1. Questão 1.

a) Devemos encontrar $C \in [0,1]$, tal que $\sum_x P(X = x) = 1$. Assim:

$$\frac{1}{5} + \frac{3}{5} + C = 1 \to C = 1 - \frac{4}{5} = \frac{1}{5}$$

b) Temos que

$$\mathcal{E}(X) = -\frac{1}{5} + \frac{6}{5} + \frac{3}{5} = \frac{8}{5} = 1,60 \text{ e } \mathcal{E}(X^2) = \frac{1}{5} + \frac{12}{5} + \frac{9}{5} = \frac{22}{5} = 4,4.$$

Logo

$$\mathcal{V}(X) = \mathcal{E}(X^2) - \mathcal{E}^2(X) = \frac{22}{5} - \frac{64}{25} = \frac{46}{25} = 1,84$$

c) Temos que

$$F_X(x) = P(X \leq x), \forall x \in \mathcal{R},$$

mais especificamente,

- Se $x < -1$, $F_X(x) = 0$.
- Se $x \in [-1,2)$, $F_X(x) = P(X = -1) = \frac{1}{5}$.
- Se $x \in [2,3)$, $F_X(x) = P(X = -1) + P(X = 2) = \frac{4}{5}$.
- Finalmente, de $x \in [3,\infty)$, então $P(X \leq x) = \frac{4}{5} + \frac{1}{5} = 1$.

Logo

$$F_X(x) = \frac{1}{5}\mathbb{1}_{[-1,2)}(x) + \frac{4}{5}\mathbb{1}_{[2,3)}(x) + \mathbb{1}_{[3,\infty)}(x)$$

d) Temos que
$Mo(X) = max_x P(X = x) = 2$. Por outro lado, temos que somente o valor $X = 2$ tal que $P(X \leq x) \geq \frac{1}{2}$ e $P(X \geq x) \geq \frac{1}{2}$. Assim $Md(X) = 2$.

2. Questão 2 (Primeiramente, definamos $Y \sim \text{Poisson}(\lambda)$

a) Temos que:

$$\sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})} = \frac{1}{(1-e^{-\lambda})} \left\{ -e^{-\lambda} + \sum_{x=0}^{\infty} \underbrace{\frac{e^{-\lambda}\lambda^x}{x!}}_{\text{fdp de } Y \text{(acima)}} \right\} = \frac{1}{(1-e^{-\lambda})} \left\{ -e^{-\lambda} + 1 \right\} = 1.$$

Por outro lado, note que $e^{-\lambda}, \lambda^x, X!, (1-e^{-\lambda}) > 0$, logo $P(X = x) > 0, \forall x \in \{1, 2...\}, \forall \lambda \in \mathcal{R}^+$. Assim, por este resultado e pelo resultado acima, $P(X = x) \in (0, 1), \forall x \in \{1, 2...\}, \forall \lambda \in \mathcal{R}^+$.

b) Temos que:

$$\mathcal{E}(X) = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})} = \frac{1}{(1-e^{-\lambda})} \left\{ \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x} \right\} = \frac{1}{(1-e^{-\lambda})} \mathcal{E}(Y) = \frac{\lambda}{(1-e^{-\lambda})}.$$

Além disso, temos que

$$\begin{aligned} \mathcal{E}(X(X-1)) &= \sum_{x=1}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})} = \frac{1}{(1-e^{-\lambda})} \left\{ \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} \right\} \\ &= \frac{1}{(1-e^{-\lambda})} \mathcal{E}(Y(Y-1)) = \frac{\mathcal{E}(Y^2) - \mathcal{E}(Y)}{(1-e^{-\lambda})} = \frac{\mathcal{V}(Y) + \mathcal{E}^2(Y) - \mathcal{E}(Y)}{(1-e^{-\lambda})} \\ &= \frac{\lambda^2}{(1-e^{-\lambda})} \rightarrow \mathcal{E}(X^2) = \frac{\lambda^2}{(1-e^{-\lambda})} + \frac{\lambda}{(1-e^{-\lambda})} \end{aligned}$$

Finalmente, temos que:

$$\begin{aligned} \mathcal{V}(X) &= \frac{\lambda^2}{(1-e^{-\lambda})} + \frac{\lambda}{(1-e^{-\lambda})} - \frac{\lambda^2}{(1-e^{-\lambda})^2} \\ &= \frac{\lambda}{(1-e^{-\lambda})^2} \left(\lambda - \lambda e^{-\lambda} + 1 - e^{-\lambda} - \lambda \right) = \frac{\lambda}{(1-e^{-\lambda})^2} \left(1 - \lambda e^{-\lambda} - e^{-\lambda} \right) \end{aligned}$$

3. Questão 3

a) Temos que se $x \in [-1,0)$, então $f_X(x) = x^2 > 0$, se $x \in [0,1]$, então $f_X(x) = 2x^2 \geq 0$ e $f_X(x) = 0$, caso contrário. Assim $f_X(x) \geq 0, \forall x \in \mathcal{R}$. Além disso:

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-1}^{0} x^2 dx + \int_{0}^{1} 2x^2 dx = \left.\frac{x^3}{3}\right|_{-1}^{0} + \left.\frac{2x^3}{3}\right|_{0}^{1} = \frac{1}{3} + \frac{2}{3} = 1$$

b) Temos que
Se $x < -1$, então $F_X(x) = 0$ e se $x > 1$, então $F_X(x) = 1$. Por outro lado,
Se $x \in [-1,0)$, então

$$F_X(x) = \int_{-1}^{x} t^2 dt = \left.\frac{t^3}{3}\right|_{-1}^{x} = \frac{x^3+1}{3}$$

Se $x \in [0,1]$, então

$$F_X(x) = \int_{-1}^{x} f_X(t)dt = \frac{t^3}{3}|_{-1}^{0} + \frac{2t^3}{3}|_{0}^{x} = \frac{1+2x^3}{3}$$

Logo:

$$F_X(x) = \frac{x^3+1}{3}\mathbb{1}_{[-1,0)}(x) + \frac{1+2x^3}{3}\mathbb{1}_{[0,1]}(x) + \mathbb{1}_{(1,\infty)}(x)$$

c) Note que

$$\begin{aligned}\mathcal{E}(X^k) &= \int_{-\infty}^{\infty} x^k f_X(x)dx = \int_{-1}^{0} x^{k+2}dx + \int_{0}^{1} 2x^{k+2}dx = \left.\frac{x^{k+3}}{k+3}\right|_{-1}^{0} + \left.\frac{2x^{k+3}}{k+3}\right|_{0}^{1} \\ &= \frac{-(-1)^{k+3}+2}{k+3}\end{aligned}$$

Logo

$$\begin{aligned}\mathcal{E}(X) &= \frac{-(-1)^4+2}{4} = \frac{1}{4} = 0,25 \\ \mathcal{E}(X^2) &= \frac{-(-1)^5+2}{5} = \frac{3}{5} = 0,6 \\ \mathcal{V}(X) &= \frac{3}{5} - \frac{1}{16} = \frac{48-5}{80} = \frac{43}{80} = 0,54\end{aligned}$$

d) Note que, para a $Md(x) \in [0,1]$, pois $PX(\leq 0) = \frac{1}{3}$. Logo

$$\frac{1+2x^3}{3} = \frac{1}{2} \rightarrow x^3 = \frac{1}{4} \rightarrow x = 0,63 = \left(\frac{1}{4}\right)^{1/3}$$

Para a moda, note que, se $x \in [-1,0)$, $f'_x(x) = x < 0$, portanto f_X é decrescente e, se $x \in [0,1]$, $f'_X(x) = 2x \geq 0$ é não decrescente (sendo crescente a partir de $x > 0$). Além disso,

$$f_X(-1) = 1 < f_X(1) = 2$$

Logo, $Mo(X) = 1$.

4. Questão 4

a) Temos que

i)

$$
\begin{aligned}
\lim_{x\to-\infty} G(x) &= \frac{2\lim\limits_{x\to-\infty} F_X(x)}{1+\lim\limits_{x\to-\infty} F_X(x)} = \frac{0}{1+0} = 0 \\
\lim_{x\to\infty} G(x) &= \frac{2\lim\limits_{x\to\infty} F_X(x)}{1+\lim\limits_{x\to\infty} F_X(x)} = \frac{2}{1+1} = 1,
\end{aligned} \tag{1}
$$

ii)

Note que $G(x) = \dfrac{2}{1+\dfrac{1}{F_X(x)}}$. Por outro lado, se $a < b$, então

$$
\begin{aligned}
F_X(a) &\le F_X(b) \leftrightarrow 1+\frac{1}{F_X(a)} \ge 1+\frac{1}{F_X(b)} \leftrightarrow \frac{2}{1+\frac{1}{F_X(a)}} \le \frac{2}{1+\frac{1}{F_X(b)}} \\
&\leftrightarrow G(a) \le G(b)
\end{aligned}
$$

iii)

Finalmente,

$$
\lim_{0<h,h\to0} G(x+h) = \frac{2\lim\limits_{0<h,h\to0} F_X(x+h)}{1+\lim_{0<h,h\to0} F_X(x+h)} = \frac{2F_X(x)}{1+F_X(x)} = G(x)
$$

b) Temos que

$$
g(x) = G'(x) = \frac{2}{[1+F_X(x)]^2}\left(f_X(x)(1+F_X(x)) - f_X(x)F_X(x)\right) = \frac{2f_X(x)}{[1+F_X(x)]^2}
$$