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EXPONENTIAL OF A 2x2 MATRIX

$$\exp[M] = \sum_{0}^{\infty} {}_k \frac{M^k}{k!}$$

$$M^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \quad M^2 = \begin{pmatrix} 13 & 12 \\ 12 & 13 \end{pmatrix} \quad M^3 = \begin{pmatrix} 62 & 63 \\ 63 & 62 \end{pmatrix}$$

$$\exp\left[\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 13 & 12 \\ 12 & 13 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 62 & 63 \\ 63 & 62 \end{pmatrix} + \dots$$

$$\frac{1}{5!} M^5 = \frac{1}{120}\begin{pmatrix} 1562 & 1563 \\ 1563 & 1562 \end{pmatrix} \sim \begin{pmatrix} 13.017 & 13.025 \\ 13.025 & 13.017 \end{pmatrix}$$

$$\frac{1}{10!} M^{10} = \frac{1}{3\,628\,800}\begin{pmatrix} 4\,882\,813 & 4\,882\,812 \\ 4\,882\,812 & 4\,882\,813 \end{pmatrix} \sim \begin{pmatrix} 1.346 & 1.346 \\ 1.346 & 1.346 \end{pmatrix}$$

$$\sum_{0}^{m} {}_k \frac{M^k}{k!}$$

$$m=5 \begin{pmatrix} 45.89 & 45.52 \\ 45.52 & 45.89 \end{pmatrix} \quad m=6 \begin{pmatrix} 56.74 & 56.38 \\ 56.38 & 56.74 \end{pmatrix}$$

$$m=7 \begin{pmatrix} 64.49 & 64.13 \\ 64.13 & 64.49 \end{pmatrix} \quad m=8 \begin{pmatrix} 69.34 & 68.97 \\ 68.97 & 69.34 \end{pmatrix}$$

$$m=9 \begin{pmatrix} 72.03 & 71.66 \\ 71.66 & 72.03 \end{pmatrix} \quad m=10 \begin{pmatrix} 73.37 & 73.01 \\ 73.01 & 73.37 \end{pmatrix}$$

TWO POSSIBILITIES FOR THE EIGENVALUES OF A 2x2 MATRIX

(1) $\lambda_1 \neq \lambda_2$

(2) $\lambda_1 = \lambda_2 = \lambda$

(1) $M = S D S^{-1}$ $\quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ DIAGONAL FORM

(2) $M = S J S^{-1}$ $\quad J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ JORDAN FORM

$$W = \{D, J\} \quad M = SWS^{-1} \quad M^2 = SW\underbrace{S^{-1}S}_{1}WS^{-1} = SW^2S^{-1}$$

$$e^M = \sum_{0}^{\infty} {}_k \frac{(SWS^{-1})^k}{k!} = S \sum_{0}^{\infty} {}_k \frac{W^k}{k!} S^{-1} = S e^W S^{-1}$$

$$e^D = 1 + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \dots = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

$$e^J = 1 + J + \frac{J^2}{2!} + \frac{J^3}{3!} + \dots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix} + \dots$$

$$= \begin{pmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} e^{\lambda}$$

$$M=\begin{pmatrix}2&3\\3&2\end{pmatrix} \qquad \det(M-\lambda\mathbb{1})=0 \Rightarrow (2-\lambda)^2-9=0$$

$$2-\lambda=\pm 3$$

$$D=\begin{pmatrix}-1&0\\0&5\end{pmatrix} \qquad M=SDS^{-1}$$

$$\Downarrow$$

$$MS=SD$$

$$\begin{pmatrix}2&3\\3&2\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}-1&0\\0&5\end{pmatrix}$$

$$\begin{pmatrix}2a+3c&2b+3d\\3a+2c&3b+2d\end{pmatrix}=\begin{pmatrix}-a&5b\\-c&5d\end{pmatrix}$$

$$\begin{pmatrix}3a+3c&-3b+3d\\3a+3c&3b-3d\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix} \Rightarrow \begin{matrix}a=-c\\b=d\end{matrix}$$

$$c=d=1 \Rightarrow S=\begin{pmatrix}-1&1\\1&1\end{pmatrix} \Rightarrow S^{-1}=\frac{1}{2}\begin{pmatrix}-1&1\\1&1\end{pmatrix}$$

$$\exp\left[\begin{pmatrix}2&3\\3&2\end{pmatrix}\right]=\begin{pmatrix}-1&1\\1&1\end{pmatrix}\begin{pmatrix}e^{-1}&0\\0&e^5\end{pmatrix}\frac{1}{2}\begin{pmatrix}-1&1\\1&1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}-1/e&e^5\\1/e&e^5\end{pmatrix}\begin{pmatrix}-1&1\\1&1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}e^5+\frac{1}{e}&e^5-\frac{1}{e}\\e^5-\frac{1}{e}&e^5+\frac{1}{e}\end{pmatrix}\sim\begin{pmatrix}74.39&74.02\\74.02&74.39\end{pmatrix}$$

$$\sum_{0}^{10}{}_{k}\frac{M^k}{k!}\sim\begin{pmatrix}73.37&73.01\\73.01&73.37\end{pmatrix}$$

$$M=\begin{pmatrix}2&0\\3&2\end{pmatrix} \qquad (2-\lambda)^2=0 \qquad \lambda_1=\lambda_2=2$$

$$e^M=S\begin{pmatrix}1&1\\0&1\end{pmatrix}S^{-1}e^2$$

$$MS=SJ \qquad \begin{pmatrix}2&0\\3&2\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}2&1\\0&2\end{pmatrix}$$

$$\begin{pmatrix}2a&2b\\3a+2c&3b+2d\end{pmatrix}=\begin{pmatrix}2a&a+2b\\2c&c+2d\end{pmatrix}$$

$$\begin{pmatrix}0&-a\\3a&3b-c\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix} \Rightarrow a=0 \qquad c=3b$$

$a=0,\ b=1,\ c=3b=3,\ d=0$ $\quad S=\begin{pmatrix}0&1\\3&0\end{pmatrix} \Rightarrow S^{-1}=\frac{1}{3}\begin{pmatrix}0&1\\3&0\end{pmatrix}$

$$\exp\left[\begin{pmatrix}2&0\\3&2\end{pmatrix}\right]=\begin{pmatrix}0&1\\3&0\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}e^2\,\frac{1}{3}\begin{pmatrix}0&1\\3&0\end{pmatrix}$$
$$=\begin{pmatrix}0&1\\3&3\end{pmatrix}\begin{pmatrix}0&1\\3&0\end{pmatrix}\frac{e^2}{3}$$
$$=\begin{pmatrix}3&0\\9&3\end{pmatrix}\frac{e^2}{3}=\begin{pmatrix}1&0\\3&1\end{pmatrix}e^2$$

$M=\begin{pmatrix}2&-3\\3&2\end{pmatrix}$ $\quad (2-\lambda)^2+9=0$ $\quad \lambda=2\pm3i$

$MS=SD$ $\quad \begin{pmatrix}2&-3\\3&2\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}2+3i&0\\0&2-3i\end{pmatrix}$

$$\begin{pmatrix}2a-3c&2b-3d\\3a+2c&3b+2d\end{pmatrix}=\begin{pmatrix}2a+3ia&2b-3ib\\2c+3ic&2d-3id\end{pmatrix}$$
$$\begin{pmatrix}-3c-3ia&-3d+3ib\\3a-3ic&3b+3id\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$a=ic$ $\quad d=ib$

$c=b=1 \Rightarrow a=d=i$

$S=\begin{pmatrix}i&1\\1&i\end{pmatrix} \Rightarrow S^{-1}=\frac{1}{2}\begin{pmatrix}-i&1\\1&-i\end{pmatrix}$

$$\exp\left[\begin{pmatrix}2&-3\\3&2\end{pmatrix}\right]=\begin{pmatrix}i&1\\1&i\end{pmatrix}\begin{pmatrix}e^2e^{3i}&0\\0&e^2e^{-3i}\end{pmatrix}\frac{1}{2}\begin{pmatrix}-i&1\\1&-i\end{pmatrix}$$
$$=\begin{pmatrix}i&1\\1&i\end{pmatrix}\begin{pmatrix}e^{3i}&0\\0&e^{-3i}\end{pmatrix}\begin{pmatrix}-i&1\\1&-i\end{pmatrix}\frac{e^2}{2}$$
$$=\begin{pmatrix}ie^{3i}&e^{-3i}\\e^{3i}&ie^{-3i}\end{pmatrix}\begin{pmatrix}-i&1\\1&-i\end{pmatrix}\frac{e^2}{2}$$
$$=\begin{bmatrix}e^{3i}+e^{-3i}&i(e^{3i}-e^{-3i})\\-i(e^{3i}-e^{-3i})&e^{3i}+e^{-3i}\end{bmatrix}\frac{e^2}{2}$$
$$=\begin{pmatrix}\cos 3&-\sin 3\\\sin 3&\cos 3\end{pmatrix}\frac{e^2}{2}$$

$$\exp\left[\begin{pmatrix}2&3\\3&2\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}e^5+\frac{1}{e}&e^5-\frac{1}{e}\\e^5-\frac{1}{e}&e^5+\frac{1}{e}\end{pmatrix} \qquad \exp\left[\begin{pmatrix}2&0\\3&2\end{pmatrix}\right]=\begin{pmatrix}1&0\\3&1\end{pmatrix}e^2$$

$$\exp\left[\begin{pmatrix}2&-3\\3&2\end{pmatrix}\right]=\begin{pmatrix}\cos 3&-\sin 3\\\sin 3&\cos 3\end{pmatrix}\frac{e^2}{2}$$

PREPARING EXERCISES

$$\begin{pmatrix} 2 & b \\ c & 4 \end{pmatrix}$$

EIGENVALUES EQUATION

$$(2-\lambda)(4-\lambda) - bc = 0$$
$$\lambda^2 - 6\lambda + 8 - bc = 0$$
$$\lambda_{1,2} = 3 \pm \sqrt{1+bc}$$

$bc = 0$ $\quad \lambda_{1,2} = 3 \pm 1$ $\quad \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

$bc = 8$ $\quad \lambda_{1,2} = 3 \pm 3$ $\quad \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 8/3 \\ 3 & 4 \end{pmatrix}$

$bc = -1$ $\quad \lambda_{1,2} = 3$ $\quad \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1/2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1/3 \\ 3 & 4 \end{pmatrix}$

$bc = -2$ $\quad \lambda_{1,2} = 3 \pm i$ $\quad \begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -2/3 \\ 3 & 4 \end{pmatrix}$

$$\begin{pmatrix} -2 & b \\ c & -4 \end{pmatrix}$$

EIGENVALUES EQUATION

$$(2+\lambda)(4+\lambda) - bc = 0$$
$$\lambda^2 + 6\lambda + 8 - bc = 0$$
$$\lambda_{1,2} = -3 \pm \sqrt{1+bc}$$

$bc = 0$ $\quad \lambda_{1,2} = -3 \pm 1$ $\quad \begin{pmatrix} -2 & 0 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 3 & -4 \end{pmatrix}$

$bc = 8$ $\quad \lambda_{1,2} = -3 \pm 3$ $\quad \begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} -2 & 8/3 \\ 3 & -4 \end{pmatrix}$

$bc = -1$ $\quad \lambda_{1,2} = -3$ $\quad \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -2 & -1/2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} -2 & -1/3 \\ 3 & -4 \end{pmatrix}$

$bc = -2$ $\quad \lambda_{1,2} = -3 \pm i$ $\quad \begin{pmatrix} -2 & -2 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} -2 & -2/3 \\ 3 & -4 \end{pmatrix}$