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SYSTEM OF FIRST ORDER DIFFERENTIAL EQUATIONS IN THE PLANE

$\{x(t), y(t)\}$

$$\dot{x}(t) = \frac{7}{2}x(t) + \frac{3}{2}y(t)$$
$$\dot{y}(t) = \frac{3}{2}x(t) + \frac{7}{2}y(t)$$

$$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = \begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{7}{2}\end{pmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$\{x(0), y(0)\} = \{2, 0\}$$

$$\dot{V}(t) = M\,V(t)$$

$$V(t) = e^{Mt}\,V(0)$$

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = \exp\left[\begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{7}{2}\end{pmatrix}t\right]\begin{bmatrix}2\\ 0\end{bmatrix}$$

EIGENVALUES: $\left(\frac{7}{2}-\lambda\right)^2 - \frac{9}{4} = 0 \;\rightarrow\; (7-2\lambda)^2 = 9$

$$7-2\lambda = \pm 3$$

$$\lambda_1 = 2 \qquad \lambda_2 = 5$$

$$M = SDS^{-1}$$

$$\begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{7}{2}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 5\end{pmatrix}$$

$$\begin{pmatrix}\frac{7a+3c}{2} & \frac{7b+3d}{2}\\ \frac{3a+7c}{2} & \frac{3b+7d}{2}\end{pmatrix} = \begin{pmatrix}2a & 5b\\ 2c & 5d\end{pmatrix}$$

$$\begin{aligned} a+c &= 0\\ d-b &= 0\end{aligned} \;\rightarrow\; S = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = \exp\left[\begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{7}{2}\end{pmatrix}t\right]\begin{bmatrix}2\\ 0\end{bmatrix} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\exp\left[\begin{pmatrix}2t & 0\\ 0 & 5t\end{pmatrix}\right]\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{bmatrix}2\\ 0\end{bmatrix}$$

$$= \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}e^{2t} & 0\\ 0 & e^{5t}\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}e^{2t} & e^{5t}\\ -e^{2t} & e^{5t}\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}e^{5t}+e^{2t}\\ e^{5t}-e^{2t}\end{pmatrix}$$

$$\dot{x}(t) = -\frac{7}{2}x(t) - \frac{3}{2}y(t)$$
$$\dot{y}(t) = -\frac{3}{2}x(t) - \frac{7}{2}y(t)$$

$$\{x(0), y(0)\} = \{2, 0\}$$

$$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = -\begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{7}{2}\end{pmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$\dot{V}(t) = -M\,V(t)$$

$$V(t) = e^{-Mt}\,V(0)$$

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = \exp\left[-\begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{7}{2}\end{pmatrix}t\right]\begin{bmatrix}2\\ 0\end{bmatrix}$$

EIGENVALUES: $\left(\frac{7}{2}-\lambda\right)^2 - \frac{9}{4} = 0 \rightarrow (7-2\lambda)^2 = 9$

$$-7-2\lambda = \pm 3$$

$$\lambda_1 = -2 \qquad \lambda_2 = -5$$

$$-M = -SDS^{-1}$$

$$-\begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{7}{2}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}-2 & 0\\ 0 & -5\end{pmatrix}$$

$$\begin{pmatrix}\frac{7a+3c}{2} & \frac{7b+3d}{2}\\ \frac{3a+7c}{2} & \frac{3b+7d}{2}\end{pmatrix} = \begin{pmatrix}2a & \\ 2c & 5d\end{pmatrix}$$

$$\begin{matrix}a+c=0\\ d-b=0\end{matrix} \rightarrow S = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = \exp\left[-\begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ \frac{3}{2} & \frac{7}{2}\end{pmatrix}t\right]\begin{bmatrix}2\\ 0\end{bmatrix} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\exp\left[-\begin{pmatrix}2t & 0\\ 0 & 5t\end{pmatrix}\right]\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{bmatrix}2\\ 0\end{bmatrix}$$

$$= \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}e^{-2t} & 0\\ 0 & e^{-5t}\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

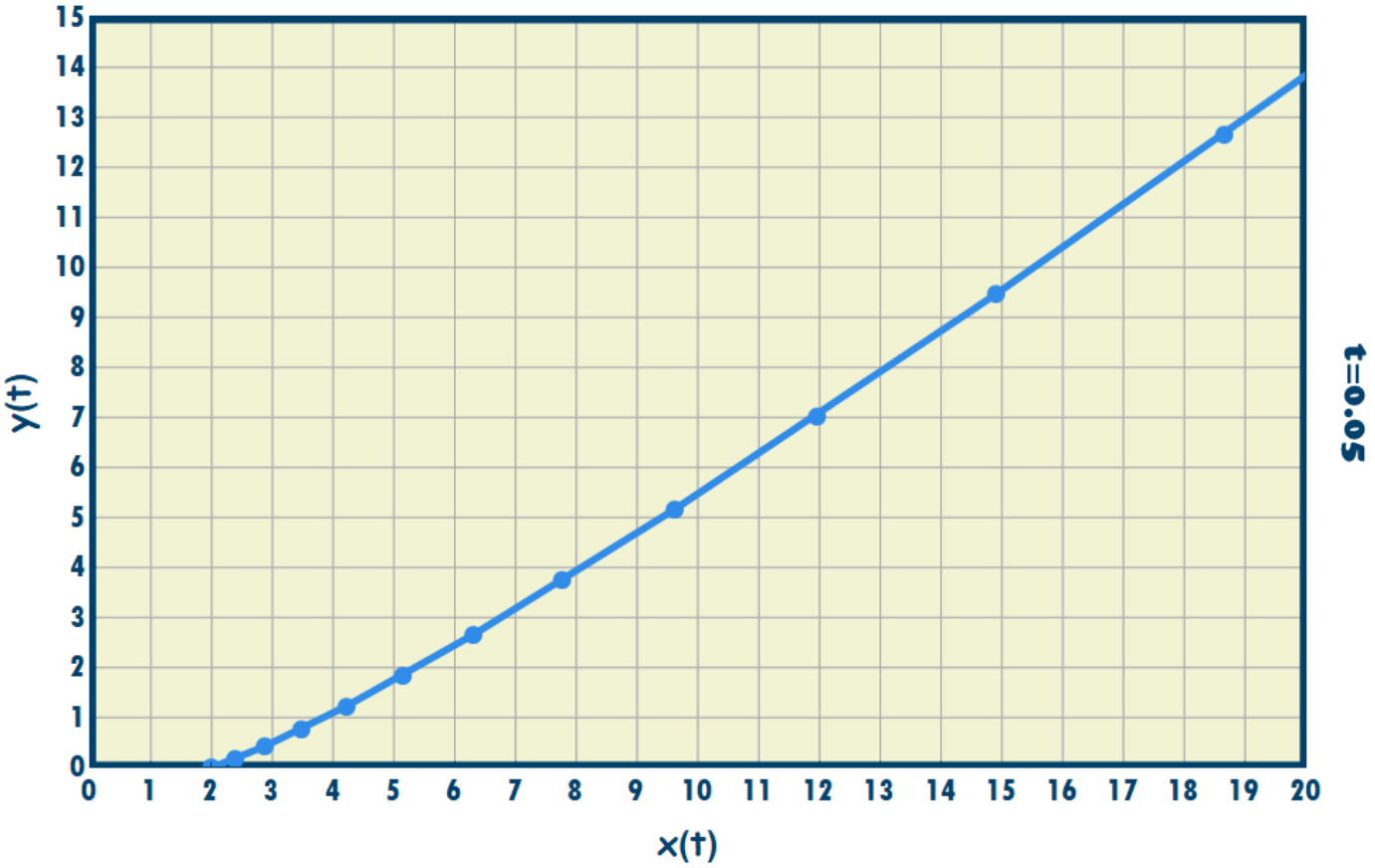
$$= \begin{pmatrix}e^{-2t} & e^{-5t}\\ -e^{-2t} & e^{-5t}\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}e^{-5t} + e^{-2t}\\ e^{-5t} - e^{-2t}\end{pmatrix}$$

$$x(t) = e^{5t} + e^{2t}$$
$$y(t) = e^{5t} - e^{2t}$$

$$x(t) = e^{-5t} + e^{-2t}$$
$$y(t) = e^{-5t} - e^{-2t}$$

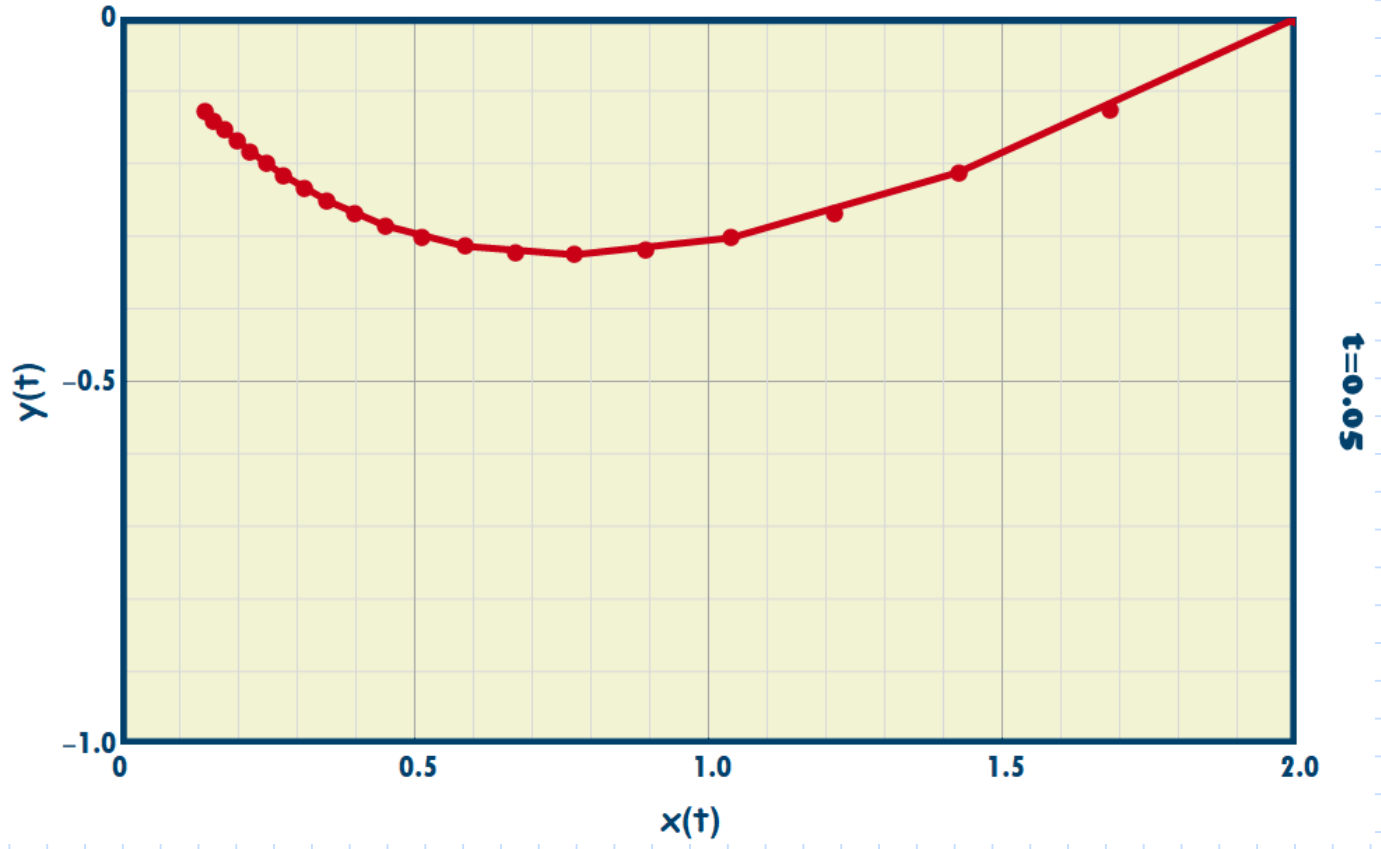
$$\dot{x}(t) = +\frac{5}{2}x(t) + \frac{3}{2}y(t)$$
$$\dot{y}(t) = +\frac{3}{2}x(t) + \frac{5}{2}y(t)$$

$$\{e^{2t} + e^{5t}, -e^{2t} + e^{5t}\}$$



$$\dot{x}(t) = -\frac{5}{2}x(t) - \frac{3}{2}y(t)$$
$$\dot{y}(t) = -\frac{3}{2}x(t) - \frac{5}{2}y(t)$$

$$\{e^{-5t} + e^{-2t}, e^{-5t} - e^{-2t}\}$$



$$\dot{x}(t) = \frac{7}{2}x(t)$$
$$\dot{y}(t) = \frac{3}{2}x(t) + \frac{7}{2}y(t)$$

$$\begin{pmatrix} \frac{7}{2}-\lambda & 0 \\ \frac{3}{2} & \frac{7}{2}-\lambda \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = \frac{7}{2}$$

$$\begin{pmatrix} \frac{7}{2} & 0 \\ \frac{3}{2} & \frac{7}{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{7}{2} & 1 \\ 0 & \frac{7}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{7a}{2} & \frac{7b}{2} \\ \frac{3a+7c}{2} & \frac{3b+7d}{2} \end{pmatrix} = \begin{pmatrix} \frac{7a}{2} & \frac{2a+7b}{2} \\ \frac{7c}{2} & \frac{2c+7d}{2} \end{pmatrix}$$

$a = 0 \qquad 3b = 2c$

$\forall d$

$d = 0$

$c = \frac{1}{2} \qquad b = \frac{1}{3}$

$$S = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$$

$$\exp\left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}t\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}t + \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}\frac{t^2}{2!} + \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^2 \end{pmatrix}\frac{t^3}{3!} + \dots$$

$$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}e^{\lambda t}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}e^{\frac{7}{2}t}\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{t}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 6 \end{pmatrix}e^{\frac{7}{2}t}$$

$$= \begin{bmatrix} 2e^{\frac{7}{2}t} \\ 3te^{\frac{7}{2}t} \end{bmatrix}$$

$$\left\{2, 3t\right\}e^{\frac{7}{2}t}$$

$$\dot{x}(t) = -\frac{7}{2}x(t)$$
$$\dot{y}(t) = -\frac{3}{2}x(t) - \frac{7}{2}y(t)$$

$$x(0) = 2$$
$$y(0) = 0$$

$$\begin{pmatrix} -\frac{7}{2}-\lambda & 0 \\ -\frac{3}{2} & -\frac{7}{2}-\lambda \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = -\frac{7}{2}$$

$$\begin{pmatrix} -\frac{7}{2} & 0 \\ -\frac{3}{2} & -\frac{7}{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -\frac{7}{2} & 1 \\ 0 & -\frac{7}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{7a}{2} & -\frac{7b}{2} \\ -\frac{3a+7c}{2} & -\frac{3b+7d}{2} \end{pmatrix} = \begin{pmatrix} -\frac{7}{2}a & \frac{2a-7b}{2} \\ -\frac{7}{2}c & \frac{2c-7d}{2} \end{pmatrix}$$

$a = 0 \qquad 3b = -2c$

$c = \frac{1}{2} \qquad b = -\frac{1}{3}$

$$S = \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}e^{-\frac{7}{2}t}\begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

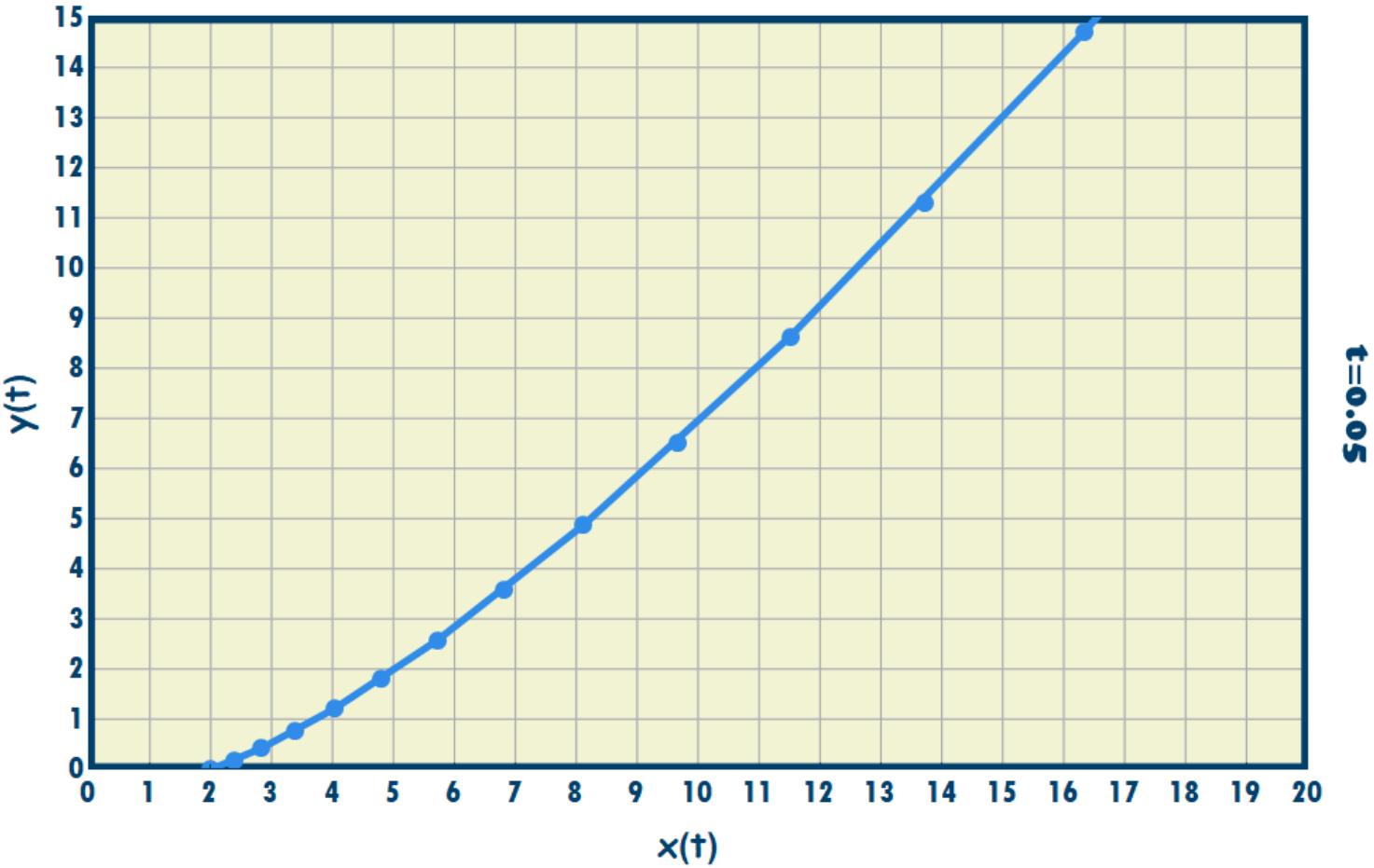
$$= \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{2} & \frac{t}{2} \end{pmatrix}\begin{pmatrix} 0 \\ -6 \end{pmatrix}e^{-\frac{7}{2}t}$$

$$= \begin{bmatrix} 2e^{-\frac{7}{2}t} \\ -3te^{-\frac{7}{2}t} \end{bmatrix}$$

$$\left\{2, -3t\right\}e^{-\frac{7}{2}t}$$

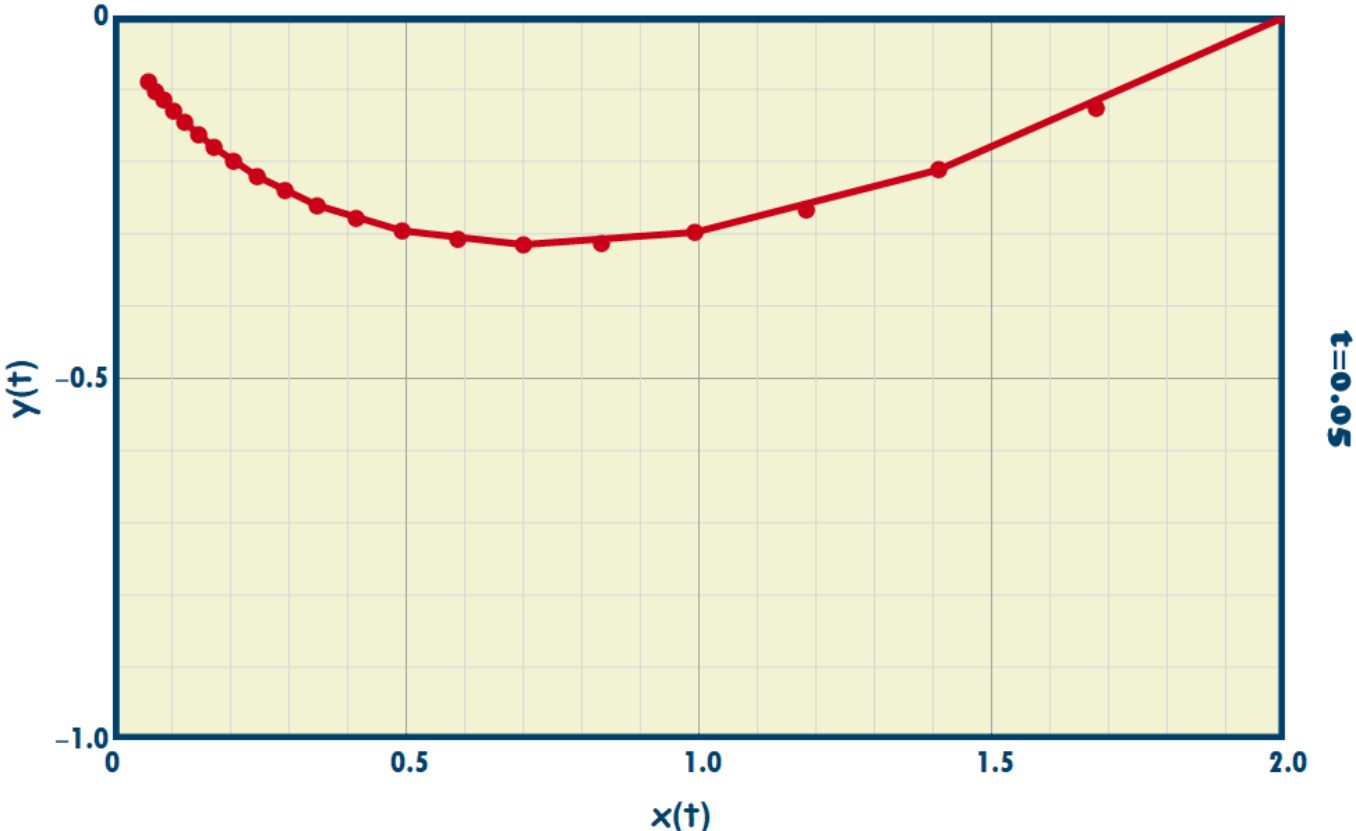
$$\dot{x}(t) = \frac{7}{2}x(t)$$
$$\dot{y}(t) = \frac{3}{2}x(t) + \frac{7}{2}y(t)$$

$\{2e^{7t/2}, 3e^{7t/2}t\}$



$$\dot{x}(t) = -\frac{7}{2}x(t)$$
$$\dot{y}(t) = -\frac{3}{2}x(t) - \frac{7}{2}y(t)$$

$\{2e^{-7t/2}, -3e^{-7t/2}t\}$



$$\dot{x}(t) = \frac{7}{2}x(t) - \frac{3}{2}y(t)$$
$$\dot{y}(t) = \frac{3}{2}x(t) + \frac{7}{2}y(t)$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\left(\frac{7}{2}-\lambda\right)^2 + \frac{9}{2} = 0 \Rightarrow \lambda_{1,2} = \frac{7}{2} \pm \frac{3}{2}i$$

$$\begin{pmatrix} \frac{7}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{7}{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{7}{2}-\frac{3}{2}i & 0 \\ 0 & \frac{7}{2}+\frac{3}{2}i \end{pmatrix}$$

$$\begin{pmatrix} 7a-3c & 7b-3d \\ 3a+7c & 3b+7d \end{pmatrix} = \begin{pmatrix} 7a-3ai & 7b+3bi \\ 7c-3ci & 7d+3di \end{pmatrix}$$

$c = ai \qquad -d = bi$
$a = -ci \qquad b = di$

$$S = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \qquad S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} e^{-\frac{3}{2}it} & 0 \\ 0 & e^{\frac{3}{2}it} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\frac{e^{\frac{7}{2}t}}{2}$$

$$= \begin{pmatrix} e^{-\frac{3}{2}it} & ie^{\frac{3}{2}it} \\ ie^{-\frac{3}{2}it} & e^{\frac{3}{2}it} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\frac{e^{\frac{7}{2}t}}{2}$$

$$= \begin{pmatrix} 2\cos\frac{3}{2}t & -2\sin\frac{3}{2}t \\ 2\sin\frac{3}{2}t & 2\cos\frac{3}{2}t \end{pmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{\frac{7}{2}t}$$

$$= 2\begin{bmatrix} \cos\frac{3}{2}t \\ \sin\frac{3}{2}t \end{bmatrix}e^{\frac{7}{2}t}$$

$$\dot{x}(t) = -\frac{7}{2}x(t) + \frac{3}{2}y(t)$$
$$\dot{y}(t) = -\frac{3}{2}x(t) - \frac{7}{2}y(t)$$

$$\left(-\frac{7}{2}-\lambda\right)^2 + \frac{9}{2} = 0 \Rightarrow \lambda_{1,2} = -\frac{7}{2} \pm \frac{3}{2}i$$

$$\begin{pmatrix} -\frac{7}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{7}{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -\frac{7}{2}-\frac{3}{2}i & 0 \\ 0 & -\frac{7}{2}+\frac{3}{2}i \end{pmatrix}$$

$$\begin{pmatrix} -7a+3c & -7b+3d \\ -3a-7c & -3b-7d \end{pmatrix} = \begin{pmatrix} -7a-3ai & -7b+3bi \\ -7c-3ci & -7d+3di \end{pmatrix}$$

$c = -ai \qquad d = bi$
$a = ci \qquad -b = di$

$$S = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \qquad S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} 1-i \\ -i & 1 \end{pmatrix}\begin{pmatrix} e^{-\frac{3}{2}it} & 0 \\ 0 & e^{\frac{3}{2}it} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\frac{e^{-\frac{7}{2}t}}{2}$$

$$= \begin{pmatrix} e^{-\frac{3}{2}it} & -ie^{\frac{3}{2}it} \\ -ie^{-\frac{3}{2}it} & e^{\frac{3}{2}it} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\frac{e^{-\frac{7}{2}t}}{2}$$

$$= \begin{pmatrix} 2\cos\frac{3}{2}t & 2\sin\frac{3}{2}t \\ -2\sin\frac{3}{2}t & 2\cos\frac{3}{2}t \end{pmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{-\frac{7}{2}t}$$

$$= 2\begin{bmatrix} \cos\frac{3}{2}t \\ -\sin\frac{3}{2}t \end{bmatrix}e^{-\frac{7}{2}t}$$

$$\dot{x}(t) = \frac{7}{2}x(t) + y(t)$$
$$\dot{y}(t) = -\frac{29}{2}x(t) - \frac{7}{2}y(t)$$

$$\left(\frac{7}{2}-\lambda\right)\left(-\frac{7}{2}-\lambda\right) + \frac{29}{2} = 0 \Rightarrow \lambda^2 - \frac{49}{4} + \frac{29}{2} = 0 \Rightarrow \lambda_{1,2} = \pm\frac{3}{2}i$$

$x(0) = 2 \qquad y(0) = 0$

$$\begin{pmatrix} \frac{7}{2} & 1 \\ -\frac{29}{2} & -\frac{7}{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -\frac{3}{2}i & 0 \\ 0 & \frac{3}{2}i \end{pmatrix} \Rightarrow \begin{pmatrix} 7 & 2 \\ -29 & -7 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -3i & 0 \\ 0 & 3i \end{pmatrix}$$

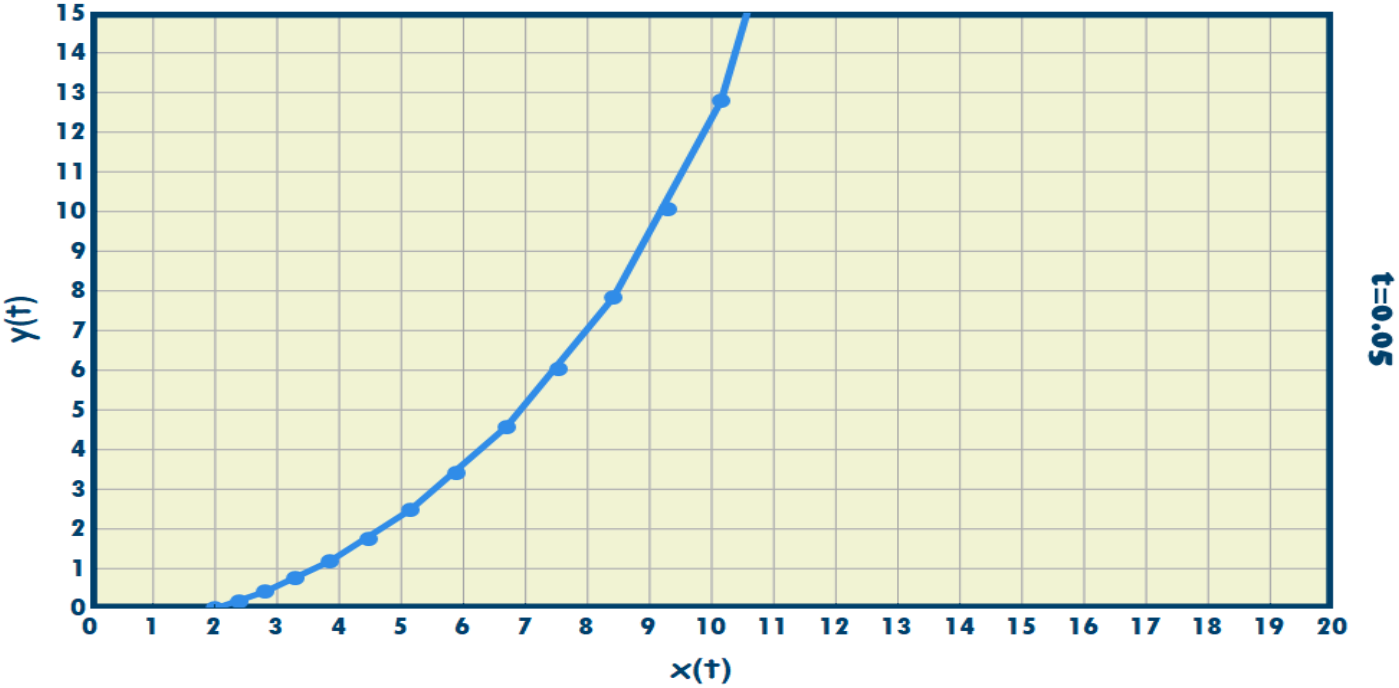
$$\begin{pmatrix} 7a+2c & 7b+2d \\ -29a-7c & -29b-7d \end{pmatrix} = \begin{pmatrix} -3ai & 3bi \\ -3ci & 3di \end{pmatrix}$$

$a = -2 \qquad c = 7+3i$
$b = 2 \qquad d = 3i-7$

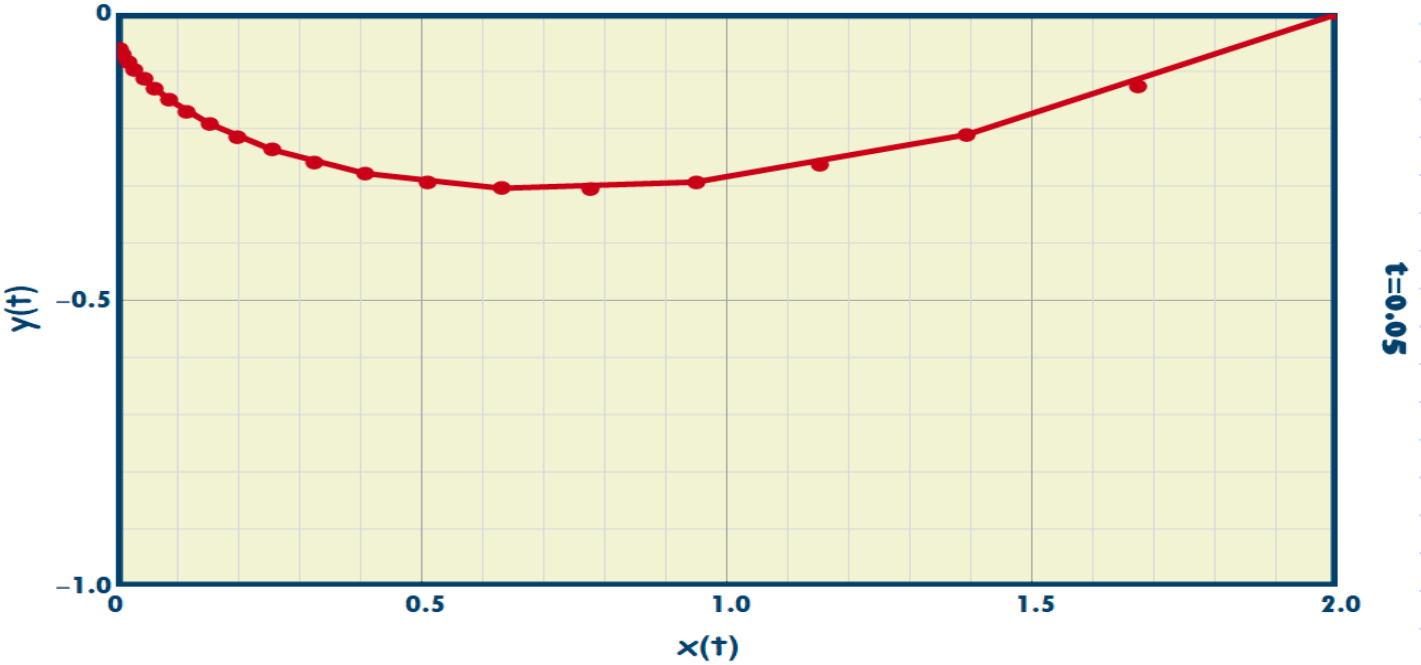
$$S = \begin{pmatrix} -2 & 2 \\ 7+3i & 3i-7 \end{pmatrix} \qquad \begin{pmatrix} e^{-\frac{3}{2}it} & 0 \\ 0 & e^{\frac{3}{2}it} \end{pmatrix} \qquad S^{-1} = \frac{1}{12i}\begin{pmatrix} 7-3i & 2 \\ 7+3i & 2 \end{pmatrix} \qquad \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} -2e^{-\frac{3}{2}it} & 2e^{\frac{3}{2}it} \\ (7+3i)e^{-\frac{3}{2}it} & (3i-7)e^{\frac{3}{2}it} \end{pmatrix}\begin{pmatrix} 14-6i \\ 14+6i \end{pmatrix}\frac{1}{12i} = \begin{bmatrix} 2\cos\left(\frac{3}{2}t\right) + \frac{14}{3}\sin\left(\frac{3}{2}t\right) \\ -\frac{58}{3}\sin\left(\frac{3}{2}t\right) \end{bmatrix}$$

$\{2\,e^{7\,t/2}\,\mathrm{Cos}[\frac{3\,t}{2}],\ 2\,e^{7\,t/2}\,\mathrm{Sin}[\frac{3\,t}{2}]\}$



$\{2\,e^{-7\,t/2}\,\mathrm{Cos}[\frac{3\,t}{2}],\ -2\,e^{-7\,t/2}\,\mathrm{Sin}[\frac{3\,t}{2}]\}$



$\{2\,\mathrm{Cos}[\frac{3\,t}{2}] + \frac{14}{3}\,\mathrm{Sin}[\frac{3\,t}{2}],\ -\frac{58}{3}\,\mathrm{Sin}[\frac{3\,t}{2}]\}$