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Holling-type functional responses of fuzzy population models under cross operations

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Abstract. This paper provides an introductory study on Holling-type functional responses, namely, Holling-types I and II, described in a fuzzy environment. The environment considered is the space of $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy numbers, established from the structure of vector spaces embedded in the class of fuzzy numbers. The arithmetic structure is given by the induced, and cross, arithmetic operations. The parameters involved are written as fuzzy quantities, as well as the population. A brief analysis of the choice of the underlying strongly linearly independent set is made considering the fuzziness associated with the Holling term encloses the work.

Key words: *Holling functional responses; fuzzy parameters; strong linear independence; $\mathcal{S}$-linearly correlated fuzzy numbers; cross operations.*

1. Introduction

Multi-species interactions are known for having very complex dynamics. The classical prey-predator model, originally proposed by Lotka (1925) and Volterra (1926), have been extensively studied in the last decades from several points of view. The original model predicts that the predator population increases proportionally to the encounter with the prey population which, in turn, decreases proportionally to this encounter. As a result, the solution to

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the model depicts fluctuating populations in an ecosystem (May and McLean, 2007). Several rereadings on the prey-predator dynamics were done all over the years. Okubo and Levin (2001), for instance, dealt with two and multispecies population dynamics with spatial dispersion - including prey-predator model. His focus were mainly for the ecological point of view of the phenomenon.

Holling (1959b,a, 1965, 1966) introduced the families called Holling-type functional responses by considering the response of the consumption of prey by individual predators to changes of prey density. His studies involved the sawfly cocoons and small mammals dynamics, and the mathematical expressions:

$$f(x) = cx, \quad x > 0 \tag{1.1}$$

and

$$f(x) = \frac{c}{h + x}, \quad x > 0 \tag{1.2}$$

were provided, being denoted by Holling-Type I response, and Holling-Type II response, respectively. In both equations (1.1) and (1.2), x denotes the prey density. Holling's contributions still represent a remarkable contribution from the mathematical ecology until nowadays, once some dynamics, such as parasitoids in an ecosystem, for instance, are only well-described with Holling interactions (see (Fernández-Arhex and Corley, 2003) for details). Justified by its scientific relevance, Pervez et al. (2018) presented a study which explains how environmental factors produce different functional response of predators in a multi-especies dynamics.

Peixoto et al. (2008), as one of the first studies on prey-predator dynamics using tools from fuzzy sets theory, used a fuzzy rule-based system to study the Holling-Tanner model, assuming a Holling Type II functional response. More recent contributions considered functional responses with uncertainty. Using granular derivative for fuzzy-valued functions, Das et al. (2022) considered prey-predator model under Ivlev's functional response - also known as Type V Holling response. Mondal et al. (2022) considered Holling- type II functional response and interval uncertainty to study prey-pradator model. Sukarsih et al. (2023) used Zadeh's extension principle applied to a Runge-Kutta method to provide numerical solutions to the dynamics.

This manuscript aims to study Holling-type functional responses considering the population and parameters as fuzzy quantities. To this end, a linear algebra approach is used, so that all fuzzy quantities are assumed to belong to vector spaces generated by a *Strongly linearly independent set* (Esmi et al., 2021). The arithmetic operations employed are called $\psi-$*arithmetic operations*, given by induced and cross operations (Esmi et al., 2021; Laiate et al., 2021b). The structure is organized as follows: The section 2 presents the basic concepts on the algebra used all long the text. The section 3 derives the analytical expressions to represent both population and parameters in the Holling-type I and II functional responses. The section 4 provides a brief analysis on the basis of the vector space considered. The section 5 encloses the paper by presenting some final considerations.

2. Fuzzy sets theory

A fuzzy set A of a topological space $\mathcal{U}$ is characterized by a function

$$\mu_A : \mathcal{U} : [0,1],$$

where $A(x)$ represents the membership degree of each $x \in \mathcal{U}$ belongs to A. The application $\mu_A(\cdot)$ is called the *membership function* of A, and the set of all fuzzy sets of $\mathcal{U}$ is denoted by $\mathcal{F}(\mathcal{U})$. By notational convenience, $\mu_A(\cdot)$ is usually written as $A(\cdot)$.

For a given topological space $\mathcal{U}$, each $A \in \mathcal{F}(\mathcal{U})$ is completely described by the so-called $\alpha-$levels (or levelsets) of A, given by the relation:

$$[A]_\alpha = \begin{cases} \{x \in \mathcal{U} : A(x) \geq \alpha\}, & \alpha \in [0,1) \\ \overline{\{x \in \mathcal{U} : A(x > 0)\}}, & \alpha = 0 \end{cases},$$

where $\overline{X}$ denotes the closure of $X \subset \mathbb{R}$. Note that the levelsets of $A \in \mathcal{F}(\mathcal{U})$ consist on classical subsets of $\mathcal{U}$, i.e, $[A]_\alpha \in \mathcal{P}(\mathcal{U})$, $\forall \alpha \in [0,1]$.

A fuzzy number is a normal fuzzy subset of $\mathbb{R}$ whose $\alpha-$levels are compact intervals of $\mathbb{R}$ (Barros et al., 2017). More specifically, A is a fuzzy number if the following conditions are fulfilled:

i) There exists $x \in \mathbb{R}$ such that $A(x) = 1$ (A is normal);

ii) $[A]_\alpha \in \mathcal{K}_c$, where $\mathcal{K}_c$ is the set of all compact subsets of $\mathbb{R}$;

iii) The set $\text{supp}(A) = \{x \in \mathbb{R} : A(x) > 0\}$ is compact.

The set of all fuzzy numbers is denoted by $\mathbb{R}_\mathcal{F}$. It is immediate from definition above that for each $A \in \mathbb{R}_\mathcal{F}$, there exist $a^-_\alpha \leq a^+_\alpha$ such that $[A]_\alpha = [a^-_\alpha, a^+_\alpha]$, $\forall \alpha \in [0,1]$.

As we shall see in the following, the study of arithmetic operations on fuzzy numbers with the arithmetic operations on intervals.

The usual arithmetic operations on fuzzy numbers are given in terms of Zadeh's Extension Principle. In brief words, the standard arithmetic in $\mathbb{R}_{\mathcal{A}}$ inherits the usual interval arithmetic operations in the following sense: let $A, B \in \mathbb{R}_{\mathcal{A}}$ be fuzzy numbers given levelwise by $[A]_\alpha = [a^-_\alpha, a^+_\alpha]$ and $[B]_\alpha = [b^-_\alpha, b^+_\alpha]$, for all $\alpha \in [0,1]$, and $\lambda \in \mathbb{R}$. Then, the sum, subtraction and scalar multiplication are given, respectively, by

$$\begin{aligned}[A+B]_\alpha &= [A]_\alpha + [B]_\alpha = \left[a^-_\alpha + b^-_\alpha, a^+_\alpha + b^+_\alpha\right] \\ [A-B]_\alpha &= [A]_\alpha + [B]_\alpha = \left[a^-_\alpha - b^+_\alpha, a^+_\alpha - b^-_\alpha\right]\end{aligned}, \tag{2.3}$$

and

$$[\lambda A]_\alpha = \begin{cases} [\lambda a^-_\alpha, \lambda a^+_\alpha], & \lambda \geq 0 \\ [\lambda a^+_\alpha, \lambda a^-_\alpha], & \lambda < 0 \end{cases}, \tag{2.4}$$

$\forall \alpha \in [0,1]$. From (2.3) and (2.4), we conclude that the space of fuzzy numbers is quasilinear, so that $(\lambda+\mu)A \leq \lambda A + \mu A$ for all scalar $\mu, \lambda \in \mathbb{R}$, where the equality holds whenever $\mu\lambda \geq 0$ (Bede, 2013). In the meantime, the operations of product and division are given respectively by

$$\begin{aligned}[A \cdot B]_\alpha &= [A]_\alpha \cdot [B]_\alpha = [\min P_\alpha, \max P_\alpha] \\ [A \div B]_\alpha &= [A]_\alpha \div [B]_\alpha = [\min Q_\alpha, \max Q_\alpha]\end{aligned}$$

where $P_\alpha = \{a^-_\alpha b^-_\alpha, a^+_\alpha b^-_\alpha, a^-_\alpha b^+_\alpha, a^+_\alpha b^+_\alpha\}$ and $Q_\alpha = \left\{\frac{a^-_\alpha}{b^-_\alpha}, \frac{a^+_\alpha}{b^-_\alpha}, \frac{a^-_\alpha}{b^+_\alpha}, \frac{a^+_\alpha}{b^+_\alpha}\right\}$, $\alpha \in [0,1]$.

As pointed out by several papers, these arithmetic operations are computationally expensivel because of the computation of the indexed sets P_α and Q_α for all α. In addition, as well as the usual sum, the usual product do not have an inverse in general, so that both relations $A - A \neq 0$ and $A \div A \neq 1$ hold in general for $A \in \mathbb{R}_{\mathcal{F}}$. Other drawbacks are associated to the usual operation of product, including the impossibility of controlling the width, and the changing shape of the resulting fuzzy number.

The next subsection provides an alternative arithmetic structure for specific subclasses of fuzzy numbers contained in $\mathbb{R}_{\mathcal{F}}$.

2.1. Induced fuzzy arithmetic operations: a vector space structure

As presented previously, the class of fuzzy numbers is not linear under the usual arithmetic operations of sum and scalar multiplication. In order to establish finite-dimensional vector spaces of fuzzy numbers, we recall the notion of symmetry of a fuzzy number.

Let $A \in \mathbb{R}_{\mathcal{F}}$ be a fuzzy number given. If there exists $x \in \mathbb{R}$ so that $A(x-y) = A(x+y)$, we say that A is symmetric with respect to $x \in \mathbb{R}$ (or simply *symmetric*), and we denote it by $(A|x)$.

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\} \subset \mathbb{R}_{\mathcal{F}}$ be a set of m fuzzy numbers given. The set of all linear combinations (or *Minkowski combinations*) of A_i is denoted by the equation (2.5)

$$\mathcal{S}(\mathcal{A}) = \{q_1 A_1 + \ldots + q_m A_m \,:\, q_1, \ldots, q_m \in \mathbb{R}\}. \tag{2.5}$$

If $B \in \mathcal{S}(\mathcal{A})$, then we can write by the equation (2.6)

$$[B]_\alpha = q_1 [A_1]_\alpha + \ldots + q_n [A_m]_\alpha, \quad \alpha \in [0,1]. \tag{2.6}$$

Next, we recall the notion of *Strong Linear Independence* of a set of fuzzy numbers.

Definition 2.1 ((Esmi et al., 2021)) *Let $\mathcal{A} = \{A_1, \ldots, A_m\} \subset \mathbb{R}_{\mathcal{F}}$ be a set of fuzzy numbers given and $B \in \mathcal{S}(\mathcal{A})$ be given by $B = q_1 A_1 + \ldots + q_m A_m$. The set $\mathcal{A}$ is Strongly Linearly Independent (SLI, for short) if, and only if, the following implication holds:*

$$(B \,|\, 0) \Rightarrow q_1 = \ldots = q_m = 0.$$

The next theorem is particularly useful to identify if a finite $\mathcal{A}$ of $\mathbb{R}_{\mathcal{F}}$ is SLI.

Theorem 2.1 (Esmi et al. (2021)) *The set $\mathcal{A} = \{A_1, \ldots, A_m\} \subset \mathbb{R}_{\mathcal{F}}$ is SLI if and only if the function*

$$\psi : \mathbb{R}^m \to \mathcal{S}(\mathcal{A})$$

given by

$$\psi(x_1, \ldots, x_m) = x_1 A_1 + \ldots + x_m A_m \tag{2.7}$$

is an isomorphism, where "$+$" and "$q_i A_i$" stand for the usual operations of sum and scalar multiplication in $\mathbb{R}_{\mathcal{F}}$, respectively.

From Theorem 2.1, we can say that if $\mathcal{A} \subset \mathbb{R}_{\mathcal{F}}$ is SLI and $B \in \mathcal{S}(\mathcal{A})$, then there exists a real-vector $q = (q_1, \ldots, q_m) \in \mathbb{R}^m$ such that $B = \psi(q_1, \ldots, q_m) = q_1 A_1 + \ldots + q_m A_m$. In this case, B is called an $\mathcal{S}(\mathcal{A})$-*linearly correlated fuzzy number.*

Theorem 2.1 allows us to define arithmetic operations induced from isomorphism ψ on $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy numbers. Let $B, C \in \mathcal{S}(\mathcal{A})$ and $\lambda \in \mathbb{R}$. Define

$$\begin{aligned} B +_\psi C &\doteq \psi\left(\psi^{-1}(B) + \psi^{-1}(C)\right) \\ \lambda \cdot_\psi B &\doteq \psi\left(\psi^{-1}(\lambda B)\right) \end{aligned}. \tag{2.8}$$

For $B = \psi(q_1, \ldots, q_m)$ and $C = \psi(p_1, \ldots, p_m)$, (2.8) yields the following three well-defined operations of sum, subtraction and scalar multiplication:

$$\begin{aligned} B +_\psi C &= (q_1 + p_1)A_1 + \ldots (q_m + p_m)A_m \\ B -_\psi C &= (q_1 - p_1)A_1 + \ldots (q_m - p_m)A_m. \\ \lambda \cdot_\psi B &= (\lambda q_1)A_1 + \ldots (\lambda q_m)A_m \end{aligned} \tag{2.9}$$

We observe from equation (2.9) that computing the induced arithmetic operations on $\mathcal{S}(\mathcal{A})$ is equivalent to computing the arithmetic operations on the real-valued coordinates of the fuzzy numbers involved. Moreover, the space $(\mathcal{S}(\mathcal{A}), +_\psi, \cdot_\psi)$ turns to be a vector space, with dimension equals to m (more details on this methodology can be seen in Esmi et al. (2022)).

We henceforth assume that $\mathbb{R} \subseteq \mathcal{S}(\mathcal{A}) \subset \mathbb{R}_{\mathcal{F}}^{\wedge}$, where

$$\mathbb{R}_{\mathcal{F}}^{\wedge} = \{A \in \mathbb{R}_{\mathcal{F}} : [A]_1 \text{ has a unique element}\} \subset \mathbb{R}_{\mathcal{F}}.$$

Denote $\text{core}(B) = [B]_1 = b$ and $\text{core}(C) = [C]_1 = c$ with $c \neq 0$. Note that b and c exist and are unique since $\mathcal{S}(\mathcal{A}) \subset \mathbb{R}_{\mathcal{F}}^{\wedge}$. The operations of product and division $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy numbers are given by

$$\begin{aligned} B \odot_\psi C &= c \cdot_\psi B +_\psi b \cdot_\psi C -_\psi b \cdot_\psi c \\ B \div_\psi C &= B \odot_\psi C_\psi^{-1} \end{aligned}, \tag{2.10}$$

where $C_\psi^{-1} = \left(\frac{2}{a_1 c} - \frac{p_1}{c^2}\right) A_1 - \frac{p_1}{c^2} A_2 - \ldots - \frac{p_m}{c^2} A_m$ and $a_1 = \text{core}(A_1)$ (Laiate et al., 2021b). The arithmetic operations defined by (2.10) can be seen as linearized operations of product and division (Laiate et al., 2021a), an ideal feature to make $\mathcal{S}(\mathcal{A})$ closed under the $\psi-$cross product $\odot_\psi$ and the $\psi-$cross division $\div_\psi$.

The so-called $\psi-$*arithmetic operations* $\otimes_\psi \in \{+_\psi, -_\psi, \odot_\psi, \div_\psi\}$ are direct extensions of the corresponding arithmetic operations in the classical case, i.e., $\chi_{\{a\}} \otimes_\psi \chi_{\{b\}} = a \otimes b$ whenever $a, b \in \mathbb{R}$, which are regarded as singletons. The next lemma offers a different point of view to this argument:

Lemma 2.1 *(Adapted from Laiate et al. (2021b)) Let $\mathcal{A} \subset \mathbb{R}^{\wedge}_{\mathcal{F}}$ be SLI, $B, C \in \mathcal{S}(\mathcal{A})$ and $\lambda \in \mathbb{R}$. Then the following equality holds:*

$$[B \otimes_\psi C]_1 = [B]_1 \otimes [C]_1 \quad \textit{and} \quad [\lambda \cdot_\psi B]_1 = \lambda [B]_1$$

for all $\otimes_\psi \in \{+_\psi, -_\psi, \odot_\psi, \div_\psi\}$.

For notational convenience, we henceforth omit the symbol '$\cdot_\psi$' when the scalar multiplication is referred in $\mathcal{S}(\mathcal{A})$. The next section presents a brief study on the Holling-type functional responses as functions of the form $f : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$.

3. Holling-type functional responses in $\mathcal{S}(\mathcal{A})$

In this subsection, we assume the populations involved in the functional response are given by $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy functions. By hypothesis, let $\mathcal{A} = \{1, A_2, \ldots, A_m\} \subset \mathbb{R}^{\wedge}_{\mathcal{F}}$ be an SLI set, and denote $\text{core}(A_i) = a_i$, for each $i = 1, \ldots, m$. In addition, let $N : [0, T] \to \mathcal{S}(\mathcal{A})$ the population of predators in a prey-predator dynamic. Thus, we are assuming there exists a real-vector valued function $n : [0, T] \to \mathbb{R}^m$ given by

$$n(t) = (n_1(t), \ldots, n_m(t)) \in \mathbb{R}^m, \quad t \in [0, T]$$

such that

$$N(t) = (\psi \circ n)(t) = n_1(t) + n_2(t)A_2 + \ldots + n_m(t)A_m \qquad \forall t \in [0, T],$$

where $\psi : \mathbb{R}^m \to \mathcal{S}(\mathcal{A})$ is given by (2.7).

Since the prey density of the ecosystem only assumes positive values in classical population dynamics, we consider that the coordinates of $N(t)$ in the vector space $(\mathcal{S}(\mathcal{A}), +_\psi, \cdot_\psi)$ are positive in all instants $t \in [0, T]$, i.e., $n_i(t) > 0$, $i = 1, \ldots, m$. A similar argument can justify the non-negativity of the coordinates corresponding to the fuzzy rate predation, given by (3.12), and the fuzzy half saturation constant, given by (3.17).

3.1. Fuzzy Holling-type I functional response

Let $C, A \in \mathcal{S}(\mathcal{A})$ be fuzzy quantities given, or estimated according to some data set. The fuzzy Holling-type I functional response, given as in (1.1) for crisp populations, is described by the function $f : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$ given by

$$f(N) = C \odot_\psi N, \tag{3.11}$$

where the *consumption rate of predator to prey* is given by the $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy number

$$C = c_1 + c_2 A_2 + \ldots + c_m A_m \in \mathcal{S}(\mathcal{A}) \tag{3.12}$$

for some positive constants $c_1, \ldots, c_m > 0$ given. Let us denote $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$. From (3.12), we have that $C = \psi \circ c = \psi(c_1, \ldots, c_m)$. In addition, from equation (2.6), we can write

$$[N(t)]_1 = n_1(t)[A_1]_1 + n_2(t)[A_2]_1 + \ldots + n_m(t)[A_m]_1 = \sum_{i=1}^{m} n_i(t) a_i,$$

that is,

$$[N(t)]_1 = \langle n(t), \overline{a} \rangle, \tag{3.13}$$

where $\overline{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ is the vector whose entries are given by $a_i = \text{core}(A_i)$, $i = 1, \ldots, m$. A similar reasoning leads us to

$$[C]_1 = \sum_{i=1}^{m} c_i a_i = \langle c, \overline{a} \rangle. \tag{3.14}$$

Since $A_1 = 1 \in \mathbb{R}$, from (3.13) and (3.14), we can write (3.11) as

$$\begin{aligned} f(N) &= [N(t)]_1 C +_\psi [C]_1 N -_\psi [N]_1 [C]_1 \\ &= \langle n(t), \overline{a} \rangle C +_\psi \langle c, \overline{a} \rangle N(t) -_\psi \langle n(t), \overline{a} \rangle \langle c, \overline{a} \rangle \\ &= \langle n(t), \overline{a} \rangle \psi(c_1, \ldots, c_m) +_\psi \langle c, \overline{a} \rangle \psi(n_1(t), \ldots, n_m(t)) -_\psi \langle n(t), \overline{a} \rangle \langle c, \overline{a} \rangle \end{aligned}$$

Therefore, (3.11) is written, in coordinates in $\mathcal{S}(\mathcal{A})$, as

$$\begin{aligned} f(N) = &\left(\langle n(t), \overline{a} \rangle c_1 + \langle c, \overline{a} \rangle n_1(t) - \langle n(t), \overline{a} \rangle \langle c, \overline{a} \rangle \right) \\ &+ \left(\langle n(t), \overline{a} \rangle c_2 + \langle c, \overline{a} \rangle n_2(t) \right) A_2 \\ &+ \ldots \\ &+ \left(\langle n(t), \overline{a} \rangle c_m + \langle c, \overline{a} \rangle n_m(t) \right) A_m \end{aligned} . \tag{3.15}$$

Equation (3.15) reveals the Holling-type I functional response as a function in $\mathcal{S}(\mathcal{A})$ can be decomposed into a Minkowski sum of a crisp contribution with a fuzzy contribution. In fact, (3.15) can be written as

$$f(N) = \underbrace{(b_1(t) - \langle n(t), \overline{a}\rangle\langle c, \overline{a}\rangle)}_{\in\mathbb{R}} + \underbrace{\sum_{i=2}^{m} b_i(t) A_i}_{\in\mathbb{R}_{\mathcal{F}}\setminus\mathbb{R}},$$

where $b_i(t) = (\langle n(t), \overline{a}\rangle c_i + \langle c, \overline{a}\rangle n_i(t))$, for each $i = 1, \ldots, m$. It follows that, for each $t \in [0, T]$, we have

$$\operatorname{diam}(f(N)) = \sum_{i=2}^{m} |b_i(t)| \operatorname{diam}(A_i),$$

that is, the real-valued function $b_i : [0, T] \to \mathbb{R}$ are direct related to the diameter of the holling-type I response. Since each b_i is a function of c and $\overline{a}$, we conclude that the choice of the coordinates of $C \in \mathcal{S}(\mathcal{A})$ and the SLI set $\mathcal{A}$ corresponding to the core-vector $\overline{a} = (a_1, \ldots, a_m)$ determines intrinsically the uncertainty represented by the Holling term.

3.2. Fuzzy Holling-type II functional response

Similarly to the previous case, the fuzzy Holling-type II functional response, given as in (1.2) for crisp populations, is described by the function $f : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$ given by

$$f(N) = C \div_{\psi} (H +_{\psi} N), \tag{3.16}$$

where $C \in \mathcal{S}(\mathcal{A})$ is given as in (3.12) and the *half saturation constant* is given by the $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy number

$$H = h_1 + h_2 A_2 + \ldots + h_m A_m \in \mathcal{S}(\mathcal{A}), \tag{3.17}$$

for some positive constants $h_1, \ldots h_m > 0$ given. Let us denote $h = (h_1, \ldots, h_m) \in \mathbb{R}^m$, so that, from (3.17), we have $H = \psi \circ h = \psi(h_1, \ldots, h_m)$. From (2.10), we have that (3.16) is equivalent to

$$\begin{aligned} f(N) &= C \odot_{\psi} (H +_{\psi} N)_{\psi}^{-1} \\ &= \left[(H +_{\psi} N(t))_{\psi}^{-1}\right]_1 C +_{\psi} [C]_1 (H +_{\psi} C)_{\psi}^{-1} - \left[(H +_{\psi} N(t))_{\psi}^{-1}\right]_1 [C]_1 . \end{aligned} \tag{3.18}$$

Using the fact that

$$\left[(H +_\psi N(t))_\psi^{-1}\right]_1 = \frac{1}{[H +_\psi N(t)]_1} = \frac{1}{[H]_1 + [N(t)]_1},$$

a reasoning similar to that used in section 3.1 leads us to

$$[H]_1 = \langle h, \overline{a} \rangle,$$

and hence, from the linearity of the inner product $\langle \cdot, \cdot \rangle$, we have

$$\left[(H +_\psi N(t))_\psi^{-1}\right]_1 = \frac{1}{\langle h, \overline{a} \rangle + \langle n(t), \overline{a} \rangle} = \frac{1}{\langle h + n(t), \overline{a} \rangle}.$$

Equation (2.10) assures that

$$(H +_\psi N(t))_\psi^{-1} = \psi\left(\frac{2}{\langle h + n(t), \overline{a} \rangle} - \frac{h_1 + c_1}{\langle h + n(t), \overline{a} \rangle^2}, \ \ldots, \ -\frac{h_m + c_m}{\langle h + n(t), \overline{a} \rangle^2}\right).$$

Since $A_1 = 1 \in \mathbb{R}$ and $A_i \in \mathbb{R} \backslash \mathbb{R}$ for $i = 2, \ldots, m$, equation (3.18) can be written, in coordinates in $\mathcal{S}(\mathcal{A})$, as

$$\begin{aligned} f(N) = & \frac{1}{\langle h + n(t), \overline{a} \rangle} \psi(c_1, \ldots, c_m) \\ & +_\psi \langle c, \overline{a} \rangle \psi\left(\frac{2}{\langle h + n(t), \overline{a} \rangle} - \frac{h_1 + c_1}{\langle h + n(t), \overline{a} \rangle^2}, \ \ldots, \ -\frac{h_m + c_m}{\langle h + n(t), \overline{a} \rangle^2}\right) \\ & -_\psi \psi\left(\frac{\langle c, \overline{a} \rangle}{\langle h + n(t), \overline{a} \rangle}, 0, \ldots, 0\right) \end{aligned}$$

or, equivalently,

$$\begin{aligned} f(N) = & \left(\frac{c_1 + \langle c, \overline{a} \rangle}{\langle h + n(t), \overline{a} \rangle} - \frac{h_1 + c_1}{\langle h + n(t), \overline{a} \rangle^2}\right) \\ & - \frac{h_2 + c_2}{\langle h + n(t), \overline{a} \rangle^2} A_2 - \ldots - \frac{h_m + c_m}{\langle h + n(t), \overline{a} \rangle^2} A_m \end{aligned}. \tag{3.19}$$

In resemblance to fuzzy Holling-type I functional response, equation (3.19) reveals the Holling-type II functional response as a function in $\mathcal{S}(\mathcal{A})$ can be decomposed in a Minkowski sum of a crisp contribution with a fuzzy contribution. In fact, (3.19) can be written as

$$f(N) = \underbrace{\left(\frac{c_1 + \langle c, \overline{a} \rangle}{\langle h + n(t), \overline{a} \rangle} - w_1(t)\right)}_{\in \mathbb{R}} - \underbrace{\sum_{i=2}^{m} w_i(t) A_i}_{\in \mathbb{R}_{\mathcal{F}} \backslash \mathbb{R}},$$

where $w_i(t) = \frac{h_i + c_i}{\langle h+n(t),\overline{a}\rangle^2}$ for each $i = 1,\ldots,m$. It follows that, for each $t \in [0,T]$, we have

$$\operatorname{diam}(f(N)) = \sum_{i=2}^{m} |w_i(t)| \operatorname{diam}(A_i), \tag{3.20}$$

that is, the real-valued function $w_i : [0,T] \to \mathbb{R}$ are direct related to the diameter of the holling-type I response. Since each w_i is a function of c, h, and $\overline{a}$, we conclude that the choice of the coordinates of C and $H \in \mathcal{S}(\mathcal{A})$, as well as the SLI set $\mathcal{A}$ corresponding to the core-vector $\overline{a} = (a_1,\ldots,a_m)$ determines intrinsically the uncertainty represented by the Holling term.

4. Analysis on the choice of SLI sets

SLI sets of fuzzy numbers can be built using the power hedges of a non-symmetric trapezoidal fuzzy number via fuzzy modifiers or Zadeh's Extension Principle. In fact, for a given fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$ trapezoidal and non-symmetric, the sets given by

$$\left\{A^i\right\}_{i=0,1,\ldots,m} \quad \text{and} \quad \left\{\hat{f}_i(A)\right\}_{i=0,1,\ldots,m} \tag{4.21}$$

are SLI for all $m \in \mathbb{N}$ (Esmi et al., 2021).

We can say that (4.21) shows us that vector spaces generated by linear combinations of SLI sets can be seen as a structure completely determined by a single fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$, when it is choosen to be non-symmetric. In this case, the following definition holds:

Definition 4.1 *(Adapted from Laiate et al. (2023)) Let $\mathcal{A} \subset \mathbb{R}_{\mathcal{F}}$ be an SLI set given as in equation* (4.21)*, for some $A \in \mathbb{R}_{\mathcal{F}}$, and consider the first-order system of FIVPs*

$$\begin{cases} X_i'(t) = f_i(t, X_1,\ldots,X_p) \\ X_i(t_0) = X_0 \in \mathcal{S}(\mathcal{A}) \end{cases}, \tag{4.22}$$

where $X_i : [a,b] \to \mathcal{S}(\mathcal{A})$ for $i = 1,\ldots,p$. Then the fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$ is called a fuzzy basal number of (4.22)*.*

The Holling-type functional responses, presented as functions in $\mathcal{S}(\mathcal{A})$ in (3.11) and (3.16), aim to be a tool to be used in fuzzy differential equations for $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy functions. Thus, assuming the SLI sets are generated for some fuzzy basal number $A \in \mathbb{R}_{\mathcal{F}}$, we observe from (3.1) and (3.20) that the choice of A determines the fuzziness of Holling-term of both types. Moreover, the fuzzy basal number determinates the core-vector $\overline{a} \in \mathbb{R}^m$ which, intuitively, shall be considered in first place from possible data set given. Note that the curve corresponding to the core of the fuzzy curve of a FDE under the $\psi-$arithmetic operations coincides with the classical solution to the corresponding differential equation. It is noteworthy that, in this case, the fuzzy basal number represents the source of uncertainty of an ecological phenomena in which the Holling-type functional response acts.

5. Final considerations

This manuscript provided a brief study on Holling-type functional responses as functions of the form $f : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$, where

$$\mathcal{A} = \{A_1, \ldots, A_m\}$$

is a given finite SLI set. The expressions of the functional responses were deduced using the $\psi-$arithetic operations, well defined in the space of $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy numbers.

Both Type I and Type II Holling terms were expressed in an expression decomposable into a Minkowski sum of a crisp term with a fuzzy term. Analytical relations between these expressions and the coordinates of the parameters involved were made. Here, the parameters and the population were considered as fuzzy quantities. In addition, an introductory analysis on the underlying SLI set was made from the point of view of the *fuzzy basal number* (Laiate et al., 2023). In this case, the fuzzy basal number appeared to be the source of uncertainty of the ecological phenomenon known as Holling-functional responses, widely studied in the literature in the last decades.

This work brings some theoretical novelties from the point of view of fuzzy arithmetic. The $\psi-$arithmetic operations maintain the relation called $\mathcal{S}(\mathcal{A})$-linearly correlation, well-defined between some families of fuzzy numbers. Thus, fuzzy differential equations associated to these families of functions can model ecological phenomena using the content of this paper, either from the point of view of Fréchet derivative (Esmi et al., 2022; Laiate, 2023), Hilger derivative in time scales (Shahidi et al., 2023), fuzzy fractional derivative (Son et al., 2021), or from control theory (Son et al., 2023).

Lastly, from the point of view of ecological modeling, a contribution is provided since some uncertainties and imprecisions associated with natural phenomena are partially described by parameters and/or populations given by fuzzy quantities. Future papers should include different Holling-types functional responses with numerical simulations to solutions of fuzzy differential equations associated to $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy functions.

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