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STABILITY AND DECAY PROPERTIES OF SOLITARY-WAVE SOLUTIONS TO THE GENERALIZED BO–ZK EQUATION

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Abstract. Studied here is the generalized Benjamin-Ono–Zakharov-Kuznetsov equation

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} + \varepsilon u_{xyy} = 0, \quad (x, y) \in \mathbb{R}^2, \;\; t \in \mathbb{R}^+,$$

in two space dimensions. Here, $\mathscr{H}$ is the Hilbert transform and subscripts denote partial differentiation. We classify when this equation possesses solitary-wave solutions in terms of the signs of the constants α and ε appearing in the dispersive terms and the strength of the nonlinearity. Regularity and decay properties of these solitary wave are determined and their stability is studied.

1. Introduction

This paper is concerned with existence and non-existence, stability and some decay properties of solitary-wave solutions of the two-dimensional, generalized Benjamin-Ono–Zakharov-Kuznetsov equation (BO–ZK equation henceforth),

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} + \varepsilon u_{xyy} = 0, \quad (x, y) \in \mathbb{R}^2, \;\; t \in \mathbb{R}^+. \tag{1.1}$$

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Here, $p > 0, \alpha$ and ε are non-zero real constants with ε normalized to ± 1 by appropriately rescaling the y-variable while $\mathscr{H}$ is the Hilbert transform

$$\mathscr{H}u(x,y,t) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{u(z,y,t)}{x-z}\,dz,$$

in the x-variable; p.v. denotes the Cauchy principal value.

When $p = 1$, this equation arises as a model for electromigration in thin nanoconductors on a dielectric substrate (see [32]) and in modeling propagation of internal waves in the presence of weak lateral dispersion of the Zakharov-Kuznetsov variety (*e.g.* [38]).

Equation (1.1) may be viewed as one of the natural, two-dimensional generalizations of the one-dimensional generalized Benjamin-Ono equation

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} = 0, \qquad x \in \mathbb{R}, \;\; t \in \mathbb{R}^+. \tag{1.2}$$

When $p = 1$, this is the Benjamin-Ono equation which arose in Benjamin's study [5] of internal waves propagating along the interface between two fluid layers of different densities. The BO–ZK equation can also be considered as a non-local version of the generalized Zakharov-Kuznetsov equation

$$u_t + u^p u_x + \alpha u_{xxx} + \varepsilon u_{xyy} = 0, \quad (x,y) \in \mathbb{R}^2, \;\; t \in \mathbb{R}^+. \tag{1.3}$$

Again, when $p = 1$, this model was introduced by Zakharov and Kuznetsov in [52] as a higher-dimensional extension of the Korteweg-de Vries model of surface wave propagation. It was originally derived as a model for ion acoustic waves in a plasma under the influence of a planar external magnetic field. In the form (1.3), it comprises a two-dimensional version of the generalized Korteweg-de Vries equation

$$u_t + u^p u_x + u_{xxx} = 0. \tag{1.4}$$

Well-posedness issues for the pure initial-value problem for (1.2) began with [9] and [1] and have attracted a lot of interest recently (see, *e.g.* [18, 35, 36, 47, 48, 51]). Well-posedness for (1.3) was studied in [28, 29, 30, 41, 46, 50].

Both (1.2) and (1.4) have traveling-wave solutions called solitary waves. These special solutions appear to play a distinguished role in the long-term asymptotics of finite energy initial disturbances. Questions about the stability or instability of solitary waves goes back to the pioneering work of Benjamin [6] (and see also [10]) in the context of the Korteweg-de Vries equation itself. The existence and stability of solitary waves for (1.2) has been investigated in many subsequent works, *e.g.* [2, 3, 4, 9, 33]. Numerical simulations indicating instability and singularity formation when $p > 2$ appear in [11]. As far as we know, the only results about existence and stability of solitary-wave solutions of (1.3) were provided in [14].

It is worth note that the so-called Benjamin equation (see [7, 8])

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} + \beta u_{xxx} = 0, \quad x \in \mathbb{R}, \ \ t \in \mathbb{R}^+, \tag{1.5}$$

for propagation of internal waves, which is valid in the same deep-water limits as is the Benjamin-Ono equation, but which takes into account surface tension effects between the two layers of fluid, could be generalized in a similar way. In this case, a model of the form

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} + \beta u_{xxx} + \varepsilon u_{xyy} = 0, \quad (x, y) \in \mathbb{R}^2, \ \ t \in \mathbb{R}^+, \tag{1.6}$$

would emerge. It could be derived as a model for internal waves under the same conditions that would arise in obtaining (1.1) in this context. Existence theory for solitary-wave solutions of (1.5) appears in [21]. A numerical study of the spatial structure and stability of these waves can be found in [33].

Since solitary-wave solutions of equations like (1.2) and (1.4) turn out to be important ingredients of general solutions, it seems not unlikely the same is true for the more complex, two-spatial dimensional wave equations mentioned here. Hence, an extended investigation of their solitary waves will most probably provide information about solutions of the equations emanating from general initial data.

The solitary-wave solutions of interest here have the form $u(x, y, t) = \varphi(x - ct, y)$, where $c \neq 0$ is the speed of propagation and u belongs to a natural function space denoted $\mathscr{Z}$ and introduced presently. Substituting this form into (1.1), integrating once with respect to the variable $z = x - ct$ and assuming $\varphi(z, y)$ decays suitably for large values of $|z|$, it transpires that φ must satisfy

$$-c\varphi + \tfrac{1}{p+1}\varphi^{p+1} + \alpha \mathscr{H} \varphi_x + \varepsilon \varphi_{yy} = 0, \tag{1.7}$$

where we have replaced the variable z by x.

Remark 1.1. When it is convenient, it may be assumed that (1.7) has the normalized form

$$-\varphi + \tfrac{1}{p+1}\varphi^{p+1} + \mathscr{H} \varphi_x \pm \varphi_{yy} = 0, \tag{1.8}$$

by scaling the independent and dependent variables, *viz.*

$$u(x, y, t) = av(bx, dy, et),$$

where $a^p = c$, $e = b = c/\alpha$ and $d = \varepsilon/c^2$. If instead, we insist that $d > 0$, so $\varepsilon = +1$, then equation (1.7) may be taken in the form

$$-\varphi + \tfrac{1}{p+1}\varphi^{p+1} \pm \mathscr{H} \varphi_x + \varphi_{yy} = 0. \tag{1.9}$$

Of course, throughout, it will be presumed that the power p appearing in the nonlinearity is rational and has the form k/m where k and m are relatively prime and m is odd. This restriction allows us to define a branch of the mapping $w \mapsto w^{\frac{1}{m}}$ that is real on the real axis.

Attention is now turned to the structure of the paper. The theory begins by examining when solitary-wave solutions of (1.1) exist. As pointed out in [38], no exact formulas are known for solitary-wave solutions to (1.1), so an existence theory logically precedes questions of stability. Pohozaev-type identities are used to show that solitary-wave solutions do not exist for certain values of p and signs of ε and α. In some of the cases where such solutions are not prohibited by elementary inequalities, a suitable minimization problem can be solved using P.-L. Lions' concentration-compactness principle [42, 43] (see Theorem 2.1). For example, our results imply there are solitary-wave solutions when $c > 0$, $\alpha < 0$, $\varepsilon > 0$ and $0 < p < 4$. Moreover, these solutions are shown to be ground states.

With solitary waves in hand, their orbital stability is at issue. The variational approach of Cazenave and Lions [20] comes to the fore in Section 3 in establishing stability for the case $\alpha\varepsilon < 0$, $c\alpha < 0$, and $0 < p < 4/3$. Complementary instability results appear in [25] for the same conditions on c, α and ε, but with $4/3 < p < 4$.

The regularity and spatial asymptotics of the solitary-wave solutions shown to exist in Section 2 are developed in Sections 4 and 5. Solitary-wave solutions are shown to be positive and real analytic. (Of course, in the case where $p = k/m$, k, m relatively prime with m odd and k even, if ϕ is a solution of (1.7), then so is $-\phi$.) They are symmetric about their peak with respect to both the direction of propagation and the transverse direction and decay to zero algebraically in the direction of propagation and exponentially in the transverse direction. Some of the results in Section 4 inform the analysis of instability in [25].

In the theory developed here, the issue of well-posedness is not addressed. The presumption throughout is that suitable well-posedness obtains for these models. Detailed analysis of the initial-value problem that is more than sufficient for our theory here has recently appeared in [23]. Complementary ill-posedness results are available in [27].

Remark 1.2. The scale-invariant Sobolev spaces for the BO–ZK equation (1.1) are $\dot{H}^{s_1,s_2}(\mathbb{R}^2)$, where $2s_1 + s_2 = \frac{3}{2} - \frac{2}{p}$ (see the definitions below). Hence, a reasonable framework for studying local well-posedness of the BO–ZK equation (1.1) is the family of spaces $H^{s_1,s_2}(\mathbb{R}^2)$ with $2s_1 + s_2 \geq \frac{3}{2} - \frac{2}{p}$.

Remark 1.3. An n-dimensional version of (1.1) is

$$u_t + u^p u_{x_1} + \alpha \mathscr{H} u_{x_1 x_1} + \sum_{i=2}^{n} \varepsilon_i u_{x_1 x_i x_i} = 0, \tag{1.10}$$

where $t \in \mathbb{R}^+$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha, \varepsilon_i \in \mathbb{R}$, $i = 2, \ldots, n$. The theory developed here has natural analogs for (1.10) which will be developed elsewhere.

Notation and Preliminaries. As already mentioned, the exponent p in (1.1) is taken to be a rational number of the form $p = k/m$, where m and k are relatively prime and m is odd. This allows the nonlinearity to be given a definition that is real-valued. The notation $\widehat{f} = \widehat{f}(\xi, \eta)$ connotes the Fourier transform,

$$\widehat{f}(\xi, \eta) = \int_{\mathbb{R}^2} e^{-\mathrm{i}(x\xi + y\eta)} f(x, y)\, dxdy,$$

of $f = f(x, y)$. For any $s \in \mathbb{R}$, the space $H^s := H^s(\mathbb{R}^2)$ denotes the usual, isotropic, $L^2(\mathbb{R}^2)$-based, Sobolev space. For $s_1, s_2 \in \mathbb{R}$, the anisotropic Sobolev space $H^{s_1,s_2} := H^{s_1,s_2}(\mathbb{R}^2)$ is the set of all distributions f such that

$$\|f\|^2_{H^{s_1,s_2}} = \int_{\mathbb{R}^2} \left(1 + \xi^2\right)^{s_1} \left(1 + \eta^2\right)^{s_2} |\widehat{f}(\xi, \eta)|^2 \, d\xi d\eta < \infty.$$

The fractional Sobolev-Liouville spaces $H_p^{(s_1,s_2)} := H_p^{(s_1,s_2)}(\mathbb{R}^2)$, $1 \le p < \infty$, are the set of all functions $f \in L^p(\mathbb{R}^2)$ such that

$$\|f\|_{H_p^{(s_1,s_2)}} = \|f\|_{L^p(\mathbb{R}^2)} + \sum_{i=1}^{2} \left\|D_{x_i}^{s_i} f\right\|_{L^p(\mathbb{R}^2)} < \infty,$$

where $D_{x_i}^{s_i} f$ denotes the Bessel derivative of order s_i with respect to x_i (see *e.g.* [37], [44]). For short, $H_p^{(s)}(\mathbb{R}^2)$ denotes the space $H_p^{(s,s)}(\mathbb{R}^2)$.

The particular space $\mathscr{Z} := H^{\frac{1}{2},0}\left(\mathbb{R}^2\right) \cap H^{0,1}\left(\mathbb{R}^2\right) = H^{\left(\frac{1}{2},1\right)}\left(\mathbb{R}^2\right)$ arises naturally in the analysis to follow. It can be characterized alternatively as the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|\varphi\|^2_{\mathscr{Z}} = \|\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\varphi_y\|^2_{L^2(\mathbb{R}^2)} + \left\|D_x^{1/2}\varphi\right\|^2_{L^2(\mathbb{R}^2)}, \tag{1.11}$$

where $D_x^{1/2}\varphi$ denotes the fractional derivative of order $1/2$ with respect to x. The operator $D_x^{1/2}$ is a Fourier multiplier operator defined via its Fourier transform by

$$\widehat{D_x^{1/2}\varphi}(\xi, \eta) = |\xi|^{1/2} \widehat{\varphi}(\xi, \eta),$$

just as the Hilbert transform can be defined by

$$\widehat{\mathscr{H} f}(\xi,\eta) = -\mathrm{i}\,\mathrm{sgn}(\xi)\hat{f}(\xi,\eta).$$

Remark 1.4. By combining fractional Gagliardo-Nirenberg and Hölder's inequalities, one can deduce the existence of a positive constant C such that

$$\|u\|_{L^{p+2}}^{p+2} \leq C\|u\|_{L^2}^{(4-p)/2}\|D_x^{1/2}u\|_{L^2}^{p}\|u_y\|_{L^2}^{p/2}, \quad 0 \leq p < 4. \tag{1.12}$$

This in turn implies the continuous embedding

$$\mathscr{Z} \hookrightarrow L^{p+2}\left(\mathbb{R}^2\right), \quad 0 \leq p < 4. \tag{1.13}$$

2. Solitary waves

This section is devoted to establishing existence and non-existence results for solitary-wave solutions of the BO-ZK equations. We begin with a theorem about non-existence.

Theorem 2.1. *Equation* (1.7) *cannot have a smooth non-trivial solitary-wave solution unless either*

(i) $\varepsilon = 1$, $c > 0$, $\alpha < 0$, $p < 4$,
(ii) $\varepsilon = -1$, $c < 0$, $\alpha > 0$, $p < 4$,
(iii) $\varepsilon = 1$, $c < 0$, $\alpha < 0$, $p > 4$, *or*
(iv) $\varepsilon = -1$, $c > 0$, $\alpha > 0$, $p > 4$.

Remark 2.2. By 'smooth', we mean that the functions have sufficient regularity that all the integrals displayed below exist.

It is worth note that only the case (i) with $p = 1$ has known physical relevance. Higher values of the homogeneity of the nonlinearity without corresponding lower-order terms seem not to arise in practice. And the model itself is derived only for unidirectional waves (see Remark 2.6), so $c > 0$ is mandated.

Cases (ii) and (iv) are the same as (i) and (iii), respectively except that the sign of the nonlinearity is reversed. The sign of the nonlinearity plays no role in the analysis to follow.

Proof. These conclusions follow from some Pohozaev-type identities. If (1.7) is multiplied by φ, $x\varphi_x$ and $y\varphi_y$ and the results integrated over $\mathbb{R}^2$, then the identities

$$\int_{\mathbb{R}^2}\Big(-c\varphi^2 + \alpha\varphi\mathscr{H}\varphi_x - \varepsilon\varphi_y^2 + \frac{1}{p+1}\varphi^{p+2}\Big)dxdy = 0, \tag{2.1}$$

$$\int_{\mathbb{R}^2}\Big(c\varphi^2 + \varepsilon\varphi_y^2 - \frac{2}{(p+1)(p+2)}\varphi^{p+2}\Big)dxdy = 0, \tag{2.2}$$

$$\int_{\mathbb{R}^2} \Big(c\varphi^2 - \alpha\varphi\mathscr{H}\varphi_x - \varepsilon\varphi_y^2 - \frac{2}{(p+1)(p+2)}\varphi^{p+2} \Big) dxdy = 0, \tag{2.3}$$

emerge. These formulas follow from the elementary properties of the Hilbert transform together with suitably chosen formal integrations by parts. They can be justified for functions of the minimal regularity required for them to make sense by first establishing them for smooth solutions and then using a standard truncation argument as in [15].

Summing (2.1) and (2.2) leads to

$$\int_{\mathbb{R}^2} \Big(\alpha\varphi\mathscr{H}\varphi_x + \frac{p}{(p+1)(p+2)}\varphi^{p+2} \Big) dxdy = 0, \tag{2.4}$$

whilst adding (2.2) and (2.3) yields

$$\int_{\mathbb{R}^2} \Big(c\varphi^2 - \frac{\alpha}{2}\varphi\mathscr{H}\varphi_x - \frac{2}{(p+1)(p+2)}\varphi^{p+2} \Big) dxdy = 0. \tag{2.5}$$

If the integral of φ^{p+2} is eliminated between (2.4) and (2.5), there appears

$$\int_{\mathbb{R}^2} \Big(2pc\varphi^2 + \alpha(4-p)\varphi\mathscr{H}\varphi_x \Big) dxdy = 0. \tag{2.6}$$

On the other hand, adding (2.1) and (2.3) gives

$$\int_{\mathbb{R}^2} \Big(2\varepsilon\varphi_y^2 - \frac{p}{(p+1)(p+2)}\varphi^{p+2} \Big) dxdy = 0. \tag{2.7}$$

Finally, substituting (2.2) into (2.7), there obtains

$$\int_{\mathbb{R}^2} \Big(pc\varphi^2 + \varepsilon(p-4)\varphi_y^2 \Big) dxdy = 0. \tag{2.8}$$

The advertised results follow immediately from (2.6) and (2.8). □

For cases (i) and (ii) of Theorem 2.1, the existence of solitary-wave solutions of (1.1) is established in the next result.

Theorem 2.3. *Let $\alpha\varepsilon, c\alpha < 0$ and $p = \frac{k}{m} < 4$, where $m \in \mathbb{N}$ is odd and m and k are relatively prime. Then equation* (1.7) *admits a non-trivial solution $\varphi \in \mathscr{Z}$.*

Proof. The proof is based on the concentration-compactness principle [42, 43]. Suppose that $\alpha < 0$ (the proof for $\alpha > 0$ is similar). Without loss of generality, assume that $\alpha = -1$ and $c = 1$ so that $\varepsilon = +1$ (see Remark 1.1) and the equation for a solitary wave has the form

$$-\varphi + \tfrac{1}{p+1}\varphi^{p+1} - \mathscr{H}\varphi_x + \varphi_{yy} = 0. \tag{2.9}$$

Consider the minimization problem

$$I_\lambda = \inf\Big\{I(\varphi)\ ;\ \varphi \in \mathscr{Z}\ ,\ J(\varphi) = \int_{\mathbb{R}^2} \varphi^{p+2} dxdy = \lambda\Big\}, \tag{2.10}$$

where $\lambda \neq 0$ and

$$I(\varphi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\varphi^2 + \varphi \mathscr{H}\varphi_x + \varphi_y^2\right) dxdy = \frac{1}{2}\|\varphi\|_{\mathscr{Z}}^2. \tag{2.11}$$

Clearly, $I_\lambda < \infty$ if there are elements $\varphi \in \mathscr{Z}$ such that $\int_{\mathbb{R}^2} \varphi^{p+2}\, dxdy = \lambda$.[1] The embedding (1.13) allows us to adduce a positive constant C such that

$$0 < |\lambda| = \Big|\int_{\mathbb{R}^2} \varphi^{p+2}\, dxdy\Big| \leq C\|\varphi\|_{\mathscr{Z}}^{p+2} = CI(\varphi)^{\frac{p+2}{2}},$$

from which one concludes that $I_\lambda \geq (\frac{|\lambda|}{C})^{\frac{2}{p+2}} > 0$.

For suitable λ let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for I_λ. For $n = 1, 2, \cdots$ and $r > 0$, define the concentration function $Q_n(r)$ associated to φ_n by

$$Q_n(r) = \sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{B_r(\widetilde{x},\widetilde{y})} \rho_n\, dxdy,$$

where

$$\rho_n = |\varphi_n|^2 + \left|D_x^{1/2}\varphi_n\right|^2 + |\partial_y\varphi_n|^2,$$

and $B_r(x, y)$ denotes the ball of radius $r > 0$ centered at $(x, y) \in \mathbb{R}^2$. If evanescence of the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ occurs, which is to say, for any $r > 0$,

$$\lim_{n\to+\infty} \sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{B_r(\widetilde{x},\widetilde{y})} \rho_n\, dxdy = 0,$$

then the embedding (1.13) implies that $\lim_{n\to\infty} \|\varphi_n\|_{L^{p+2}} = 0$, which contradicts the constraint imposed for the minimization problem. Thus, according to the concentration-compactness theorem, either dichotomy or compactness must occur for the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$.

The splitting lemma proved next enables us to rule out the possibility of dichotomy occurring in the present context. Suppose that

$$\gamma \in (0, I_\lambda), \tag{2.12}$$

[1] Depending on p, this might require that $\lambda > 0$. Of course, I_λ is a number, but we will sometimes use it to refer to the minimization problem. For example, the phrase "$\{\phi_n\}$ is a minimizing sequence for the problem I_λ" means that $J(\phi_n) = \lambda$ for all n and $I(\phi_n) \to I_\lambda$ as $n \to \infty$.

where it is assumed that

$$\gamma = \lim_{r\to+\infty} \lim_{n\to+\infty} \sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{B_r(\widetilde{x},\widetilde{y})} \rho_n \, dxdy.$$

Lemma 2.4. *For every $\epsilon > 0$, there is an $n_0 \in \mathbb{N}$ and sequences $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ in $\mathscr{Z}$ satisfying*

$$\mathrm{dist}\big(\mathrm{supp}(g_n), \mathrm{supp}(h_n)\big) \to +\infty, \quad \text{as} \quad n \to +\infty, \tag{2.13}$$

and, for $n \geq n_0$,

$$|I(\varphi_n) - I(g_n) - I(h_n)| \leq C\epsilon, \tag{2.14}$$

$$|I(g_n) - \gamma| \leq C\epsilon, \quad |I(h_n) - I_\lambda + \gamma| \leq C\epsilon, \tag{2.15}$$

$$|J(\varphi_n) - J(g_n) - J(h_n)| \leq C\epsilon. \tag{2.16}$$

The constants $C > 0$ appearing above are independent of ϵ and $n \geq n_0$.

The following commutator estimate is helpful in proving Lemma 2.4.

Lemma 2.5 ([19, 22])**.** *Let $g \in C^\infty(\mathbb{R})$ with $g' \in L^\infty(\mathbb{R})$. Then $[\mathscr{H}, g]\partial_x$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ with*

$$\|[\mathscr{H}, g]\partial_x f\|_{L^2(\mathbb{R})} \leq C\|g'\|_{L^\infty(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}.$$

Proof of Lemma 2.4. Because of (2.12), for a given $\epsilon > 0$, there exist $r_0 > 0$, $r_n > 0$ with $r_n \to +\infty$, as $n \to \infty$, $n_0 \in \mathbb{N}$ and $\{(\widetilde{x}_n, \widetilde{y}_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ such that

$$\gamma \geq \int_{B_{r_0}(\widetilde{x}_n,\widetilde{y}_n)} \rho_n \, dxdy > \gamma - \epsilon \quad \text{and} \quad Q_n(2r_n) < \gamma + \epsilon,$$

for $n \geq n_0$. It follows that

$$\int_{r_0 \leq |(x,y)-(\widetilde{x}_n,\widetilde{y}_n)| \leq 2r_n} \rho_n \, dxdy \leq 2\epsilon.$$

Let ϕ, ψ lie in $C^\infty(\mathbb{R}^2)$ and suppose

- supp $\phi \subset B_2(0,0)$, $\phi \equiv 1$ on $B_1(0,0)$ and $0 \leq \phi \leq 1$,
- supp $\psi \subset \mathbb{R}^2 \setminus B_1(0,0)$, $\psi \equiv 1$ on $\mathbb{R}^2 \setminus B_2(0,0)$ and $0 \leq \psi \leq 1$.

Define the sequences $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ by

$$g_n(x,y) = \phi_n(x,y)\varphi_n(x,y) \quad \text{and} \quad h_n(x,y) = \psi_n(x,y)\varphi_n(x,y),$$

where

$$\phi_n(x,y) = \phi\Big(\frac{(x,y) - (\widetilde{x}_n,\widetilde{y}_n)}{r_0}\Big) \quad \text{and} \quad \psi_n(x,y) = \psi\Big(\frac{(x,y) - (\widetilde{x}_n,\widetilde{y}_n)}{r_n}\Big).$$

Clearly, the functions g_n and h_n lie in $\mathscr{Z}$ and (2.16) holds. It follows that for all $n \geq n_0$,

$$\begin{aligned}
2I(g_n) &= \int_{\mathbb{R}^2} \phi_n^2 \left[\varphi_n^2 + \varphi_n \partial_x \mathscr{H} \varphi_n + (\partial_y \varphi_n)^2\right] dxdy \\
&\quad + 2\int_{\mathbb{R}^2} \phi_n \varphi_n (\partial_y \phi_n)(\partial_y \varphi_n) dxdy \\
&\quad + \int_{\mathbb{R}^2} \left[(\partial_y \phi_n)^2 \varphi_n^2 + \varphi_n \phi_n \mathscr{H}(\varphi_n \partial_x \phi_n)\right] dxdy \\
&\quad + \int_{\mathbb{R}^2} \varphi_n \phi_n [\mathscr{H}, \phi_n] \partial_x \varphi_n dxdy \\
2I(h_n) &= \int_{\mathbb{R}^2} \psi_n^2 \left[\varphi_n^2 + \varphi_n \partial_x \mathscr{H} \varphi_n + (\partial_y \varphi_n)^2\right] dxdy \\
&\quad + 2\int_{\mathbb{R}^2} \psi_n \varphi_n (\partial_y \psi_n)(\partial_y \varphi_n) dxdy \\
&\quad + \int_{\mathbb{R}^2} \left[(\partial_y \psi_n)^2 \varphi_n^2 + \varphi_n \psi_n \mathscr{H}(\varphi_n \partial_x \psi_n)\right] dxdy \\
&\quad + \int_{\mathbb{R}^2} \varphi_n \psi_n [\mathscr{H}, \psi_n] \partial_x \varphi_n dxdy.
\end{aligned}$$

Lemma 2.5 and the definition of ϕ_n and ψ_n imply that

$$\left| I(g_n) - \frac{1}{2}\int_{\mathbb{R}^2} \phi_r^2 \Big[\varphi_n^2 + \varphi_n \partial_x \mathscr{H} \varphi_n + (\partial_y \varphi_n)^2\Big]\, dxdy \right| \leq C\epsilon,$$

and

$$\left| I(h_n) - \frac{1}{2}\int_{\mathbb{R}^2} \psi_r^2 \Big[\varphi_n^2 + \varphi_n \partial_x \mathscr{H} \varphi_n + (\partial_y \varphi_n)^2\Big]\, dxdy \right| \leq C\epsilon.$$

These inequalities imply (2.13); inequality (2.14) can be established in a similar way. Inequality (2.15) follows from (2.13), the fact $\mathrm{supp}(g_n) \cap \mathrm{supp}(h_n) = \emptyset$ for large enough n and the injection of $\mathscr{Z}$ into L^{p+2}. □

Attention is now returned to the proof that dichotomy cannot occur. By Lemma 2.4, it may be presumed that there is a $\rho(\epsilon)$ and $\widetilde{\rho}(\epsilon)$ such that

$$\lim_{n\to+\infty} J(g_n) = \rho(\epsilon), \quad \lim_{n\to+\infty} J(h_n) = \widetilde{\rho}(\epsilon),$$

with $\lim_{\epsilon \downarrow 0} |\lambda - \rho(\epsilon) - \widetilde{\rho}(\epsilon)| = 0$. If $\lim_{\epsilon\downarrow 0} \rho(\epsilon) = 0$, then for ϵ sufficiently small and n large enough, it must be that $J(h_n) > 0$. Hence, by considering $\left(\widetilde{\rho}(\epsilon) J(h_n)^{-1}\right)^{\frac{1}{p+2}} h_n$, and noting that

$$J\left(\left(\widetilde{\rho}(\epsilon) J(h_n)^{-1}\right)^{\frac{1}{p+2}} h_n\right) = \widetilde{\rho}(\epsilon),$$

it transpires that

$$I_{\widetilde{\rho}(\epsilon)} \leq \liminf_{n\to+\infty} I(h_n) \leq I_\lambda - \gamma + C\epsilon,$$

which leads to a contradiction since $\lim_{\epsilon\downarrow 0}\widetilde{\rho}(\epsilon) = \lambda$. Therefore we must have $\lim_{\epsilon\downarrow 0}|\rho(\epsilon)| > 0$ and similarly $\lim_{\epsilon\downarrow 0}|\widetilde{\rho}(\epsilon)| > 0$. It follows that

$$I_{|\rho|} + I_{|\widetilde{\rho}|} \leq \liminf_{n\to+\infty} I(g_n) + \liminf_{n\to+\infty} I(h_n) \leq I_\lambda + C\epsilon.$$

But $I_\lambda = \lambda^{\frac{2}{p+2}} I_1$ from which it follows that I_λ is subadditive as a function of λ. However, upon letting $\epsilon \downarrow 0$ in the last display, it is concluded that I_λ cannot be subadditive. This contradiction rules out dichotomy.

The remaining case in the concentration-compactness principle is local compactness. Thus, there exists a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ such that for any $\epsilon > 0$, there are finite values $R > 0$ and $n_0 > 0$ with

$$\int_{B_R(x_n,y_n)} \rho_n\, dxdy \geq \iota_\lambda - \epsilon,$$

for all $n \geq n_0$, where

$$\iota_\lambda = \lim_{n\to+\infty} \int_{\mathbb{R}^2} \rho_n\, dxdy.$$

This in turn implies that for n large enough,

$$\int_{B_R(x_n,y_n)} |\varphi_n|^2 dxdy \geq \int_{\mathbb{R}^2} |\varphi_n|^2 dxdy - 2\epsilon.$$

Since φ_n is bounded in the Hilbert space $\mathscr{X}$, there exists $\varphi \in \mathscr{X}$ and a subsequence of $\{\varphi_n(\cdot - (x_n, y_n))\}_{n\in\mathbb{N}}$ (denoted by $\{\psi_n(x,y)\}_{n\in\mathbb{N}}$) which converges weakly in $\mathscr{X}$ and in $L^2(\mathbb{R}^2)$ to φ. It follows that

$$\begin{aligned}\int_{\mathbb{R}^2} |\varphi|^2 dxdy &\leq \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\varphi_n|^2 dxdy \\ &\leq \liminf_{n\to+\infty} \int_{B_R(x_n,y_n)} |\varphi_n|^2 dxdy + 2\epsilon = \liminf_{n\to+\infty} \int_{B_R(0,0)} \left|\psi_n\right|^2 dxdy + 2\epsilon.\end{aligned}$$

But, when restricted to the bounded set $B_R((0,0)$ in $\mathbb{R}^2$, $\mathscr{X}$ is compactly embedded into L^2. Consequently, a further subsequence of $\{\psi_n\}_{n\in\mathbb{N}}$ may be presumed to converge strongly to φ in $L^2(B_R((0,0)))$. Consequently, as $n \to +\infty$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |\varphi_n|^2 dxdy &\leq \int_{B_R((0,0))} |\psi_n|^2 dxdy + 2\epsilon \longrightarrow \int_{B_R((0,0))} |\varphi|^2 dxdy + 2\epsilon \\ &\leq \int_{\mathbb{R}^2} |\varphi|^2 dxdy + 2\epsilon \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |\varphi_n|^2 dxdy + 4\epsilon,\end{aligned}$$

It follows that $\{\psi_n\}_{n\in\mathbb{N}} = \{\varphi_n(\cdot-(x_n,y_n))\}_{n\in\mathbb{N}}$ converges strongly in $L^2(\mathbb{R}^2)$. Because of inequality (1.12), it also converges to φ strongly in $L^{p+2}(\mathbb{R}^2)$, whence $J(\varphi)=\lambda$. In consequence,

$$I_\lambda \le I(\varphi) \le \liminf_{n\to+\infty} I(\varphi_n) = I_\lambda,$$

which is to say, φ is a solution of the minimization problem I_λ.

The Lagrange Multiplier Theorem asserts there exists $\theta\in\mathbb{R}$ such that

$$\varphi + \mathscr{H}\varphi_x - \varphi_{yy} = \theta(p+2)\varphi^{p+1}, \tag{2.17}$$

as an equation in $\mathscr{Z}'$ (the dual space of $\mathscr{Z}$ in L^2-duality). Multiplying the last equation by $\widetilde{\varphi}$ and integrating over $\mathbb{R}^2$ yields, after an integration by parts,

$$\int_{\mathbb{R}^2} \big(\varphi^2 + \varphi_y^2 + \varphi\mathscr{H}\varphi_x\big)dxdy = \frac{\theta}{p+1}J(\varphi). \tag{2.18}$$

Thus, $\theta \neq 0$ and a simple change of scale yields a $\widetilde{\varphi}$ which satisfies (2.9). □

Remark 2.6. Theorem 2.3 shows the existence of solitary-wave solutions of (1.1) in the cases (i) and (ii) in Theorem 2.1. The question of existence or nonexistence of solitary waves in cases (iii) and (iv) is currently open.

We point out that in the original work of Zakharov and Kuznetsov [52], their equation for propagation of ion acoustic waves in a plasma in the presence of a magnetic field has the form

$$u_t + au_x + buu_x + du_{xxx} + eu_{xyy} = 0,$$

where $a>0$ is the speed of sound in the environment and b,d and e are positive constants determined by various properties of the medium of propagation and the orientation and strength of the magnetic field. This equation governs only waves propagating to the right, as interactions between right- and left-going waves have been ignored. The equation one usually sees in the mathematical literature is written, as in (1.1), in traveling coordinates. The physically relevant version of the BO equation (1.2) has $\alpha<0$. Hence, a physically relevant version of *BO-ZK*, written in laboratory coordinates, will have an au_x–term, $p=1$, $\alpha<0$ while $\epsilon>0$. It follows from Theorems 2.1 and 2.3 that solitary waves exist in this situation exactly when the solitary-wave speed c is larger than the 'sound speed' a.

The next lemma shows that there exists a $\lambda>0$ such that every element in the set of minimizers

$$M_\lambda = \{\varphi\in\mathscr{Z};\ I(\varphi)=I_\lambda,\ J(\varphi)=\lambda\}, \tag{2.19}$$

exactly satisfies (2.9). This will be useful presently.

Lemma 2.7. *If* $\lambda = \big(2(p+1)I_1\big)^{\frac{p+2}{p}}$ *in the minimization problem* (2.10), *then any* $\varphi \in M_\lambda$ *is a solitary-wave solution of* (2.9).

Proof. As above, if u is a member of M_λ, then Lagrange's Theorem implies there is a real value θ such that

$$u - u_{yy} + \mathscr{H}u_x = \frac{\theta}{p+1}u^{p+1}. \tag{2.20}$$

As in (2.18), it must be the case that $\theta \neq 0$. Because u satisfies (2.20) with $\theta \neq 0$, it follows that for an appropriate choice of the constant γ, $u = \gamma\Phi$ where $\Phi \in M_1$. Consequently, we must have $\lambda = J(u) = \gamma^{p+2}J(\Phi) = \gamma^{p+2}$ and $I_\lambda = \gamma^2 I_1$. Equation (2.18), in terms of γ, is

$$2\gamma^2 I_1 = \theta(p+2)\gamma^{p+1}.$$

The choice $\lambda = \big(2(p+1)I_1\big)^{\frac{p+2}{p}}$ causes (2.20) to agree with (2.9). □

Definition 2.8. *A solution* φ *of equation* (1.7) *is called a ground state, if* φ *minimizes the action* $\mathcal{S}(u) = \mathscr{E}(u) + c\mathscr{F}(u)$ *among all non-trivial solutions of* (1.7), *where*

$$\mathscr{F}(u) = \frac{1}{2}\int_{\mathbb{R}^2} u^2\, dxdy, \tag{2.21}$$

and

$$\mathscr{E}(u) = \frac{1}{2}\int_{\mathbb{R}^2}\Big(\varepsilon u_y^2 - \alpha u\mathscr{H}u_x - \frac{2}{(p+1)(p+2)}\,u^{p+2}\Big)\,dxdy. \tag{2.22}$$

The functionals in (2.22) and (2.21) are invariants of the motion when applied to solutions u of BO–ZK which lie at least in $C(0,T;\mathscr{Z})$.

Next, it is established that the minima obtained in Theorem 2.3 are precisely the ground-state solutions of (1.7). The proof is inspired by that of Lemma 2.1 in the work of de Bouard and Saut in [16]. The result is stated for the scaled version (2.9) of (1.7).

Theorem 2.9. *In the context of equation* (2.9) *for solitary-wave solutions of the BO-ZK equation, let*

$$\mathcal{K}(u) = \frac{1}{2}\int_{\mathbb{R}^2}(u^2 + u_y^2)dxdy - \frac{1}{(p+1)(p+2)}J(u),$$

with

$$J(u) = \int u^{p+2}dxdy,$$

as in (2.10)*. Then, the positive value* $\lambda^* = \big(2(p+1)I_1\big)^{\frac{p+2}{p}}$ *is such that the following assertions about a function* $u^* \in \mathscr{Z}$ *are, up to a change of scale, equivalent:*

(i) *The functional* J *has the value* $J(u^*) = \lambda^*$ *and* u^* *is a minimizer of* I_{λ^*}*,*

(ii) $\mathcal{K}(u^*) = 0$ *and*

$$\inf\left\{\int_{\mathbb{R}^2} u\mathscr{H}u_x\,dxdy:\ u\in\mathscr{Z},\ u\neq 0,\ \mathcal{K}(u)=0\right\} = \int_{\mathbb{R}^2} u^*\mathscr{H}u^*_x\,dxdy,$$

(iii) u^* *is a ground state,*

(iv) $\mathcal{K}(u^*) = 0$ *and*

$$\inf\left\{\mathcal{K}(u):\ u\in\mathscr{Z},\ u\neq 0,\ \int_{\mathbb{R}^2} u\mathscr{H}u_x\,dxdy = \int_{\mathbb{R}^2} u^*\mathscr{H}u^*_x\,dxdy\right\} = 0.$$

Proof. (i) $\Rrightarrow$ (ii) : Assume that (i) holds for u^*. Then $\mathcal{K}(u^*) = 0$ because of (2.2) and the fact that u^* satisfies (2.9) by Lemma 2.7. It follows that $J(u^*) > 0$. Let $u \in \mathscr{Z}$ with $u \neq 0$ and $\mathcal{K}(u) = 0$, so that $J(u) > 0$ also. Let μ be given by

$$\mu = \frac{J(u^*)}{J(u)} \quad \text{and define} \quad u_\mu(x,y) = u(\tfrac{x}{\mu}, y).$$

The constant μ is arranged so that $J(u_\mu) = J(u^*) = \lambda^*$ and $\mathcal{K}(u_\mu) = 0$. Since u^* is a minimum of I_{λ^*},

$$\begin{aligned}\mathcal{K}(u^*) + C_pJ(u^*) + \frac{1}{2}\int_{\mathbb{R}^2} u^*\mathscr{H}u^*_x\,dxdy &= I(u^*)\\ &\le I(u_\mu) = \mathcal{K}(u_\mu) + C_pJ(u_\mu) + \frac{1}{2}\int_{\mathbb{R}^2} u_\mu\mathscr{H}(u_\mu)_x\,dxdy,\end{aligned}$$

where $C_p = \frac{1}{(p+1)(p+2)}$. This in turn implies that

$$\int_{\mathbb{R}^2} u^*\mathscr{H}u^*_x\,dxdy \le \int_{\mathbb{R}^2} u\mathscr{H}u_x\,dxdy,$$

and (ii) holds.

(ii) $\Rrightarrow$ (iii): The identity

$$S(u) = \mathcal{K}(u) + \frac{1}{2}\int_{\mathbb{R}^2} u\mathscr{H}u_x dxdy,$$

shows that if u is a solution of (2.9), then

$$S(u) = \frac{1}{2}\int_{\mathbb{R}^2} u\mathscr{H}u_x\,dxdy \ge \frac{1}{2}\int_{\mathbb{R}^2} u^*\mathscr{H}u^*_x\,dxdy = S(u^*), \tag{2.23}$$

because of (ii). Also, since u^* satisfies (ii), there is a Lagrange multiplier θ such that

$$u^* - u^*_{yy} + \theta \mathscr{H} u^*_x - \frac{1}{p+1}(u^*)^{p+1} = 0. \tag{2.24}$$

By multiplying the above equation by u^* , integrating by parts as in (2.18) and using that $\mathcal{K}(u^*) = 0$, it is deduced that θ is positive. Hence, the scale change

$$u_*(x,y) = u^*(x/\theta, y), \tag{2.25}$$

will satisfy equation (2.9). On the other hand, this change of scale has the property that

$$\int_{\mathbb{R}^2} u^* \mathscr{H} u^*_x dx = \int_{\mathbb{R}^2} u_* \mathscr{H} u_{*x} dx.$$

Hence, up to the change of scale (2.25), u^* is a ground state.

(iii) $\Rrightarrow$ (i) : From (2.2) in Theorem 2.1, one sees that if u is a solution of (2.9), then $\mathcal{K}(u) = 0$. Moreover, equation (2.4) and the fact that $\alpha = -1$ here implies

$$I(u) = \tfrac{1}{2}(1 + \tfrac{2}{p}) \int_{\mathbb{R}^2} u \mathscr{H} u_x \, dxdy. \tag{2.26}$$

Hence, if u^* is a ground state, then u^* minimizes both $I(u)$ and

$$\int_{\mathbb{R}^2} u \mathscr{H} u_x dxdy,$$

among all non-trivial solutions of (2.9). Let $\lambda = J(u)$ and $\widetilde{u}$ be a minimum of I_λ. Then

$$I_\lambda = I(\widetilde{u}) \leq I(u^*), \tag{2.27}$$

and there is a positive number θ such that

$$\widetilde{u} - \widetilde{u}_{yy} + \mathscr{H} \widetilde{u}_x = \frac{\theta}{p+1} \widetilde{u}^{p+1}.$$

Using the equations satisfied by $\widetilde{u}$ and u^*, inequality (2.27) is written as

$$I_\lambda = \frac{\lambda\theta}{p+1} \leq \frac{\lambda}{p+1},$$

from which it is deduced immediately that $\theta \leq 1$. On the other hand, $u_* = \theta^p \widetilde{u}$ satisfies equation (2.9), and since u^* is a ground state, it must be the case that

$$I(u^*) \leq I(u_*) = \theta^{2p} I(\widetilde{u}),$$

so that $\theta \geq 1$. In consequence, $u^* = \widetilde{u}$ is a minimum of I_λ. The fact that $\lambda = \lambda^*$ now follows as in the proof of Lemma 2.7.

(ii) ⇛ (iv) : Let $u \in \mathscr{Z}$ with

$$\int_{\mathbb{R}^2} u\mathscr{H}u_x dxdy = \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* dxdy.$$

Suppose that $\mathcal{K}(u) < 0$. Since $\mathcal{K}(\tau u) > 0$ for $\tau > 0$ sufficiently small, there is a $\tau_0 \in (0,1)$ such that $\mathcal{K}(\tau_0 u) = 0$. By setting $\widetilde{u} = \tau_0 u$, one has $\widetilde{u} \in \mathscr{Z}$, $\mathcal{K}(\widetilde{u}) = 0$ and

$$\int_{\mathbb{R}^2} \widetilde{u}\mathscr{H}\widetilde{u}_x \, dxdy < \int_{\mathbb{R}^2} u\mathscr{H}u_x \, dxdy = \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* \, dxdy,$$

which contradicts (ii) and shows that u^* satisfies (iv) because $\mathcal{K}(u^*) = 0$.

(iv) ⇛ (ii) : Let $u \in \mathscr{Z}$ with $\mathcal{K}(u) = 0$ and $u \neq 0$. Suppose that

$$\int_{\mathbb{R}^2} u\mathscr{H}u_x \; dxdy < \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* \; dxdy.$$

Since $\mathcal{K}(\tau u) < 0$ for $\tau > 1$, there is a $\tau_0 > 1$ with

$$\int_{\mathbb{R}^2} (\tau_0 u)\mathscr{H}(\tau_0 u)_x \, dxdy = \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* \, dxdy,$$

and $\mathcal{K}(\tau_0 u) < 0$. This contradicts (iv). Hence,

$$\int_{\mathbb{R}^2} u\mathscr{H}u_x \; dxdy \geq \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* \; dxdy,$$

and (ii) holds. □

Remark 2.10. In fact, what is true is that (i) and (iii) are equivalent and imply (ii) and (iv), which are also equivalent. The converse holds modulo a change of scale. To see this, we simply need to check that (i) ⇛ (iii) without a change of scale. This follows readily from (2.23), (2.24) and Lemma 2.7.

3. STABILITY

This section is devoted to establishing the stability of solitary-wave solutions of the BO-ZK equations in the case where we are assured they exist on account of the theory in the preceding section.

Some of the arguments below can be found in [4] where the stability of solitary waves for the generalized BO equation has been established. Hereafter, without loss of generality, we take $\alpha = -1$ so that $\varepsilon = +1$. However, we leave the wave speed $c > 0$ unscaled. The equation for the solitary wave now has the form

$$-c\varphi + \frac{1}{p+1}\varphi^{p+1} - \mathscr{H}\varphi_x + \varphi_{yy} = 0, \tag{3.1}$$

while the functional $I = I_c$ whose minima subject to the constraint $J = \lambda$ provides solitary-wave solutions is

$$I_c(\varphi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(c\varphi^2 + \varphi\mathscr{H}\varphi_x + \varphi_y^2\right) dxdy. \tag{3.2}$$

The following theorem is a consequence of Theorem 2.3. It will be used presently.

Theorem 3.1. *Let $\lambda \neq 0$ and $c > 0$.*

(i) *Every minimizing sequence for the problem $I_\lambda = I_{c,\lambda}$ converges, up to translations, in the topology of $\mathscr{Z}$ to an element in the set*

$$M_{c,\lambda} = \{\varphi \in \mathscr{Z};\ I_c(\varphi) = I_{c,\lambda},\ J(\varphi) = \lambda\},$$

of minimizers for I_c.

(ii) *Let $\{\varphi_n\}$ be a minimizing sequence for $I_{c,\lambda}$. Then, it must be the case that*

$$\lim_{n\to+\infty} \inf_{\psi\in M_{c,\lambda},\ z\in\mathbb{R}^2} \|\varphi_n(\cdot + z) - \psi\|_{\mathscr{Z}} = 0, \tag{3.3}$$

$$\lim_{n\to+\infty} \inf_{\psi\in M_{c,\lambda}} \|\varphi_n - \psi\|_{\mathscr{Z}} = 0. \tag{3.4}$$

Proof. Part (i) follows immediately from the proof of Theorem 2.3. The equality (3.3) is proved by contradiction. Indeed, if (3.3) does not hold, then there exists a subsequence of the sequence $\{\varphi_n\}$, denoted again by $\{\varphi_n\}$, and an $\epsilon > 0$ such that

$$\varpi = \inf_{\psi\in M_{c,\lambda}, r\in\mathbb{R}^2} \|\varphi_n(\cdot + r) - \psi\|_{\mathscr{Z}} \geq \epsilon,$$

for all n sufficiently large. On the other hand, since $\{\varphi_n\}$ is a minimizing sequence for I_λ, part (i) implies that there exists a sequence $\{r_n\} \subset \mathbb{R}^2$ such that, up to a subsequence, $\varphi_n(\cdot + r_n) \to \varphi$ in $\mathscr{Z}$, as $n \to \infty$. Hence, for n large enough, it is inferred that

$$\frac{\epsilon}{2} \geq \|\varphi_n(\cdot + r_n) - \varphi\|_{\mathscr{Z}} \geq \varpi \geq \epsilon,$$

which is a contradiction. The proof of (3.4) follows from (3.3), the fact that if $\psi \in M_{c,\lambda}$, then $\psi(\cdot + r) \in M_{c,\lambda}$ for all $r \in \mathbb{R}^2$, and the equalities

$$\inf_{\psi\in M_{c,\lambda}} \|\varphi_n - \psi\|_{\mathscr{Z}} = \inf_{\psi\in M_{c,\lambda}, r\in\mathbb{R}^2} \|\varphi_n - \psi(\cdot - r)\|_{\mathscr{Z}} = \inf_{\psi\in M_{c,\lambda}, r\in\mathbb{R}^2} \|\varphi_n(\cdot + r) - \psi\|_{\mathscr{Z}}.$$

This completes the proof of the theorem. □

For $\lambda_c = \big(2(p+1)I_{c,1}\big)^{\frac{p+2}{p}}$ as in Lemma 2.7, define the set

$$\mathscr{N}_c = \{\varphi \in \mathscr{Z};\ J(\varphi) = 2(p+1)I_c(\varphi) = \lambda_c\}\,.$$

It is clear from the choice of λ_c that $M_{c,\lambda} = \mathscr{N}_c$; the latter notation emphasizes the dependence upon the wave speed c. Next, for any $c > 0$ and any $\varphi \in \mathscr{N}_c$, define the function $d : \mathbb{R} \to \mathbb{R}$ by

$$d(c) = \mathscr{E}(\varphi) + c\mathscr{F}(\varphi). \tag{3.5}$$

Lemma 3.2. *The function d in (3.5) is constant on $\mathscr{N}_c$. As a function of c, it is twice differentiable and strictly increasing for $c > 0$. Moreover, $d''(c) > 0$ if and only if $0 < p < \frac{4}{3}$.*

Proof. It is straightforward to check that

$$\begin{aligned} d(c) &= I_c(\varphi) - \frac{1}{(p+1)(p+2)}J(\varphi) = \frac{p}{2(p+1)(p+2)}J(\varphi) \\ &= \frac{p\big(2(p+1)\big)^{\frac{2}{p}}}{p+2} I_{c,1}^{\frac{p+2}{p}}. \end{aligned} \tag{3.6}$$

It is plain from this formula that d is constant on $\mathscr{N}_c$. From the second equality in (3.6) and the definition of J, one obtains

$$d(c) = \frac{p}{2(p+1)(p+2)} c^{\frac{2}{p}-\frac{1}{2}} J(\psi), \tag{3.7}$$

where $\psi(x,y) = c^{-\frac{1}{p}}\varphi(\frac{x}{c}, \frac{y}{\sqrt{c}})$. Note that ψ satisfies (3.1), with $c = 1$. But, from (2.4) and (2.6), one infers that

$$\frac{1}{(p+1)(p+2)}J(\varphi) = \frac{4c}{4-p}\mathscr{F}(\varphi).$$

Thus, from (3.7) follows the formula $d'(c) = c^{(\frac{2}{p}-\frac{3}{2})}\mathscr{F}(\psi)$, which is strictly positive. This further entails that

$$d''(c) = (\tfrac{2}{p} - \tfrac{3}{2})c^{(\frac{2}{p}-\frac{5}{2})}\mathscr{F}(\psi),$$

thereby proving the lemma. □

A study is initiated of the behavior of d in a neighborhood of the set $\mathscr{N}_c$. As d is C^1 and strictly increasing, it has a C^1 inverse. Formula (3.7) shows that the mapping $d : (0,+\infty) \to (0,+\infty)$ is a surjection when $0 < p < \frac{4}{3}$.

Lemma 3.3. *Let $c > 0$. Then, there exists a positive number ϵ and a C^1-map $v : \mathscr{B}_\epsilon(\mathscr{N}_c) \to (0, +\infty)$ defined by*

$$v(u) = d^{-1}\Big(\frac{p}{2(p+1)(p+2)}J(u)\Big),$$

such that $v(\varphi) = c$ for every $\varphi \in \mathscr{N}_c$. Here,

$$\mathscr{B}_\epsilon\left(\mathscr{N}_c\right) = \Big\{\varphi \in \mathscr{Z}\ ;\ \inf_{\psi\in\mathscr{N}_c} \|\varphi - \psi\|_{\mathscr{Z}} < \epsilon\Big\}.$$

Proof. By definition, $\mathscr{N}_c$ is a bounded set in $\mathscr{Z}$. Let $r > 0$ be such that $\mathscr{N}_c \subset B_r(0) \subset \mathscr{Z}$. where $B_r(0)$ is the ball of radius $r > 0$ centered at the origin in $\mathscr{Z}$. Since the function $u \mapsto J(u)$ is uniformly continuous on bounded subsets of $\mathscr{Z}$, given $\iota > 0$, say $\iota < \frac{1}{2}d(c)$, there exists $\epsilon > 0$ such that if $u, v \in B_r(0)$ and $\|u - v\|_{\mathscr{Z}} < 2\epsilon$, then $|J(u) - J(v)| < \iota$. Considering the neighborhoods $\mathscr{I} = (d(c)-\iota, d(c)+\iota)$ of $d(c)$ and $\mathscr{B}_\epsilon(\mathscr{N}_c)$ of $\mathscr{N}_c$, respectively, it transpires that if $u \in \mathscr{B}_\epsilon(\mathscr{N}_c)$, then $\frac{p}{2(p+1)(p+2)}J(u) \in \mathscr{I}$. Hence, v is well defined on $\mathscr{B}_\epsilon(\mathscr{N}_c)$ and satisfies $v(\varphi) = c$, for all $\varphi \in \mathscr{N}_c$. □

Here is the crucial inequality in the study of stability.

Lemma 3.4. *Let $c > 0$ and suppose that $d''(c) > 0$. Then for $\epsilon > 0$ sufficiently small, $u \in \mathscr{B}_\epsilon(\mathscr{N}_c)$ and any $\varphi \in \mathscr{N}_c$,*

$$\mathscr{E}(u) - \mathscr{E}(\varphi) + v(u)\big(\mathscr{F}(u) - \mathscr{F}(\varphi)\big) \geq \tfrac{1}{4}d''(c)|v(u) - c|^2.$$

Proof. First, let $\epsilon > 0$ be small enough that v is well defined on $\mathscr{B}_\epsilon(\mathscr{N}_c)$. As in (3.2), for $\omega > 0$, let I_ω be the functional

$$I_\omega(\varphi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\omega\varphi^2 + \varphi\mathscr{H}\varphi_x + \varphi_y^2\right) dxdy.$$

It follows that

$$\mathscr{E}(u) + v(u)\mathscr{F}(u) = I_{v(u)}(u) - \frac{1}{(p+1)(p+2)}J(u).$$

Let φ_ω stand for any element of $\mathscr{N}_\omega$. Notice that

$$J(u) = J\left(\varphi_{v(u)}\right),$$

because $d(v(u)) = \frac{p}{2(p+1)(p+2)}J(u)$ for $u \in \mathscr{B}_\epsilon(\mathscr{N}_c)$ and also $d(v(u)) = \frac{p}{2(p+1)(p+2)}J\left(\varphi_{v(u)}\right)$ from (3.6). Since φ_ω minimizes I_ω over elements $u \in \mathscr{Z}$ with $J(u) = J\left(\varphi_\omega\right)$, it transpires that

$$I_{v(u)}(u) \geq I_{v(u)}\left(\varphi_{v(u)}\right).$$

Thus, for ϵ small enough, a Taylor expansion of d around c and the fact that $d''(c) > 0$ implies that

$$\begin{aligned}\mathscr{E}(u) + v(u)\mathscr{F}(u) &\geq I_{v(u)}\left(\varphi_{v(u)}\right) - \frac{1}{(p+1)(p+2)} J\left(\varphi_{v(u)}\right) \\ &= d(v(u)) \geq d(c) + \mathscr{F}(\varphi)(v(u)-c) + \frac{1}{4} d''(c)|v(u)-c|^2 \\ &= \mathscr{E}(\varphi) + v(u)\mathscr{F}(\varphi) + \frac{1}{4} d''(c)|v(u)-c|^2,\end{aligned}$$

thereby establishing the lemma. □

Before proving stability, a well-posedness result for (1.1) is stated. This can be proved in several standard ways, for example by using a parabolic regularization (see [31] and [23]). It is worth noting that $H^s(\mathbb{R}^2) \hookrightarrow \mathscr{Z}$, for all $s \geq 1$.

Theorem 3.5. *Let $s > 2$. Then for any $u_0 \in H^s(\mathbb{R}^2)$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R}^2))$ of equation (1.1) with $u(0) = u_0$. In addition, $u(t)$ depends continuously on u_0 in the H^s-norm and satisfies $\mathscr{E}(u(t)) = \mathscr{E}(u_0)$, $\mathscr{F}(u(t)) = \mathscr{F}(u_0)$, for all $t \in [0,T)$. Moreover, if $p < 4/3$, the solution exists on $[0,T]$ for any $T > 0$ and*

$$\sup_{t>0} \|u(t)\|_{\mathscr{Z}} \leq C(\mathscr{F}(u_0), \mathscr{E}(u_0)).$$

When $0 < p < \frac{4}{3}$, the stability in $\mathscr{Z}$ of the set of minimizers $\mathscr{N}_c$ is the next topic of conversation.

Theorem 3.6. *Let $c > 0$, $s > 2$, $0 < p < \frac{4}{3}$ and $\lambda = \left(2(p+1)I_{c,1}\right)^{\frac{p+2}{p}}$. Then the set $\mathscr{N}_c = M_{c,\lambda}$ is $\mathscr{Z}$-stable with regard to the flow of the BO-ZK equation. That is, for any positive ϵ, there is a positive $\delta = \delta(\epsilon)$ such that if $u_0 \in H^s$ and $\inf_{\varphi \in \mathscr{N}_c} \|u_0 - \varphi\|_{\mathscr{Z}} \leq \delta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies*

$$\inf_{\varphi \in \mathscr{N}_c} \|u(t) - \varphi\|_{\mathscr{Z}} \leq \epsilon,$$

for any $t \in [0,T]$.

Proof. The proof follows standard lines. Assume that $\mathscr{N}_c$ is $\mathscr{Z}$-unstable with regard to the flow of the BO-ZK equation. Then, there is an $\epsilon > 0$ and a sequence of initial data $u_k(0)$ such that

$$\inf_{\varphi \in \mathscr{N}_c} \|u_k(0) - \varphi\|_{\mathscr{Z}} \leq \frac{1}{k}, \tag{3.8}$$

and positive times $\{t_k\}$ such that

$$\inf_{\varphi\in\mathscr{N}_c} \|u_k\,(t_k)-\varphi\|_{\mathscr{X}}=\frac{\epsilon}{2}. \tag{3.9}$$

Here, $u_k(t)$ denotes the solution of (1.1) with initial data $u_k(0)$. Since $\mathscr{E}$ and $\mathscr{F}$ are conserved quantities, (3.8) implies that there are $\varphi_k\in\mathscr{N}_c$ such that that

$$|\mathscr{E}(u_k(t_k))-\mathscr{E}(\varphi_k)|=|\mathscr{E}(u_k(0))-\mathscr{E}(\varphi_k)|\to 0, \tag{3.10}$$

$$|\mathscr{F}(u_k(t_k))-\mathscr{F}(\varphi_k)|=|\mathscr{F}(u_k(0))-\mathscr{F}(\varphi_k)|\to 0, \tag{3.11}$$

as $k\to+\infty$. In this circumstance, Lemma 3.4 implies that

$$\mathscr{E}(u_k(t_k))-\mathscr{E}(\varphi_k)+v(u_k(t_k))\big(\mathscr{F}(u_k(t_k))-\mathscr{F}(\varphi_k)\big)\geq\tfrac{1}{4}d''(c)|v(u_k(t_k))-c|^2,$$

for all k large enough. Since $\{u_k(t_k)\}$ is uniformly bounded in k, the left-hand side of the last inequality goes to zero as $k\to+\infty$ on account of (3.10) and (3.11). This in turn implies that $v(u_k(t_k))\to c$ as $k\to+\infty$. Hence, by the definition of v and continuity of d, we must have

$$\lim_{k\to+\infty} J(u_k(t_k))=\frac{2(p+1)(p+2)}{p}\;d(c). \tag{3.12}$$

On the other hand, Lemma 3.2 implies that

$$\begin{aligned}I_c(u_k(t_k))&=\mathscr{E}(u_k(t_k))+c\mathscr{F}(u_k(t_k))+\frac{1}{(p+1)(p+2)}J(u_k(t_k))\\&=d(c)+\mathscr{E}(u_k(t_k))-\mathscr{E}(\varphi_k)+c\,(\mathscr{F}(u_k(t_k))-\mathscr{F}(\varphi_k))\\&\quad+\frac{1}{(p+1)(p+2)}J(u_k(t_k)).\end{aligned}$$

The limit (3.12) then yields

$$\lim_{k\to+\infty} I_c(u_k(t_k))=\frac{p+2}{p}d(c)=\big(2(p+1)\big)^{\frac{2}{p}}\;I_{c,1}^{\frac{p+2}{p}}. \tag{3.13}$$

Define

$$\vartheta_k(t_k)=\Big(J\big(u_k(t_k)\big)\Big)^{-\frac{1}{p+2}}u_k(t_k),$$

so that $J\left(\vartheta_k(t_k)\right)=1$. Combining (3.12), (3.13) and Lemma 3.2 leads to

$$\lim_{k\to+\infty} I_c(\vartheta_k(t_k))=I_{c,1}. \tag{3.14}$$

Hence, $\{\vartheta_k(t_k)\}$ is a minimizing sequence for $I_{c,1}$. Theorem 3.1 allow us to adduce a sequence $\{\psi_k\}\subset M_{c,1}$ such that

$$\lim_{k\to+\infty} \|\vartheta_k(t_k)-\psi_k\|_{\mathscr{X}}=0. \tag{3.15}$$

Lagrange's theorem then implies there is a sequence $\{\theta_k\} \subset \mathbb{R}$ such that

$$\mathscr{H}\,(\psi_k)_x + c\psi_k - (\psi_k)_{yy} = \theta_k(p+2)\psi_k^{p+1}. \tag{3.16}$$

In other words, $2I_{c,1} = \theta_k(p+2)$, which implies that there is θ such that $\theta_k = \theta$ for all k. Write $\varphi_k = \mu\psi_k$ with

$$\mu^p = \theta(p+1)(p+2) = 2(p+1)I_{c,1}.$$

Then the φ_k satisfy (1.7) and $2(p+1)I_c(\varphi_k) = J(\varphi_k) = \mu^{p+2}$ so that $\varphi_k \in \mathscr{N}_c$ for all k. Additionally, (3.12)-(3.15) and Lemma 3.2 together allow the conclusion

$$\begin{aligned}\|u_k(t_k) - \varphi_k\|_{\mathscr{Z}} &= J\big(u_k(t_k)\big)^{\frac{1}{p+2}} \left\| J\big(u_k(t_k)\big)^{-\frac{1}{p+2}}\big(u_k(t_k) - \varphi_k\big)\right\|_{\mathscr{Z}} \\ &\leq J\big(u_k(t_k)\big)^{\frac{1}{p+2}} \left(\left\|\vartheta_k(t_k) - \mu^{-1}\varphi_k\right\|_{\mathscr{Z}} + \|\varphi_k\|_{\mathscr{Z}}|\mu^{-1} - J\big(u_k(t_k)\big)^{-\frac{1}{p+2}}|\right).\end{aligned}$$

This in turn implies that

$$\lim_{k\to+\infty} \|u_k(t_k) - \varphi_k\|_{\mathscr{Z}} = 0,$$

which contradicts (3.9) and completes the proof of the theorem. □

4. Symmetry, Decay and Regularity

To investigate the regularity and the spatial asymptotics of the solitary-wave solutions of (1.1), it is convenient to take the Fourier transform of equation (1.7) for the solitary-wave in both x and y. If (ξ, η) are the variables dual to (x, y) by way of the Fourier transform, then (1.7) implies that

$$\widehat{\varphi} = \frac{\widehat{g}}{c - \alpha|\xi| + \varepsilon\eta^2}, \quad \text{where} \quad g = -\frac{1}{p+1}\varphi^{p+1}. \tag{4.1}$$

Taking the inverse Fourier transform then yields

$$\varphi = -\frac{1}{p+1}\int_{\mathbb{R}^2} K\big(x-s, y-t\big)\varphi^{p+1}(s,t)\,dsdt. \tag{4.2}$$

Properties of the integral kernel K in (4.2) will be central in the analysis to follow. Here are few standard properties of anisotropic Sobolev spaces that will be helpful in expressing useful aspects of K.

Lemma 4.1. *If $s_i > 1/2$, for $i = 1, 2$, then H^{s_1,s_2} is a Banach algebra.*

Lemma 4.2. *Let $s_{ij}, 1 \leq i, j \leq 2$ and $\theta \in [0, 1]$ be given real numbers with $s_{1j} \leq s_{2j}$, $j = 1, 2$. Define $\varrho_j = \theta s_{1j} + (1-\theta)s_{2j}$ for $j = 1, 2$. Then, H^{ϱ_1,ϱ_2} is* *an interpolation space between $H^{s_{11},s_{12}}$ and its subspace $H^{s_{21},s_{22}}$. Moreover, if $f \in H^{s_{21},s_{22}}$, then*

$$\|f\|_{H^{\varrho_1,\varrho_2}} \le \|f\|^{\theta}_{H^{s_{11},s_{12}}} \, \|f\|^{1-\theta}_{H^{s_{21},s_{22}}}. \tag{4.3}$$

Remark 4.3. Since $\widehat{K}(\xi,\eta) = \frac{1}{c-\alpha|\xi|+\eta^2}$, the Residue Theorem allows us to write the kernel K as an integral, namely,

$$K(x,y) = K_c(x,y) = C \int_0^{+\infty} \frac{|\alpha|\sqrt{t}}{\alpha^2 t^2 + x^2} \, e^{-(ct+\frac{y^2}{4t})} \, dt, \tag{4.4}$$

where $C > 0$ is independent of α, x and y. Fubini's theorem can then be used to show that

$$\|K\|_{L^1} = C \int_0^{+\infty} \int_{\mathbb{R}^2} \frac{|\alpha|\sqrt{t}}{\alpha^2 t^2 + x^2} \, e^{-(ct+\frac{y^2}{4t})} dx dy dt = C(\alpha) \int_0^{+\infty} e^{-ct} dt.$$

In consequence of the representation (4.4), the following facts about K become clear.

Lemma 4.4. *The kernel K is positive, an even function of both x and y, monotone decreasing in both $|x|$ and $|y|$, tends to zero as $|(x,y)| \to \infty$ and is C^∞ away from the origin. Furthermore, $\widehat{K} \in L^p(\mathbb{R}^2)$ for any $p \in (3/2,+\infty]$ and $K \in L^p(\mathbb{R}^2)$, for any $p \in [1,3)$. (However, while $K(x,y)$ is symmetric in both x and y, it is not radially symmetric.)*

Lemma 4.5. *$K \in H^{s_1,0}(\mathbb{R}^2) \cap H^{0,s_2}(\mathbb{R}^2)$ for any $s_1 < \frac{1}{4}$ and $s_2 < \frac{1}{2}$. Moreover, $K \in H^{r,s}(\mathbb{R}^2) \cap H^{s_1,s_2}(\mathbb{R}^2)$, where $rs_2 + ss_1 = s_1 s_2$ and $r \in [0,1]$.*

Lemma 4.6. *The kernel K and its Fourier transform $\widehat{K}$ have the following detailed properties:*

(i) *$\widehat{K} \in H^{s_1,0}(\mathbb{R}^2) \cap H^{0,s_2}(\mathbb{R}^2)$, for any $s_1 < \frac{3}{2}$ and $s_2 \in \mathbb{R}$. Moreover, $\widehat{K} \in H^{r,s}(\mathbb{R}^2) \cap H^{(s_1,s_2)}(\mathbb{R}^2)$, where $rs_2 + ss_1 = s_1 s_2$ and $r \in [0,1]$.*

(ii) *For $p \ge 2$, $\widehat{K} \in H_p^{(s_1,s_2)}(\mathbb{R}^2)$, for any $s_1 < 1 + \frac{1}{p}$ and $s_2 \in \mathbb{R}$.*

(iii) *For $1 \le p \le \infty$, $|x|^{s_1}|y|^{s_2} K \in L^p(\mathbb{R}^2)$, for any $s_1, s_2 \ge 0$ such that $s_1 < 2 - \frac{1}{p}$ and $2s_1 + s_2 > 1 - \frac{3}{p}$.*

With these facts about K in hand, the solitary-wave solutions of the BO-ZK equation (1.1) now become the focus of attention.

Theorem 4.7. *Let p be a positive integer. Any solitary-wave solution φ of (1.1) with such a value of p belongs to $H_r^{(k)}$, for all $k \in \mathbb{N}$ and all $r \in [1,+\infty]$. In particular, the solitary-wave solutions of the BO-ZK equation are C^∞ and the solution and all its derivatives are bounded and tend to zero at infinity.*

Proof. Formula (4.1) implies that $\varphi \in H^{\frac{1}{2},1}(\mathbb{R}^2) \cap H^{0,2}(\mathbb{R}^2) \cap H^{1,0}(\mathbb{R}^2)$. Lemma 4.2 and the embedding (1.13) then imply that $\varphi \in H^{s,2(1-s)}(\mathbb{R}^2)$, for any $s \in [0,1]$. A bootstrapping argument and the use of Lemmas 4.1 and 4.2 complete the proof. □

More detailed aspects of the solitary-wave solutions of (1.1) are now addressed. Interest will focus first upon their symmetry properties. For $u : \mathbb{R}^2 \to \mathbb{R}^+$, $u^\sharp$ will denote the Steiner symmetrization of u with respect to $\{x = 0\}$ and u^* the Steiner symmetrization of u with respect to $\{y = 0\}$ (see, for example, [17, 34, 49]). Notice that $u^{\sharp *} = u^{*\sharp}$ is a function symmetric with respect to both the x- and y-axis.

Lemma 4.8. *If $f \in \mathscr{X}$, then $|f|$ lies in $\mathscr{X}$ and $I(|f|) \le I(f)$.*

Proof. If $g = |f|$, then for any $c > 0$, $\langle f, K * f\rangle \le \langle g, K * g\rangle$, since $K = K_c \ge 0$. It thus transpires that

$$\int_{\mathbb{R}^2} \widehat{K}(\xi,\eta)\big|\widehat{f}(\xi,\eta)\big|^2 d\xi d\eta = \langle f, K * f\rangle \le \langle g, K * g\rangle = \int_{\mathbb{R}^2} \widehat{K}(\xi,\eta)\,|\widehat{g}(\xi,\eta)|^2\, d\xi d\eta.$$

Since $\|\widehat{f}\|_{L^2} = \|\widehat{g}\|_{L^2}$, it follows that

$$\int_{\mathbb{R}^2} c(1 - c\widehat{K})\,|\widehat{g}(\xi,\eta)|^2\, d\xi d\eta \le \int_{\mathbb{R}^2} c(1 - c\widehat{K})|\widehat{f}(\xi,\eta)|^2 d\xi d\eta. \tag{4.5}$$

Taking the limit as $c \to +\infty$ on both sides of (4.5), the Monotone Convergence Theorem yields

$$\int_{\mathbb{R}^2} \left(|\xi| + \eta^2\right) |\widehat{g}(\xi,\eta)|^2 d\xi d\eta \le \int_{\mathbb{R}^2} \left(|\xi| + \eta^2\right) |\widehat{f}(\xi,\eta)|^2 d\xi d\eta, \tag{4.6}$$

which shows that $|f| \in \mathscr{X}$ and $I(|f|) \le I(f)$. □

Corollary 4.9. *For $c > 0$, there is always a non-negative solitary-wave solution φ_c of the BO-ZK equation.*

Proof. Theorem 2.9 assures that there are solitary-wave solutions ψ, say. The last result shows that if $\psi \in M_{c,\lambda}$, then so is $\varphi = |\psi|$. □

If $p = \frac{k}{m}$ where m is odd and k and m relatively prime it follows from the formula

$$\varphi = \tfrac{1}{p+1} K * \varphi^{p+1}, \tag{4.7}$$

that if k is odd, then necessarily all solitary-wave solutions are non-negative. This is false if k is even, however. Indeed, in this case, if φ is a solitary wave, then so is $-\varphi$. Hence, when k is even, there are always at least two solitary-wave solutions, one positive and one negative. Of course, when k is even, it is also the case that $J(|f|) = J(f)$.

Lemma 4.10. *If $f \in \mathscr{Z}$ is non-negative, its Steiner symmetrizations $f^\sharp$ and f^* also lie in $\mathscr{Z}$. Moreover, $I(f^\sharp) \leq I(f)$ and $I(f^*) \leq I(f)$.*

Proof. Remark first that $K^\sharp = K = K^*$. The Riesz-Sobolev rearrangement inequality (see [17, 34, 49]) implies that

$$\int_{\mathbb{R}^4} f(x,y)f(s,t)K(x-s,y-t)ds\,dt\,dx\,dy \leq \int_{\mathbb{R}^4} f^\sharp(x,y)f^\sharp(s,t)K(x-s,y-t)ds\,dt\,dx\,dy.$$

In the Fourier transformed variables, this amounts to

$$\int_{\mathbb{R}^2} \widehat{K}(\xi,\eta)\left|\widehat{f}(\xi,\eta)\right|^2 d\xi d\eta \leq \int_{\mathbb{R}^2} \widehat{K}(\xi,\eta)\left|\widehat{f^\sharp}(\xi,\eta)\right|^2 d\xi d\eta.$$

On the other hand, the fact that symmetrization does not change the measure theoretic properties of f implies that

$$\|\widehat{f}\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)} = \|f^\sharp\|_{L^2(\mathbb{R}^2)} = \|\widehat{f^\sharp}\|_{L^2(\mathbb{R}^2)}.$$

This together with the analysis in Lemma 4.8 shows that $f^\sharp \in \mathscr{Z}$ and that $I(f^\sharp) \leq I(f)$. The same argument applies to f^*. □

Corollary 4.11. *There are non-negative, solitary-wave solutions of the BO-ZK equation* (1.1) *that are symmetric with respect to both the propagation direction and the transverse direction and are monotone decreasing in both $|x|$ and $|y|$.*

Proof. By Theorems 2.3 and 4.7, there is a non-negative function φ satisfying (1.7). Since Steiner symmetrization preserves the $L^{p+2}-$norm, it follows that $J(\varphi) = J(\varphi^\sharp) = J(\varphi^{\sharp *})$. On the other hand, because of Lemma 4.10, the double rearrangement $\varphi^{\sharp *}$ has the property that

$$I_{c,\lambda} \leq I_c(\varphi^{\sharp *}) \leq I_c(\varphi^\sharp) \leq I(\varphi) = I_{c,\lambda}.$$

Therefore, $\varphi^{\sharp *}$ is a non-negative solitary-wave solution of equation (1.1) which is symmetric with respect to both $\{x = 0\}$ and $\{y = 0\}$ and which is monotone decreasing with respect to both $|x|$ and $|y|$. □

Remark 4.12. One may also obtain symmetry properties of the solitary-wave solutions of (1.1) by using the reflection method and a unique continuation argument (see [45] and [26]).

5. Spatial Asymptotics

Attention is now turned to the spatial decay properties of the solitary-wave solutions of (1.1). In this analysis, we rely upon the ideas put forward in [13].

Lemma 5.1. *Let $j \in \mathbb{N}$. Suppose also that ℓ and m are two constants satisfying $0 < \ell < m - j$. Then there exists $C > 0$, depending only on ℓ and m, such that for all $\epsilon \in (0, 1]$,*

$$\int_{\mathbb{R}^j} \frac{|a|^\ell}{(1+\epsilon|a|)^m(1+|b-a|)^m}\,\mathrm{d}a \le \frac{C\,|b|^\ell}{(1+\epsilon|b|)^m}, \quad \forall\, b \in \mathbb{R}^j,\ |b| \ge 1, \tag{5.1}$$

$$\int_{\mathbb{R}^j} \frac{\mathrm{d}a}{(1+\epsilon|a|)^m(1+|b-a|)^m} \le \frac{C}{(1+\epsilon|b|)^m}, \qquad \forall\, b \in \mathbb{R}^j. \tag{5.2}$$

The proof of this elementary lemma is essentially the same as the proof of Lemma 3.1.1 in [13] (see also [24]).

Theorem 5.2. *Let φ be a solitary-wave solution of* (1.7).

(i) *For all $q \in (3/2, +\infty)$, $\ell \in [0, 1)$ $\varrho \ge 0$, $|x|^\ell|y|^\varrho\varphi(x, y) \in L^q\left(\mathbb{R}^2\right)$.*
(ii) *For all $q \in (3/2, +\infty)$ and any $\theta \in [0, 1)$, $|(x, y)|^\theta\varphi(x, y) \in L^q\left(\mathbb{R}^2\right)$.*
(iii) *And finally, $\varphi \in L^1\left(\mathbb{R}^2\right)$.*

Proof. (i) For $q \in (1, 3)$ and $1 - \frac{1}{q} < s_1 < 2 - \frac{1}{q}$, let $\ell \in [0, s_1 - 1 + \frac{1}{q})$. For $s_2 > 1 - \frac{1}{q}$, choose $\varrho \in [0, s_2 - 1 + \frac{1}{q})$. For $0 < \epsilon < 1$, define $\hbar_\epsilon$ by

$$\hbar_\epsilon(x, y) = \mathcal{A}(x, y)\ \varphi(x, y),$$

where

$$\mathcal{A}(x, y) = \frac{|x|^\ell|y|^\varrho}{(1+\epsilon|x|)^{s_1}(1+\epsilon|y|)^{s_2}}.$$

Then, by using the explicit representation of $\hbar_\epsilon$, it is straightforward to ascertain that $\hbar_\epsilon \in L^{q'}\left(\mathbb{R}^2\right)$, where $q' = \frac{q}{q-1}$. Hölder's inequality and (4.2) then imply that

$$|\varphi(x, y)| \le C(s_1, s_2, q)\Big(\int_{\mathbb{R}^2} |\mathcal{G}_{x,y}(z, w)|^{q'}\ dzdw\Big)^{\frac{1}{q'}},$$

where

$$\mathcal{G}_{x,y}(z, w) = \frac{g(\varphi)(z, w)}{\left(1+|x-z|\right)^{s_1}\left(1+|y-w|\right)^{s_2}},$$

$g(t) = \frac{t^{p+1}}{p+1}$ and $C := C(s_1, s_2, p) = \left\|(1+|x|)^{s_1}(1+|y|)^{s_2}K\right\|_{L^q(\mathbb{R}^2)} < \infty$. This last constant is finite thanks to Lemma 4.6. Since the solitary wave φ

converges to the rest state as $|(x,y)| \to +\infty$, it follows that for every $\delta > 0$, there exists $R_\delta > 1$ such that if $|(x,y)| \geq R_\delta$, then

$$\big|g(\varphi)(x,y)\big| \leq \delta|\varphi(x,y)|.$$

Another application of Hölder's inequality yields

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |h_\epsilon(x,y)|^{q'}dxdy &= \int_{\mathbb{R}^2\setminus B(0,R_\delta)} |h_\epsilon(x,y)|^{q'-r}\mathcal{A}^r g(x,y)|\varphi(x,y)|^r dxdy \\ &\leq C^r\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |h_\epsilon(x,y)|^{q'-r}\mathcal{A}^r(x,y)\,\|\mathcal{G}_{x,y}\|^r_{L^{q'}(\mathbb{R}^2)}(x,y)\,dxdy \\ &\leq C^r\|h_\epsilon\|^{q'-r}_{L^{q'}(\mathbb{R}^2\setminus B(0,R_\delta))}\Big\|\,\mathcal{A}\,\|\mathcal{G}_{x,y}\|_{L^{q'}(\mathbb{R}^2)}\Big\|^r_{L^{q'}(\mathbb{R}^2\setminus B(0,R_\delta))}.\end{aligned}$$

Because $h_\epsilon \in L^{q'}\left(\mathbb{R}^2\right)$, the latter inequality implies

$$\|h_\epsilon\|^r_{L^{q'}(\mathbb{R}^2\setminus B(0,R_\delta))} \leq C^r\Big\|\,\mathcal{A}\,\|\mathcal{G}_{x,y}\|_{L^{q'}(\mathbb{R}^2)}\Big\|^r_{L^{q'}(\mathbb{R}^2\setminus B(0,R_\delta))},$$

which is to say,

$$\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |h_\epsilon(x,y)|^{q'}dxdy \leq C^{q'}\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}^{q'}(x,y)\|\mathcal{G}_{x,y}\|^{q'}_{L^{q'}(\mathbb{R}^2)}dxdy.$$

Fubini's Theorem and Lemma 5.1 combine to reveal that

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}^{q'}(x,y)\|\mathcal{G}_{x,y}\|^{q'}_{L^{q'}(\mathbb{R}^2)}(x,y)\,dxdy &= \int_{\mathbb{R}^2}\big|g(\varphi)(z,w)\big|^{q'} \qquad (5.3)\\ &\times\Big(\int_{\mathbb{R}^2\setminus B(0,R_\delta)}\frac{\mathcal{A}^{q'}(x,y)}{(1+|x-z|)^{q's_1}(1+|y-w|)^{q's_2}}dxdy\Big)dzdw \\ \leq C\int_{\mathbb{R}^2\setminus B(0,R_\delta)}\big|g(\varphi)(z,w)\big|^{q'}\mathcal{A}^{q'}(z,w)\,dzdw &+ \int_{B(0,R_\delta)}\big|g(\varphi)(z,w)\big|^{q'} \\ &\times\Big(\int_{\mathbb{R}^2\setminus B(0,R_\delta)}\frac{\mathcal{A}^{q'}(x,y)}{(1+|x-z|)^{q's_1}(1+|y-w|)^{q's_2}}dxdy\Big)dzdw,\end{aligned}$$

where we used (5.1) (with $j = 1$) to show that for $|(z,w)|$ large,

$$\int_{\mathbb{R}^2\setminus B(0,R_\delta)}\frac{\mathcal{A}^{q'}(x,y)}{(1+|x-z|)^{q's_1}(1+|y-w|)^{q's_2}}dxdy \leq C\mathcal{A}^{q'}(z,w).$$

The second integral on the right-hand side of (5.3) is bounded by a constant, say C', depending on φ and R_δ, but independent of ϵ. Therefore, by using the fact that $|g(\varphi)(x,y)| \leq \delta|\varphi(x,y)|$ on $\mathbb{R}^2 \setminus B(0,R_\delta)$, there obtains

$$\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |h_\epsilon(x,y)|^{q'}dxdy \leq C^{q'}\Big(C\delta^{q'}\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |h_\epsilon(x,y)|^{q'}\,dxdy + C'\Big).$$

Choosing δ such that $C\delta C^{\frac{1}{q'}} < 1$, the last inequality entails that

$$\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{q'}\,dxdy \le C'', \tag{5.4}$$

where C'' is a constant independent of ϵ. Letting $\epsilon \to 0$ in (5.4) and applying Lebesgue's Dominated Convergence Theorem, one deduces

$$\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |x|^{\ell q'}|y|^{\varrho q'}|\varphi(x,y)|^{q'}\,dxdy \le C.$$

Hence, $|x|^\ell|y|^\varrho\varphi(x,y) \in L^{q'}\left(\mathbb{R}^2\right)$ with $q' = \frac{q}{q-1} \in (\frac{3}{2}, +\infty)$. This proves part (i) of the theorem.

(ii) This follows directly from (i).

(iii) Let $s > 1$ and $\mathcal{g}$, δ and R_δ be as defined in the proof of (i). For $\epsilon > 0$ let $\mathcal{A}$ be

$$\mathcal{A}_\epsilon(x,y) = \frac{1}{(1+\epsilon|(x,y)|)^s}.$$

Fubini's Theorem, Lemma 5.1 and the fact that φ, $\mathcal{A}_\epsilon \in L^2\left(\mathbb{R}^2\right)$ so that the product $\varphi\mathcal{A}_\epsilon \in L^1\left(\mathbb{R}^2\right)$ allow us to infer the inequalities

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\varphi(x,y)|\mathcal{A}_\epsilon(x,y)\,dxdy \\
&\le \int_{\mathbb{R}^2} \left|\mathcal{g}(\varphi)(z,w)\right| \Big(\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}_\epsilon(x,y)K(x-z,y-w)\,dxdy\Big)dzdw \\
&\le \int_{\mathbb{R}^2} \left|\mathcal{g}(\varphi)(z,w)\right| \Big(\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}_1^{-2}(x-z,y-w)K^2(x-z,y-w)\,dxdy\Big)^{\frac{1}{2}} \\
&\qquad \times \Big(\int_{\mathbb{R}^2\setminus B(0,R_\delta)} \mathcal{A}_1^2(x-z,y-w)\mathcal{A}_\epsilon^2(x,y)\;dxdy\Big)^{\frac{1}{2}} dzdw \\
&\le C(s)C^{\frac{1}{2}} \int_{\mathbb{R}^2} \left|\mathcal{g}(\varphi)(z,w)\right| \mathcal{A}_\epsilon(z,w)\,dzdw \\
&\le C(s)C^{\frac{1}{2}}\delta \int_{\mathbb{R}^2\setminus B(0,R_\delta)} |\varphi(z,w)|\mathcal{A}_\epsilon(z,w)\,dzdw \\
&\qquad + C(s)C^{\frac{1}{2}} \int_{B(0,R_\delta)} \left|\mathcal{g}(\varphi)(z,w)\right| dzdw.
\end{aligned}$$

Letting $\epsilon \to 0$, Fatou's lemma together with the restriction on δ leads to the conclusion that $\varphi \in L^1\left(\mathbb{R}^2\right)$. □

Theorem 5.2, identity (4.7) and the elementary inequality

$$|t|^\theta \leq C\left(|t-s|^\theta + |s|^\theta\right), \quad \text{for } \theta \geq 0. \tag{5.5}$$

imply the following.

Corollary 5.3. *Suppose that $\varphi \in L^\infty\left(\mathbb{R}^2\right)$ satisfies (1.7) and $\varphi \to 0$ at infinity. Then*

(i) $|x|^\ell |y|^\varrho \varphi(x,y) \in L^\infty\left(\mathbb{R}^2\right)$*, for all* $\ell \in [0,1)$ *and any* $\varrho \geq 0$*,*

(ii) $|(x,y)|^\theta \varphi(x,y) \in L^\infty\left(\mathbb{R}^2\right)$*, for all* $\theta \in [0,1)$*.*

The aim now is to display even stronger decay properties in the x-variable for solitary-wave solutions of the *BO-ZK* equation. These results are developed in a sequence of lemmas.

Lemma 5.4. $|x|^2|y|^\varrho K \in L^\infty\left(\mathbb{R}^2\right)$*, for any* $\varrho \geq 0$*.*

Proof. This is a straightforward consequence of the explicit form of K. □

In the next few results, φ always refers to a solitary-wave solution of the *BO-ZK* equation.

Corollary 5.5. *For any* ℓ *with* $0 \leq \ell \leq 2$ *and any* $\varrho \geq 0$*,* $|x|^\ell |y|^\varrho \varphi(x,y) \in L^\infty\left(\mathbb{R}^2\right)$*.*

Proof. The proof is based on a standard bootstrapping argument. Decay in the y-variable is not in question, so without loss of generality, take it that that $\varrho = 0$. Setting $\gamma_1 = \min\{2, p+1\}$ and making use of the inequality

$$|x|^{\gamma_1}|\varphi| \lesssim |x|^{\gamma_1}|K| * |g(\varphi)| + |K| * ||x|^{\gamma_1} |g(\varphi)||, \tag{5.6}$$

where $g(t) = \frac{t^{p+1}}{p+1}$, we obtain from Corollary 5.3, Lemma 5.4 and Theorem 5.2 that $|x|^{\gamma_1}\varphi \in L^\infty(\mathbb{R}^2)$. The proof is compete if $\gamma_1 = 2$. If $\gamma_1 < 2$, then define $\gamma_2 = \min\{2, (p+1)^2\}$ and repeat the above argument to show $|x|^{\gamma_2}\varphi \in L^\infty(\mathbb{R}^2)$. Continuing in this manner, one concludes that $|x|^2\varphi \in L^\infty(\mathbb{R}^2)$ after a finite number of steps. □

The following corollary follows from (5.5), Corollary 5.3 and Theorem 5.2.

Corollary 5.6. *(i)* $|x|^\ell |y|^\varrho \varphi(x,y) \in L^1\left(\mathbb{R}^2\right)$*, for all* $\ell \in [0,1)$ *and any* $\varrho \geq 0$*,*

(ii) $|(x,y)|^\theta \varphi(x,y) \in L^1\left(\mathbb{R}^2\right)$*, for all* $\theta \in [0,1)$*.*

Lemma 5.7. *For any* r *and* q *with* $1 \leq r, q < \infty$*, there is* $\sigma_0 > 0$ *such that for all* $\sigma \in [0, \sigma_0)$ *and* $s \in (\frac{1}{2} - \frac{1}{r} - \frac{1}{2q}, 2 - \frac{1}{r})$*, we have*

$$|x|^s e^{\sigma|y|} K \in L^r_x L^q_y(\mathbb{R}^2) \cap L^q_y L^r_x(\mathbb{R}^2).$$

Proof. It suffices to choose $\sigma_0 = \sqrt{\frac{c}{q}}$, where c is the wave velocity and use (4.4). □

The next result is a consequence of another of Young's inequalities, namely

$$\|f * g\|_{L^q_y L^r_x(\mathbb{R}^2)} \leq \|f\|_{L^{q_1}_y L^{r_1}_x(\mathbb{R}^2)} \|g\|_{L^{q_2}_y L^{r_2}_x(\mathbb{R}^2)},$$

where $1 \leq r, q, r_1, q_1, r_2, q_2 \leq \infty$, $1 + \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ and $1 + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Corollary 5.8. *Let q and r lie in the range $1 \leq r, q \leq +\infty$ and suppose that $\frac{1}{r} + \frac{1}{2q} > \frac{1}{2}$. Then, solitary-wave solutions φ of BO-ZK lie in the class $L^r_x L^q_y(\mathbb{R}^2) \cap L^q_y L^r_x(\mathbb{R}^2)$.*

Here is the main result about the spatial decay of the solitary-wave solutions.

Theorem 5.9. *Let φ be a solitary-wave solution of BO-ZK and let $\sigma_0 > 0$ be as in Lemma 5.7. Then, for any $\sigma \in [0, \sigma_0)$ and any s with $0 \leq s < 3/2$, it transpires that $|x|^s e^{\sigma|y|}\varphi(x,y) \in L^1\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right)$.*

Proof. Without loss of generality, assume that $s = 0$. By using Lemma 5.7 and the proof of Corollary 3.14 in [13], with natural modifications, it may be seen that there is a $\widetilde{\sigma} \geq \sigma_0$ such that $e^{\sigma|y|}\varphi(x,y) \in L^1\left(\mathbb{R}^2\right)$, for any $\sigma < \widetilde{\sigma}$. The inequality

$$|\varphi(x,y)|e^{\sigma|y|} \leq \int_{\mathbb{R}^2} |K(x-z, y-w)|e^{\sigma|y-w|}|\varphi(z,w)|e^{\sigma|w|}|\varphi(z,w)|^p \, dzdw, \tag{5.7}$$

and the facts $\varphi(x,y)e^{\sigma|y|} \in L^1(\mathbb{R}^2)$, $\varphi \in L^\infty(\mathbb{R}^2)$ and $K(x,y)e^{\sigma|y|} \in L^2(\mathbb{R}^2)$, for any $\sigma < \sigma_0$ imply that $\varphi(x,y)e^{\sigma|y|} \in L^\infty(\mathbb{R}^2)$, for the same range of σ. □

Finally, the following theorem deals with the analyticity of the solitary-wave solutions. Of course, for this, one needs to restrict p so that $z \mapsto z^p$ is analytic in a full neighborhood of the origin in $\mathbb{C}$.

Theorem 5.10. *Let $1 \leq p < 4$ be an integer and let φ be a solitary-wave solution of BO-ZK for this value of p. Then, there is a $\sigma > 0$ and a holomorphic function f of two variables z_1 and z_2, defined in the domain*

$$\mathcal{H}_\sigma = \left\{(z_1, z_2) \in \mathbb{C}^2 \; ; \; |\mathrm{Im}(z_1)| < \sigma, \; |\mathrm{Im}(z_2)| < \sigma\right\}$$

such that $f(x,y) = \varphi(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

Similar results are obtained by the same method for related evolution equations in [39] and [13]. Theory of this nature for dispersive equations made via analysis in Gevrey spaces analysis appear in [12] (and see also the references therein).

Proof. By the Cauchy-Schwarz inequality, Theorem 4.7 implies that $\widehat{\varphi} \in L^1\left(\mathbb{R}^2\right)$. Equation (1.7) implies in turn that

$$|\xi|\,|\widehat{\varphi}|\,(\xi,\eta) \leq \overbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}^{p+1}(\xi,\eta), \tag{5.8}$$

$$|\eta|\,|\widehat{\varphi}|\,(\xi,\eta) \leq \underbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}_{p+1}(\xi,\eta). \tag{5.9}$$

Denote by $\mathscr{T}_1$ the correspondence $\mathscr{T}_1(|\widehat{\varphi}|) = |\widehat{\varphi}|$ and, for $m \geq 1$, $\mathscr{T}_{m+1}(|\widehat{\varphi}|) = \mathscr{T}_m(|\widehat{\varphi}|) * |\widehat{\varphi}|$. A straightforward induction yields

$$r^m|\widehat{\varphi}|(\xi,\eta) \leq (m-1)!\,(p+1)^{m-1}\mathscr{T}_{mp+1}(|\widehat{\varphi}|)(\xi,\eta), \tag{5.10}$$

where $r = |(\xi,\eta)|$. It follows that

$$\begin{aligned} r^m|\widehat{\varphi}|(\xi,\eta) &\leq (m-1)!\,(p+1)^{m-1}\,\|\mathscr{T}_{mp+1}(|\widehat{\varphi}|)\|_{L^\infty(\mathbb{R}^2)} \\ &\leq (m-1)!\,(p+1)^{m-1}\,\|\mathscr{T}_{mp}(|\widehat{\varphi}|)\|_{L^2(\mathbb{R}^2)}\,\|\widehat{\varphi}\|_{L^2(\mathbb{R}^2)} \\ &\leq (m-1)!\,(p+1)^{m-1}\|\widehat{\varphi}\|^{mp}_{L^1(\mathbb{R}^2)}\|\widehat{\varphi}\|^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Let $a_m = \frac{(p+1)^{m-1}\|\widehat{\varphi}\|^{mp}_{L^1(\mathbb{R}^2)}\|\widehat{\varphi}\|^2_{L^2(\mathbb{R}^2)}}{m}$, so that $\frac{a_{m+1}}{a_m} \longrightarrow (p+1)\|\widehat{\varphi}\|^p_{L^1(\mathbb{R}^2)}$, as $m \to +\infty$. In consequence, the series $\sum_{m=0}^\infty t^m r^m|\widehat{\varphi}|(\xi,\eta)/m!$ converges uniformly in $L^\infty(\mathbb{R}^2)$ provided $0 < t < \sigma = \frac{1}{p+1}\|\widehat{\varphi}\|^{-p}_{L^1(\mathbb{R}^2)}$. Hence, $e^{tr}\widehat{\varphi}(\xi,\eta) \in L^\infty(\mathbb{R}^2)$, for $t < \sigma$. Now, define the function

$$f(z_1,z_2) = \int_{\mathbb{R}^2} e^{i(\xi z_1+\eta z_2)}\widehat{\varphi}(\xi,\eta)\,d\xi d\eta.$$

By the Paley-Wiener Theorem, f is well defined and analytic in $\mathcal{H}_\sigma$ while Plancherel's Theorem assures that $f(x,y) = \varphi(x,y)$ for all $(x,y) \in \mathbb{R}^2$. This proves the theorem. □

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