\hat{B}$-maximizing probability with compact support is μ_y.

Let $\ell_0 \geq 0$ be the smallest integer such that $d(\mathrm{orb}(y), \sigma^{\ell_0}(w)) > \Delta$. Recursively, given ℓ_{m-1}, let $\ell_m > \ell_{m-1}$ be the smallest integer such that $d(\mathrm{orb}(y), \sigma^{\ell_m}(w)) > \Delta$. It is now sufficient to show that the sequence (ℓ_m) can only be defined for a finite number of indices.

Let us first remark that from (12)

$$r = \frac{-\beta(\varepsilon)}{\varepsilon} \leq \frac{1}{\#\mathrm{orb}(y)}\frac{\tau}{\varepsilon}\Delta \leq \frac{\tau}{\varepsilon}\Delta < \Delta.$$

Whenever the set

$$\{\ell : \ell_m < \ell < \ell_{m+1} \quad \text{and} \quad d(\mathrm{orb}(y), \sigma^{\ell}(w)) \leq r\}$$

is empty, using (13) when necessary, note that

$$\sum_{\ell_m < \ell < \ell_{m+1}} \hat{B}(\sigma^{\ell}(w)) \leq 0.$$

Otherwise, we write k_m as the maximum of this set and we have

$$\sum_{\ell_m < \ell < \ell_{m+1}} \hat{B}(\sigma^{\ell}(w)) \leq \sum_{\ell_m < \ell \leq k_m} \hat{B}(\sigma^{\ell}(w)).$$

Since $d(\mathrm{orb}(y), \sigma^{k_m}(w)) \le r < \Delta$, the point on the orbit of y which achieves this distance is uniquely determined. If we denote it as $\sigma^{k_m}(z)$ for $z \in \mathrm{orb}(y)$, by the very definition of Δ and by the expansiveness, we obtain that

$$d(\Omega, \sigma^{\ell}(w)) = d(\sigma^{\ell}(z), \sigma^{\ell}(w)) \le \lambda^{k_m-\ell} d(\sigma^{k_m}(z), \sigma^{k_m}(w))$$

for $\ell \in \{\ell_m + 1, \dots, k_m\}$. Recalling that $\|\hat{A}\|_{\mathrm{sv}} \le 1$, we can estimate

$$\begin{aligned}\sum_{\ell_m<\ell\le k_m} \Big[\hat{B}(\sigma^{\ell}(w)) - \hat{B}(\sigma^{\ell}(z))\Big] &\le \\ &\le (1+\lambda+\ldots+\lambda^{k_m-(l_m+1)})\, r - \varepsilon \sum_{\ell_m<\ell\le k_m} d(\mathrm{orb}(y), \sigma^{\ell}(w)) \\ &\le \frac{1}{1-\lambda} r.\end{aligned}$$

Therefore the following upper bound holds:

$$\begin{aligned}\sum_{\ell_m<\ell<\ell_{m+1}} \hat{B}(\sigma^{\ell}(w)) &\le \\ &\le \frac{1}{1-\lambda} r + \left\lfloor \frac{k_m-\ell_m}{\#\mathrm{orb}(y)} \right\rfloor \#\mathrm{orb}(y) \int \hat{B}\, d\mu_y \, + \\ &\quad + (\#\mathrm{orb}(y)-1)\frac{1}{\#\mathrm{orb}(y)}\tau\Delta \\ &\le \frac{1}{1-\lambda} r + \Big(1 - \frac{1}{\#\mathrm{orb}(y)}\Big)\tau\Delta \\ &\le \Big(\frac{1}{1-\lambda}\cdot\frac{1}{\varepsilon}\cdot\frac{1}{\#\mathrm{orb}(y)} + 1 - \frac{1}{\#\mathrm{orb}(y)}\Big)\tau\Delta.\end{aligned}$$

(We use (12) to establish the first and third inequalities and we use just (11) to obtain the second one.) Obviously,

$$\begin{aligned}\hat{B}(\sigma^{\ell_{m+1}}(w)) &= \hat{A}(\sigma^{\ell_{m+1}}(w)) - \varepsilon d(\mathrm{orb}(y), \sigma^{\ell_{m+1}}(w)) - \beta(\varepsilon) \\ &\le -\varepsilon\Delta + \frac{1}{\#\mathrm{orb}(y)}\tau\Delta.\end{aligned}$$

Thus (whether or not there is k_m) we deduce that

$$\begin{aligned}\sum_{\ell_m<\ell\le\ell_{m+1}} \hat{B}(\sigma^{\ell}(w)) &\le \Big[\Big(\frac{1}{1-\lambda}\cdot\frac{1}{\varepsilon}\cdot\frac{1}{\#\mathrm{orb}(y)} + 1\Big)\tau - \varepsilon\Big]\Delta \\ &\le \Big[\Big(\frac{1}{1-\lambda}\cdot\frac{1}{\varepsilon} + 1\Big)\tau - \varepsilon\Big]\Delta.\end{aligned}$$

We then take

$$\tau := \frac{1}{2}\cdot\frac{\varepsilon}{1 + \frac{1}{1-\lambda}\cdot\frac{1}{\varepsilon}},$$

so that

$$\sum_{\ell_m<\ell\le\ell_{m+1}} \hat{B}(\sigma^{\ell}(w)) \le -\frac{\varepsilon}{2}\Delta.$$

We conclude in this way that the orbit of w has at most

$$m \leq \frac{4}{\varepsilon\,\Delta}||v|s||_\infty$$

indices such that $\sigma^{\ell_m}(w)$ lies more than Δ from the periodic orbit of y. □

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