Set-Valued and Variational Analysis
https://doi.org/10.1007/s11228-021-00573-5

Erratum to: New Constraint Qualifications and Optimality Conditions for Second Order Cone Programs

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Received: 8 September 2020 / Accepted: 13 January 2021 / Published online: 07 April 2021

Abstract
In this note we show with a counter-example that all conditions proposed in Zhang and Zhang (Set-Valued Var. Anal **27**:693–712 2019) are not constraint qualifications for second-order cone programming.

Keywords Constraint qualifications · Optimality conditions · Second-order cone programming

This comment refers to the article available online at https://doi.org/10.1007/s11228-018-0487-2.

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Mathematics Subject Classification (2010) 90C22 · 90C46 · 90C30

We consider the (nonlinear) second-order cone programming problem

$$\begin{aligned} \text{Minimize} \quad & f(x), \\ \text{s.t.} \ \ & g_j(x) \in K_{m_j}, \ \ j = 1, \dots, \ell, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$, $j = 1, \dots, \ell$ are continuously differentiable and the second-order cone K_m is defined as $K_m := \{z := (z_0, \overline{z}) \in \mathbb{R} \times \mathbb{R}^{m-1} \mid z_0 \geq \|\overline{z}\|\}$ if $m > 1$ and $K_1 := \{z \in \mathbb{R} \mid z \geq 0\}$. Here $\| \cdot \|$ is the Euclidean norm.

Given a feasible point x^*, we denote by $I_0(x^*) := \{j \in \{1, \dots, \ell\} \mid g_j(x^*) = 0\}$ the index set of constraints at the vertex of the corresponding second-order cone and by $I_B(x^*) := \{j \in \{1, \dots, \ell\} \mid [g_j(x^*)]_0 = \|\overline{g_j(x^*)}\| > 0\}$ the index set of constraints at the non-zero boundary of the corresponding second-order cone. For $j \in I_B(x^*)$ we define $\phi_j(x) := \frac{1}{2}([g_j(x)]_0^2 - \|\overline{g_j(x)}\|^2)$, with $\nabla \phi_j(x) = J_{g_j}(x)^T R_{m_j} g_j(x)$, where $J_{g_j}(x)^T$ is the $n \times m_j$ transposed Jacobian of g_j and R_m is the $m \times m$ diagonal matriz with 1 at the first position and -1 at the remaining positions.

In [11], the authors present an extension of the classical constant rank constraint qualification (CRCQ, [9]) for the second-order cone programming problem (1). It reads as follows:

Definition 1 The Constant Rank Constraint Qualification (CRCQ) as defined in [11] holds at a feasible point x^* of (1) if there exists a neighborhood V of x^* such that for any index sets $J_1 \subseteq I_0(x^*)$ and $J_2 \subseteq I_B(x^*)$, the family of matrices whose rows are the union of $J_{g_j}(x)$, $j \in J_1$ and the vector rows $\nabla \phi_j(x)^T$, $j \in J_2$ has the same rank for all $x \in V$.

When $j \in I_B(x^*)$, the conic constraint $g_j(x) \in K_{m_j}$ can be locally replaced by the nonlinear constraint $\phi_j(x) \geq 0$, which is active at x^* (see e.g. [7, Section 4] for more details). Note also that for $j \in I_0(x^*)$ such that K_{m_j} is one-dimensional, the constraint $g_j(x) \in K_{m_j}$ is also a standard nonlinear constraint. Hence, the particularity of a second-order cone lies on the fact that one may have a "multi-dimensionally active" constraint $g_j(x^*) = 0$, which must be treated accordingly since these are tipically the constraints that are hard to tackle. The first impression one has when reading Definition 1 is that there is no special treatment for these active constraints. In particular, one would expect some regularity to be assumed for each constraint $g_j(x) \in K_{m_j}$ when $j \in I_0(x^*)$. To emphasize this last point, let us consider problem (1) with a single second-order cone, that is, $\ell = 1$, with constraint $g(x) \in K_{m_1}$. Let x^* be a feasible point such that $g(x^*) = 0$. According to Definition (1), CRCQ holds at x^* when the set of vectors given by all rows of $J_g(x)$ has constant rank, i.e., the full set of gradients $\{\nabla g_0(x), \dots, \nabla g_{m_1 - 1}(x)\}$ has constant rank, and no subset of these vectors is considered. However, it is well known that the classical CRCQ for nonlinear programming requires that all subsets of active constraints possesses the constant rank property.

Despite these considerations, the example given below shows that even a strengthen definition of CRCQ, that takes all these subsets into account, is not a constraint qualification. This thus invalidates all the results proved in [11]. Therein, the authors also propose a definition for the relaxed-CRCQ (RCRCQ, [10]) and for the Constant Rank of the Subspace Component (CRSC, [6]), which, being weaker than their definition of CRCQ, are not constraint qualifications either. In particular, the definition of RCRCQ is done in such a way that only the full set of *all* gradients in $I_0(x^*)$ is considered, while every subset $J_2 \subseteq I_B(x^*)$ is considered (namely, J_1 is taken to be fixed and equal to $I_0(x^*)$ in Definition 1). However, it is easy to see that this is not a constraint qualification, since when one considers only one-dimensional cones, and consequently (1) reduces to a nonlinear programming problem, RCRCQ reads identical to the so-called Weak Constant Rank property from [1], which is not a constraint qualification. Our counter-example is discussed in the sequel.

Consider the following problem of one-dimensional variable:

$$\begin{aligned} \text{Minimize} \quad & f(x) := -x, \\ \text{s.t.} \quad & g(x) \in K_2, \end{aligned} \tag{2}$$

with

$$g(x) = \begin{pmatrix} g_0(x) \\ g_1(x) \end{pmatrix} := \begin{pmatrix} x \\ x + x^2 \end{pmatrix}.$$

The unique feasible point is $x^* = 0$, thus, it is a global solution. Since $g(x^*) = 0$, the Karush-Kuhn-Tucker conditions for this problem are given by the existence of $\mu \in K_2$ such that $\nabla f(x^*) - J_g(x^*)^T \mu = 0$, that is

$$-1 - \mu_0 - \mu_1 = 0, \tag{3}$$

with $\mu = (\mu_0, \mu_1)^T \in K_2$, or, equivalently, $\mu_0 \geq |\mu_1|$. Thus, (3) can not hold and the Karush-Kuhn-Tucker conditions fail. On the other hand $J_g(x) = \begin{pmatrix} 1 \\ 1 + 2x \end{pmatrix}$ for all x. In particular, $\nabla g_0(x) = 1$ and $\nabla g_1(x) = 1 + 2x$ for all x. Thus, all subsets of gradients

$$\{\nabla g_0(x)\}, \{\nabla g_1(x)\}, \{\nabla g_0(x), \nabla g_1(x)\}$$

have constant rank equal to 1 for all x near x^*. This shows that the definition of CRCQ from [11] is not a constraint qualification, as this property is characterized by the fact that the Karush-Kuhn-Tucker conditions hold at any local minimizer.

We next briefly point out the possible mistake in the approach followed in [11]. It is based on the proof of RCRCQ from [10], which is also similar to [1]. It is shown therein that $\mathscr{L}(x^*) \subseteq \mathscr{T}(x^*)$, for apropriate definitions of the linearized cone $\mathscr{L}(x^*)$ and tangent cone $\mathscr{T}(x^*)$ for second-order cone programming, by means of applying an implicit function-type theorem (Lyusternik's theorem [8]). This theorem allows constructing a suitable tangent curve and can be applied provided the constant rank assumption holds true. However, in the nonlinear programming context, when constraint $g_j(x^*) = 0$ is analyzed, direction $d \in \mathscr{L}(x^*)$ must be orthogonal to the gradient $\nabla g_j(x^*)$ in order to ensure the existence of a tangent curve to $\{x \mid g_j(x) = 0\}$ along the direction d. This seems to be ignored in [11].

Instead of applying the implicit function approach, constant rank constraint qualifications may be defined using the approach of sequential optimality conditions [2]. See, for instance, [4–6]. For this, one would need a proper extension of the so-called Carathéodory Lemma (see, e.g., [5]), which permits rewriting a linear combination $y := \sum_{i=1}^{m} \lambda_i v_i$ with $\lambda_i \in \mathbb{R}$ and $v_i \in \mathbb{R}^n$ for all i in the following way: $y = \sum_{i \in I} \tilde{\lambda}_i v_i$ with $I \subseteq \{1, \ldots, m\}$, $\{v_i\}_{i \in I}$ linearly independent, and $\tilde{\lambda}_i$ with the same sign of λ_i for each i. In the case of second-order cones, for which the vector of scalars $(\alpha_i)_{i=1}^m$ belongs to the second-order cone K_m, one would want to rewrite the same vector y by only using a linearly independent subset of $\{v_i\}_{i=1}^m$ and such that the new scalars still belong to the cone. However, this is not possible in general as the following examples show.

Example 1 Take $y := \beta_0 v_0 + \beta_1 v_1 + \beta_2 v_2$, with $(\beta_0, \beta_1, \beta_2) := (\sqrt{2}, 1, 1) \in K_3$, $v_0 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. There is no way of rewriting y using new scalars $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) \in K_3$ such that $\hat{\beta}_i = 0$ for some $i = 0, 1, 2$.

In the case of more than one block of constraints ($\ell > 1$), even assuming more regularity for each block, a conic variant of Carathéodory's Lemma seems not possible to obtain.

Example 2 Take $y := \beta_0 v_0 + \beta_1 v_1 + \gamma_0 w_0 + \gamma_1 w_1$ with $(\beta_0, \beta_1) := (1, 1) \in K_2$, $(\gamma_0, \gamma_1) := (1, 1) \in K_2$, and vectors

$$v_0 := \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, v_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, w_0 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{and } w_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is not possible to rewrite y with new scalars $(\hat{\beta}_0, \hat{\beta}_1) \in K_2$, $(\hat{\gamma}_0, \hat{\gamma}_1) \in K_2$ in such a way that at least one component vanishes. Note that both $\{v_0, v_1\}$ and $\{w_0, w_1\}$ are linearly independent sets, but the necessity of dealing with the product of two second-order cones makes it impossible to fulfill the desired property.

We end this erratum with the following observation. Since it is well-known that linear second-order cone programs may possess duality gap, a definition of CRCQ could not be automatically satisfied by linear problems at the vertex. In [3], a naive proposition of CRCQ is presented where the "multi-dimensionally" active constraints are treated similarly to Robinson's CQ while the remaining constraints are treated similarly to CRCQ for nonlinear programming.

Acknowledgements This work was supported by CEPID-CeMEAI (FAPESP 2013/07375-0), FAPESP (grants 2018/24293-0, 2017/18308-2, and 2017/12187-9), CNPq (grants 301888/2017-5, 303427/2018-3, and 404656/2018-8), Grant-in-Aid for scientific research (C)(19K11840) from Japan Society for the Promotion of Science, and FONDECYT (grant 1201982) and Basal Program CMM-AFB 170001, both from ANID (Chile).

Declarations

Conflict of Interests The authors declare that they have no conflict of interest.

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