MS650A - 2S04 - Lista 7

(1) Resolva os seguintes problemas de Cauchy:

(a) $\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = 0, \\ u_t(x,0) = 1. \end{cases}$ (b) $\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = \sin x, \\ u_t(x,0) = x^2. \end{cases}$ (c) $\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = x^3, \\ u_t(x,0) = x. \end{cases}$ (d) $\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = x, \\ u_t(x,0) = \sin x. \end{cases}$

(e) $\begin{cases} u_{tt} - c^2 u_{xx} = x, \\ u(x,0) = 0, \\ u_t(x,0) = 3. \end{cases}$ (f) $\begin{cases} u_{tt} - c^2 u_{xx} = \mathrm{e}^x, \\ u(x,0) = 5, \\ u_t(x,0) = x^2. \end{cases}$ (g) $\begin{cases} u_{tt} - c^2 u_{xx} = x\mathrm{e}^t, \\ u(x,0) = \sin x, \\ u_t(x,0) = 0. \end{cases}$ (h) $\begin{cases} u_{xx} + 2u_{xy} - 3u_{yy} = 0, \\ u(x,0) = \sin x, \\ u_y(x,0) = x. \end{cases}$

(2) Exercícios do livro do Edmundo, capítulo 9, do PP 9.1 ao PP 9.40.

(3) Resolva os seguintes problemas:

(a) $\begin{cases} u_{tt} = c^2 u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = 8\sin^2 x, \\ u(0,t) = 0, \quad u(\pi,t) = 0. \end{cases}$ (b) $\begin{cases} u_{tt} = c^2 u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u(x,0) = x^3, \quad u_t(x,0) = 0, \\ u(0,t) = 0, \quad u_x(\pi,t) = 0. \end{cases}$

(4) Usando coordenadas polares, resolva os problemas

(a) $\begin{cases} \nabla^2 u = 0, \quad 1 < r < 2, \quad 0 < \theta < \pi, \\ u(1,\theta) = \sin\theta, \quad u(2,\theta) = 0, \\ u(r,0) = 0, \quad u(r,\pi) = 0. \end{cases}$ (b) $\begin{cases} \nabla^2 u = 0, \quad 1 < r < 2, \quad 0 < \theta < 2\pi, \\ u_r(1,\theta) = \sin\theta, \\ u_r(2,\theta) = 0. \end{cases}$

(c) $\begin{cases} \nabla^2 u = 0, \quad r < R, \quad 0 < \theta < \pi, \\ u_r(R,\theta) = 0, \quad u(r,0) = 0, \\ u(r,\pi) = 0. \end{cases}$

(5) Usando coordenadas cilíndricas, resolva os problemas

(a) $\begin{cases} \nabla^2 u = 0, \quad r < a,\ 0 \le \theta \le 2\pi,\ 0 < z < h, \\ u(a,\theta,z) = 0, \quad u(r,\theta,0) = 0, \\ u(r,\theta,h) = V(r). \end{cases}$ (b) $\begin{cases} \nabla^2 u = 0, \quad r < a, \\ u(a,\theta,z) = V, \quad 0 < \theta < \pi, \\ u(r,\theta,z) = -V, \quad \pi < \theta < 2\pi. \end{cases}$

(6) Usando coordenadas esféricas, resolva os problemas

(a) $\begin{cases} \nabla^2 u = 0, \quad r < a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi \\ u(a,\theta,\phi) = \cos^2\theta. \end{cases}$ (b) $\begin{cases} \nabla^2 u = \frac{1}{\kappa} u_t, \quad r < a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi, \\ u(r,\theta,\phi,0) = u_0(r), \quad t = 0, \\ \left(\frac{\partial u}{\partial r} + h(u - u_1)\right)_{r=a} = 0, \quad (h = \text{cte}). \end{cases}$

(7) Usando coordenadas cartesianas, resolva os problemas

(a) $\begin{cases} \nabla^2 u = 0, \quad 0 < x < a,\ 0 < y < b,\ 0 < z < c, \\ u(0,y,z) = \sin\frac{\pi y}{b}\sin\frac{\pi z}{c}, \quad u(a,y,z) = 0, \\ u(x,0,z) = u(x,b,z) = 0, \\ u(x,y,0) = u(x,y,c) = 0. \end{cases}$

$$
\text{(b)} \begin{cases} u_{tt} = c^2 \nabla^2 u, \quad 0 < x < 1,\, 0 < y < 1,\, 0 < z < 1,\, t > 0, \\ u(x,y,z,0) = \sin \pi x \sin \pi y \sin \pi z, \quad u_t(x,y,z,0) = 0, \\ u(0,y,z,t) = u(1,y,z,t) = 0, \\ u(x,0,z,t) = u(x,1,z,t) = 0, \\ u(x,y,0,t) = u(x,y,1,t) = 0. \end{cases}
$$

(8) Resolva a equação de onda não-homogênea

$$
\begin{cases} u_{tt} = c^2 u_{xx} + h, \quad 0 < x < l,\, t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \\ u(0,t) = 0, \quad u(l,t) = 0, \end{cases}
$$

fazendo a mudança de variável $u(x,t) = v(x,t) + U(x)$, e exigindo que $v(x,t)$ satisfaça a equação de onda homogênea e $U(x)$ satisfaça a equação envolvendo o termo não-homogêneo, ou seja, $c^2 U_{xx} + h = 0$, com condições de contorno homogêneas $U(0) = U(l) = 0$.

(9) As vibrações transversais de uma viga de comprimento l são descritas pela equação $u_{tt} + a^2 u_{xxxx} = 0$, $(0 < x < l,\, t > 0)$, onde a é uma constante. Resolva o problema

$$
\begin{cases} u_{tt} + a^2 u_{xxxx} = 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le l, \\ u(0,t) = u(l,t) = 0, \quad u_{xx}(0,t) = u_{xx}(l,t) = 0. \end{cases}
$$

(10) Resolva o problema (k é uma constante)

$$
\begin{cases} \nabla^4 u = k, \quad -a/2 < x < a/2,\, 0 < y < b, \\ u(-a/2,y) = u(a/2,y) = 0, \quad u_{xx}(-a/2,y) = u_{xx}(a/2,y) = 0, \\ u(x,0) = u(x,b) = 0, \quad u_{yy}(x,0) = u_{yy}(x,b) = 0. \end{cases}
$$

Algumas Respostas (Obs: sem garantia; use-as por sua conta e risco)

$$
\text{(3-a) } u = \cos(\pi c t/2)\sin(\pi x/2),\ \text{(3-b) } u = \sum_{n=1,3,5,\ldots}^{\infty} \frac{24}{\pi n^2}\left(\pi^2 - \frac{4}{n^2}\right)\sin(n\pi/2)\cos(nct/2)\sin(nx/2),
$$

$$
\text{(4-a) } u = ((4-r^2)/3r)\sin\theta,\ \text{(4-b) } u = -((4+r^2)/3r)\sin\theta + \text{constante},\ \text{(4-c) } u = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}R^{1-n}}{n^2} r^n \sin n\theta,
$$

$$
\text{(5-a) } u = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{mn} J_n(\alpha_{mn} r/a)\sinh(\alpha_{mn} z/a)\cos m\theta,\ A_{mn} = \frac{4\delta_{n0}}{a^2 J_1^2(\alpha_{0n})\sinh(\alpha_{0n} h/a)} \int_0^a V(r) J_0(\alpha_{0n} r/a) r\, dr,
$$

$$
\text{(5-b) } u = \frac{4V}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \sin n\theta,\ \text{(6-a) } u = 1/3 + (2r^2/3a^2)P_2(\cos\theta),\ \text{(6-b) } u = u_1 + \sum_{n=1}^{\infty} A_n \frac{\sin k_n r}{k_n r} \mathrm{e}^{-\kappa k_n^2 t},
$$

$$
A_n = \frac{4k_n^3 \int_0^a [u_0(r) - u_1] r^2 j_0(k_n r)\, dr}{2k_n a - \sin 2k_n a},\ \ \frac{\tan k_n a}{k_n a} = \frac{1}{1 - ha},\ j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x)
$$

$$
\text{(7-a) } u = \frac{\sinh[(\pi/b)^2 + (\pi/c)^2]^{1/2}(a-x)}{\sinh[(\pi/b)^2 + (\pi/c)^2]^{1/2} a} \sin\frac{\pi y}{b} \sin\frac{\pi z}{c},\ \text{(7-b) } u = \sin\pi x \sin\pi y \sin\pi z \cos\sqrt{3}\pi c t,
$$

$$
\text{(8) } u = -\sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{4l^2 h}{c^2\pi^3 n^3}\right)\cos\frac{n\pi c t}{l}\sin\frac{n\pi x}{l} + \frac{hx}{2c^2}(l-x),\ \text{(9) } u = \sum_{n=1}^{\infty} [a_n \cos a(n\pi/l)^2 t + b_n \sin a(n\pi/l)^2 t] \sin\frac{n\pi x}{l},
$$

$$
a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\, dx,\ b_n = \frac{2}{al}\left(\frac{l}{n\pi}\right)^2 \int_0^l g(x)\sin\frac{n\pi x}{l}\, dx,\ \text{(10) } u = \frac{4kb^4}{\pi^5} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5}\left[1 - \frac{v_n(x)}{1 + \cosh\frac{n\pi a}{b}}\right]\sin\frac{n\pi y}{b},
$$

$$
v_n(x) = 2\cosh\frac{n\pi a}{2b}\cosh\frac{n\pi x}{b} + \frac{n\pi a}{2b}\sinh\frac{n\pi a}{2b}\cosh\frac{n\pi x}{b} - \frac{n\pi x}{b}\sinh\frac{n\pi x}{b}\cosh\frac{n\pi a}{2b}
$$