1. Questão 1.

   a) $P(X = x) = 0, \forall x \in \mathcal{R}$ se X for uma vac.

   b) Temos que $X \sim (1,70; 0,07^2)$, logo $Z = \dfrac{X - 1,70}{0,07} \sim N(0,1)$

$$P\left(X > 1,65\right) = P\left(Z > -0,71\right) = 1 - P(Z \leq -0,71) = 1 - 0,2389 = 0,7611$$

   b) Temos que

$$P\left(X < 1,80\right) = P\left(Z < 1,43\right) = 0,9236$$

   c) Temos que

$$\begin{aligned} P\left(1,65 \leq X \leq 1,80\right) &= P(X \leq 1,80) - P(X \leq 1,65) = P(Z \leq 1,43) - P(Z \leq -0,71) \\ &= 0,9236 - 0,2389 = 0,6847 \end{aligned}$$

   d) Temos que

$$\begin{aligned} P(X \leq 1,72) &= 0,6915 \leftrightarrow P(Z \leq z_1) = 0,6915 \\ P(X > 1,82) &= 0,0668 \rightarrow P(X \leq 1,82) = 0,9332 \rightarrow P(Z \leq z_2) = 0,9332 \rightarrow \end{aligned}$$

   Em que :

$$\begin{cases} z_1 = \frac{1,72-\mu}{\sigma} = 0,50 \\ z_2 = \frac{1,82-\mu}{\sigma} = 1,50 \end{cases} \rightarrow \begin{cases} \mu = -0,50\sigma + 1,72 \\ \frac{1,82-\mu}{\sigma} = 1,50 \end{cases} \rightarrow \begin{cases} \mu = -0,50\sigma + 1,72 \\ 1,50\sigma = 1,82 + 0,50\sigma - 1,72 \end{cases}$$
$$\begin{cases} \mu = -0,50\sigma + 1,72 \rightarrow = \mu = 1,67 \\ \sigma = 0,10 \rightarrow \sigma^2 = 0,01 \end{cases}$$

2. Se $Y \sim \text{binomial}(n, p)$, então $P(X = x) = \dfrac{P(Y = x)}{1 - (1-p)^n} \mathbb{1}_{\{1,2,\dots,n\}}(x)$

a) Temos que:

$$\begin{aligned}\sum_{x=1}^{n} P(X = x) &= \frac{1}{1-(1-p)^n} \sum_{x=1}^{n} P(Y = x) = \frac{1}{1-(1-p)^n} \left(\sum_{x=0}^{P} (Y = x) - (1-p)^n \right) \\ &= \frac{1-(1-p)^n}{1-(1-p)^n} = 1, \text{por outro lado,} \qquad (1)\end{aligned}$$

$$\binom{n}{p} > 0, p^x (1-p)^{n-x} > 0, 1 - (1-p)^n \in (0,1) \rightarrow P(X = x) > 0 \forall x \in \{1, 2, ..., n\}. \quad (2)$$

Logo, de (1) e (2), temos que $P(X = x) \in (0,1) \in \forall x \in \{1, 2, ..., n\}$ e $\sum_{x=1}^{n} P(X = x) = 1$. Logo $P(X = x)$ é uma legítima fdp.

b) Temos que

$$\mathcal{E}(X) = \sum_{x=1}^{n} x P(x = x) = \frac{1}{1-(1-p)^n} \sum_{x=0}^{n} x \binom{n}{p} p^x (1-p)^{n-x} = \frac{np}{1-(1-p)^n}$$

c) Primeiramente, note que

$$\mathcal{E}(Y^2) = \mathcal{V}(Y) + \mathcal{E}^2(Y) = np(1-p) + n^2p^2 = np(1-p+np) = np(p(n-1)+1).$$

Por outro lado, temos que

$$\mathcal{E}(X^2) = \sum_{x=1}^{n} x^2 P(x = x) = \frac{1}{1-(1-p)^n} \sum_{x=0}^{n} x^2 \binom{n}{p} p^x (1-p)^{n-x} = \frac{np(1-p) + n^2p^2}{1-(1-p)^n}$$

Logo

$$\begin{aligned}\mathcal{V}(X) &= \mathcal{E}(X^2) - \mathcal{E}^2(X) = \\ &= \frac{1}{(1-(1-p)^n)^2} \left(np(1-p) + n^2p^2 - (1-p)^n (np(1-p) + n^2p^2) - n^2p^2 \right) \\ &= \frac{1}{(1-(1-p)^n)^2} \left(np(1-p)(1-(1-p)^n - (1-p)^n n^2 p^2) \right)\end{aligned}$$

3. Temos que

a) Da tabela acima temos que:

$$\begin{aligned} P(X=x) &= \frac{1}{8}\mathbb{1}_{\{-1\}}(x)+\frac{5}{8}\mathbb{1}_{\{0\}}(x)+\frac{2}{8}\mathbb{1}_{\{2\}}(x) \\ P(Y=y) &= \frac{1}{2}\mathbb{1}_{\{1\}}(y)+\frac{1}{2}\mathbb{1}_{\{3\}}(y) \end{aligned}$$

b) Temos que:

$$\begin{aligned} P(X=x|Y=1) &= \frac{P(X=x,Y=1)}{P(Y=1)} = \frac{1}{4}I_{\{-1\}}(x)+\frac{2}{4}I_{\{0\}}(x)+\frac{1}{4}I_{\{2\}}(x) \\ P(Y=y|X=0) &= \frac{P(X=0,Y=y)}{P(X=0)} = \frac{2}{5}I_{\{1\}}(y)+\frac{3}{5}I_{\{3\}}(y) \end{aligned}$$

c) Do item a), temos que:

$$\begin{aligned} \mathcal{E}(X) &= \frac{-1+4}{8} = \frac{3}{8} \approx 0,38; \mathcal{E}(X^2) = \frac{1+8}{8} = \frac{9}{8} \approx 1,13; \mathcal{V}(X) = \frac{72-9}{64} = \frac{63}{64} \approx 0,98 \\ \mathcal{E}(Y) &= \frac{1+3}{2} = 2; \mathcal{E}(Y^2) = \frac{1+9}{2} = 5; \mathcal{V}(Y) = 5-4 = 1. \end{aligned}$$

Por outro lado, temos que:

$$\begin{aligned} E(XY) &= \sum_x \sum_y xyP(X=x,Y=y) = \frac{-1+2+6}{8} = \frac{7}{8} \approx 0,88; \\ Cov(X,Y) &= \frac{7}{8}-\frac{6}{8} = \frac{1}{8} \approx 0,13; Corre(X,Y) = \frac{1/8}{\sqrt{63/64}} \approx 0,13 \end{aligned}$$

4. Temos que

a) Temos que:

$$
\begin{aligned}
Cov(a_1X + b_1, a_2Y + b_2) &= \mathcal{E}[(a_1X + b_1)(a_2Y + b_2)] - \mathcal{E}(a_1X + b_1)\mathcal{E}(a_2Y + b_2) \\
&= \mathcal{E}\left[a_1a_2XY + a_1b_2X + a_2b_1Y + b_1b_2\right] \\
&- a_1a_2\mathcal{E}(X)\mathcal{E}(Y) - a_1b_2\mathcal{E}(X) - a_2b_1\mathcal{E}(Y) - b_1b_2 \\
&= a_1a_2\mathcal{E}(XY) - a_1a_2\mathcal{E}(X)\mathcal{E}(Y) + a_1b_2\mathcal{E}(X) + a_2b_1\mathcal{E}(Y) \\
&- a_1b_2\mathcal{E}(X) - a_2b_1\mathcal{E}(Y) = a_1a_2Cov(X, Y)
\end{aligned}
$$

b) Temos que:

$$
\begin{aligned}
Corre(a_1X + b_1, a_2Y + b_2) &= \frac{Cov(a_1X + b_1, a_2Y + b_2)}{\sqrt{\mathcal{V}(a_1X + b_1)\mathcal{V}(a_2Y + b_2)}} = \frac{a_1a_2Cov(X, Y)}{\sqrt{a_1^2\mathcal{V}(X)a_2^2\mathcal{V}(Y)}} \\
&= \frac{a_1a_2Cov(X, Y)}{a_1a_2\sqrt{\mathcal{V}(X)\mathcal{V}(Y)}} = Corre(X, Y)
\end{aligned}
$$