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Integer programming and pricing revisited

H. P. WILLIAMS

Faculty of Mathematical Studies, University of Southampton, Southampton SO17 1BJ, UK

Three applications of duality are mentioned: mathematical, computational, and economic. One of the earliest attempts to produce a dual of an integer programme with economic interpretations was by Gomory & Baumol in 1960. This is described together with its economic properties and some refinements and corrections. A more recent integer programming dual due to Chvátal, whose main use to date has been computational, is then described. It is shown that this can be given an economic interpretation as a generalization of Gomory & Baumol's dual which rectifies some of the deficiencies of the latter. The computational problems of calculating Chvátal's dual are remarked on.

1. Introduction

A major achievement of Steven Vajda was to explain the implications of linear programming (LP) duality in a wide variety of seemingly unrelated contexts, e.g.. combinatorics and graph theory, flows in networks, game theory, and economics. Among his many books which discuss this, it is worth mentioning *Mathematical programming* [14]. Perhaps most surprising is its application to certain problems in combinatorics, e.g. Menger's theorem and the Konig–Egervary theorem, both of which are described in Ref. 14, since these problems belong to the realm of discrete mathematics and superficially appear to have nothing to do with the continuous mathematics of LP. Of course, where these problems are cast in an LP form, the optimal solutions are integral, which results in the relevance of LP.

For more general problems, however, this will not be the case, and the solution of the LP relaxation of an integer programming (IP) model will be fractional. A number of attempts have been made to extend the concepts of LP Duality to IP. References are given in Williams [18]. There it is suggested that there are three motives for doing this. Firstly the duality theory of LP is mathematically aesthetic and possesses a symmetry: *the dual of the dual gives the original model.* It has not been possible to recapture this property in any IP duals, to the author's knowledge. Secondly duality is computationally valuable. The fact that, if an LP problem is solvable, the optimal objective value of the dual equals that of the primal provides a tight *bound on the objective value* of the original model. IP duals usually fail to produce a tight bound (there is a 'duality gap') but the bounds produced reduce the combinatorial search. In the case of Chvátal's dual, the duality gap is closed. Thirdly LP duality is economically valuable for *the pricing of resources.* It is well known that the dual solution of an LP problem enables one to obtain shadow prices for the resources representing their marginal value. We will concentrate on this last aspect in relation to IP. A survey is given by Williams [15] and applications given in Williams [16].

The earliest attempt to construct an IP dual which could be used for pricing was by Gomory & Baumol (a mathematician and an economist) in Ref. 11. While it is impossible (not surprisingly) to recapture all the properties of the LP dual in their system, their approach does have considerable merits. It can also be refined. We briefly describe the economic properties of LP duality in Section 2. We discuss the Gomory–Baumol dual in Section 3, by means of a numerical example, and remark on its properties and defects. Standard results on LP duality will be assumed. From a computational point of view, perhaps the most satisfactory general IP dual is due to Chvátal [7] (which closes the duality gap). If viewed economically, this dual can be seen as a natural extension of the Gomory–Baumol dual. It illustrates the difficulties inherent in any system of pricing in IP. Chvátal's dual is described in Section 4. It is shown how, if simplified, it reduces to the Gomory–Baumol dual.

Both the IP duals discussed in this paper only apply to pure IP. Probably of more economic importance is mixed IP. This frequently arises when fixed, as well as variable, costs are involved in a problem. The definition of a satisfactory dual with useful economic interpretations is even more difficult. It is, however, wrong to dismiss peremptorily the concept of duality here. As Appa [1] points out, for a number of mixed IP models, if we ultimately fix the values of the IP variables, the resultant IP model will, by virtue of its dual, have structural properties which are general. Therefore LP duality still gives us valuable information. Bender's decomposition [4] is an algorithmic approach to solving IP models by successive fixing of the integer variables and then making use of the dual solution of the resultant LP problem.

2. The economic properties of LP duality

In order to illustrate this paper, we will consider the following model.

$$
\begin{array}{llrr}
\text{(P1)} & \text{Maximize} & 12x_1 + 4x_2 + 3x_2 & \\
& \text{subject to} & 2x_1 + 3x_2 \qquad\quad \leqslant 7 & (1)\\
& & x_2 + 4x_3 \leqslant 8 & (2)\\
& & 3x_1 \qquad + \; x_3 \leqslant 5 & (3)\\
& & x_1 \;,\; x_2 \;,\; x_3 \geqslant 0 & (4)\\
& & x_1 \;,\; x_2 \;,\; x_3 \text{ integer.} & (5)
\end{array}
$$

Here x_1, x_2, x_3 will be thought of as the quantities to be made of three products subject to three resources modelled by (1), (2), and (3). The objective is to maximize profit. If we ignore the integrality conditions (5), we obtain the *LP relaxation*. This has the optimal solution

$$x_1 = 1\tfrac{2}{3}, \qquad x_2 = 1\tfrac{2}{9}, \qquad x_3 = 0, \qquad \text{profit} = 24\tfrac{8}{9}. \tag{6}$$

This model has a well defined (and unique) dual. The optimal dual values associated with constraints (1), (2), (3) are respectively $1\frac{1}{3}, 0, 3\frac{1}{9}$.

Economically this illustrates a number of general properties. These properties are described by Koopmans [12] in his Nobel lecture as conditions for optimal value production.

(LP1) *Every process (product) in use makes a zero profit.* For example, products 1 and 3 have imputed costs of $2 \times 1\frac{1}{3} + 3 \times 3\frac{1}{9} = 12$ and $3 \times 1\frac{1}{3} + 1 \times 0 = 4$ respectively, exactly balancing their unit profit contributions. This property is a result of the orthogonality relationship between primal and dual solutions.

(LP2) *No process (product) makes a positive profit.* For example, product 3 has an inputed cost of $4 \times 0 + 1 \times 3\frac{1}{9} = 3\frac{1}{9}$ which exceeds the unit profit contribution. This property is a result of the dual ($\geqslant$) constraint associated with product 3.

(LP3) *Every good (resource) used below the limit of its availability has a zero price.* It can be seen that resources (1) and (3) are used to their limit, but that only $1\frac{2}{9}$ units of (2) are used. The dual value associated with (2) is 0. It is known by economists as a 'free good'. If its supply is increased, no further profit results. Again this property is a result of the orthogonality relationship. If a resource has a positive price, the constraint is satisfied as an equality (no slack).

(LP4) *No good (resource) has a negative price.* It can be seen that the optimal dual values are non-negative. This is a result of the non-negativity conditions on the variables in the dual model.

(LP5) *The optimal value of the outputs equals the optimal value of the input.* The total imputed values of the resources is $7 \times 1\frac{1}{3} + 8 \times 0 + 5 \times 3\frac{1}{9} = 24\frac{8}{9}$ which equals the maximum profit attainable. This is the main result of the duality theorem of LP.

Remark. If the model has a unique solution, then the dual solution can be used to obtain the primal solution by virtue of properties LP1 and LP3. We ignore products not produced and zero-valued goods, leaving a uniquely solvable set of equations. □

(LP6) *In the absence of degeneracy, the dual values represent the effects of marginal changes in resource levels.* For example, changing resource 1 from 7 to $7 + \Delta$ increases profit by $\frac{4}{3}\Delta$.

(LP7) *In a model with n variables, at most n constraints will have positive dual values.* It is convenient in this context to include the non-negativity constraints (4) in the form $-x_j \leqslant 0$; they will be valued by their reduced costs. This result is attributed to Carathéodory [6]. In the absence of degeneracy, exactly n constraints will have positive valuations. These constraints can be shown to be *binding* in the sense that, if any one of them is removed, the optimal solution will change. If degeneracy is present, there will still be n binding constraints, but with flexibility over the choice of which n. In the example, with 3 variables, constraints (1) and (3) have positive valuations together with $-x_3 \leqslant 0$ which is valued at $\frac{1}{9}$. It can be shown that adjusting the RHS coefficient on any of these constraints results in a change in the optimal solution.

3. Gomory & Baumol's dual

In contrast to the optimal (real) solution (6) of the LP relaxation of model P1, the optimal *integer* solution is

$$x_1 = 1, \qquad x_2 = 1, \qquad x_3 = 1, \qquad \text{profit} = 19. \tag{7}$$

Gomory & Baumol seek a set of prices on the constraints which would be compatible with solution (7) and preserve as many of the properties 1 to 7 as possible.

It is easy to see that it will be impossible to preserve property 5 by a system of prices alone, since (by definition) the optimal LP dual values are those which *minimize* the total imputed value of the resources which will be equal to the optimal LP objective value. For the example, the minimum imputed valuation cannot therefore be less than $24\frac{8}{9}$, i.e. we cannot reduce it to the optimal value of 19 of the outputs; thus there is a *duality gap*.

Gomory's [10] algorithm for IP generates cutting planes which cut off fractional solutions and ultimately leads to an LP problem, with extra constraints, yielding the optimal IP solution. Gomory & Baumol initially suggest valuing these extra constraints as well leading to an LP problem with all the desirable properties. They point out, however, that these extra constraints are artificial and usually cannot be given a sensible economic interpretation. Therefore the dual values of each of the artificial constraints are 'imputed back' to the original constraints. Each of the new constraints arises, in Gomory's algorithm, from a linear combination of the original constraints (together with an integer rounding). For example, if a new constraint has a dual value of π and this constraint arises from multiples $\lambda_1, \ldots, \lambda_m$ of the original constraint values $\pi\lambda_1, \ldots, \pi\lambda_m$, then these values will be imputed back to the original constraints respectively (together with dual values arising from other new constraints and direct dual values on the original constraints).

Gomory & Baumol remark on the lack of uniqueness of this approach, depending as it does on the alternative cutting planes which might be generated. This concern seems unwarranted, since less arbitrariness may be obtained by replacing the original constraints by the *facet-defining* constraints for the convex hull of feasible integer solutions. Then the linear expression in each of the facet-defining constraints can be expressed as a non-negative linear combination of the linear expressions in the original constraints. Although there could still be alternative representations, this is no more serious than degeneracy which arises to produce alternative LP dual solutions.

In order to illustrate this version of Gomory & Baumol's dual on the example P1, we give its convex-hull representation by means of facet-defining constraints.

$$
\begin{array}{llrr}
\text{(P2)} & \text{Maximize} & 12x_1 + 4x_x + 3x_2 & \\
& \text{subject to} & x_1 + x_2 + x_3 \leqslant 3 & (8)\\
& & x_1 + x_2 \leqslant 2 & (9)\\
& & x_2 + 2x_3 \leqslant 4 & (10)\\
& & x_1 \leqslant 1 & (11)\\
& & -x_1 \leqslant 0 & (12)\\
& & -x_2 \leqslant 0 & (13)\\
& & -x_3 \leqslant 0. & (14)
\end{array}
$$

Solving P2 as an LP problem produces the solution (7) with dual values of 3, 1, 0, 8, 0, 0, and 0 respectively on (8) to (14).

It is worth pointing out that Gomory & Baumol make the assertion that these dual values 'will themselves be integer' as is the case here. This is, in fact, false in general. The following model defines a convex hull of integer points, but the optimal dual values for the constraints are $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, 0, 0, and 0 respectively.

$$
\begin{array}{ll}
\text{Maximize} & x_1 + x_2 + x_3 \\
\text{subject to} & x_1 + x_2 \leqslant 2 \\
& x_2 + x_3 \leqslant 2 \\
& x_1 + x_3 \leqslant 2 \\
& -x_1 \leqslant 0 \\
& -x_2 \leqslant 0 \\
& -x_3 \leqslant 0.
\end{array}
$$

Circumstances in which the dual, as well as the primal, solution to an IP model is integer are discussed by Giles & Pulleyblank [9].

Since the facet-defining constraints (8), (9), and (11) have positive dual values, we impute these back to the original constraints. There are a number of ways of representing the linear expression in a facet constraint, such as (8), as a non-negative combination of the original constraints. The representation used in Gomory & Baumol's method becomes apparent when the Chvátal dual is explained in Section 4. For our purposes here, we observe that the expression $x_1 + x_2 + x_3$ can be represented as

$$\tfrac{1}{4}(2x_1 + 3x_2) + \tfrac{1}{4}(x_2 + 4x_3) + \tfrac{1}{6}(3x_1 + x_3) + \tfrac{1}{6}(-x_3).$$

Therefore the dual value of 3 on (8) is imputed back as $3 \times \frac{1}{4} = \frac{3}{4}$ to (1), $3 \times \frac{1}{4} = \frac{3}{4}$ to (2), $3 \times \frac{1}{6} = \frac{1}{2}$ to (3) and $3 \times \frac{1}{6} = \frac{1}{2}$ to the non-negativity condition $-x_3 \leqslant 0$. Similarly the expression $x_1 + x_2$ can be represented as

$$\tfrac{1}{3}(2x_1 + 3x_2) + \tfrac{1}{9}(3x_1 + x_3) + \tfrac{1}{9}(-x_3).$$

Therefore the dual value of 1 on (9) is imputed back as $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{9}$ respectively to (1), (3), and $-x_3 \leqslant 0$. Finally the expression x_1 can be represented as

$$\tfrac{1}{3}(3x_1 + x_3) + \tfrac{1}{3}(-x_3).$$

Therefore the dual value of 8 on (11) is imputed back as $\frac{8}{3}$ and $\frac{8}{3}$ to (3) and $-x_3 \leqslant 0$. The resultant prices are therefore $\frac{3}{4} + \frac{1}{3} = 1\frac{1}{12}$ for (1), $\frac{3}{4}$ for (2), $\frac{1}{2} + \frac{1}{9} + \frac{8}{3} = 3\frac{5}{18}$ for (3), and $\frac{1}{2} + \frac{1}{9} + \frac{8}{3} = 3\frac{5}{18}$ for $-x_3 \leqslant 0$.

These prices exhibit a number of properties which are general and are remarked on by Gomory & Baumol. We give them an IP1 etc. to indicate the correspondence, or otherwise, with the LP properties in Section 2.

(IP1) *Every process (product) in use makes a zero profit.* It can be checked that products 1 to 2 have imputed costs equal to their unit profits but that, in relation to product 3, the price of $3\frac{5}{18}$ on $-x_3 \leqslant 0$ subtracts from the imputed cost to balance the unit profit contribution of 3 exactly. Gomory & Baumol suggest interpreting such a price as a subsidy to the product to compensate for its enforced integrality.

(IP2) *No process (product) makes a positive profit.* This is the case in the example by virtue of IP1, since all products are made at zero profit. If any were not made, then they could have a negative imputed profit.

(IP3) *If a good (resource) has a zero price, it is a free good and can be increased arbitrarily without affecting the optimal solution.* This property is a weakening of LP3.

Remark. In the example, all resources are underused in the optimal integer solution, but all have positive prices. This is not unreasonable since resources may not be used up completely by virtue of the 'lumpy' nature of the input requirements. If, however, a resource has a zero price (not illustrated by this example) then we can be sure it is non-binding in the sense that its removal will not affect the optimal solution. The converse is not, however, the case. It is possible for a resource to be given a positive valuation, yet still represent a free good in the sense that increasing its availability without limit (or removing it) will not affect the optimal solution. This is not a serious defect, since it also arises in LP when a solution is degenerate. When this happens in LP, we have alternate dual values which give two-valued shadow prices (this is discussed in Williams [17]). The upper shadow price on a free good would then be zero. A similar remedy can be applied to Gomory & Baumol's prices. Some authors, e.g. Alcaly & Klevorick [2], however, make a considerable issue of this 'defect'.

Another consequence of the fact that positively priced constraints may not be satisfied as equalities is the fact that a product may be made in the optimal solution, yet its non-negativity constraint $-x_j \leqslant 0$ receives a positive price. For the example, $x_3 = 1$ in the optimal solution, yet $-x_3 \leqslant 0$ has a positive price (subsidy). □

(IP4) *No good (resource) has a negative price.* This feature is preserved and apparent in the example.

(IP5) *The optimal value of the outputs does not generally equal the optimal value of the inputs.* It has already been pointed out that there will be a duality gap, and the prices will overvalue the resources. If, however, we restrict ourselves to only those resources used, then we can reduce the duality gap. In the example, the used qualities of the resources (1), (2), and (3) are respectively 5, 5, and 4 which give a total valuation of

$$5 \times 1\tfrac{1}{12} + 5 \times \tfrac{3}{4} + 4 \times \tfrac{59}{18} = 22\tfrac{5}{18}$$

for the used inputs.

(IP6) *Marginal changes in the resource levels will not result in continuously changing solutions.* This arises from the 'lumpy' nature of the outputs. Therefore the prices cannot be interpreted as the effect of marginal changes. From the discussion in Section 4, however, it will become apparent that they represent average rates of change in the profit above certain threshold resource levels, It is not possible to recover the optimal solution so simply from the prices. Although negatively valued products can be ignored together with zero-valued goods, the resultant system is a set of *inequalities*, since it does not follow that positively priced resources will be used to capacity.

(IP7) *In a model with n variables, at most $2^n - 1$ constraints will be binding (in the sense of nonredundant) in the optimal solution.* This was shown by a theorem of Bell [3] and explains the result in the three-variable example where more than three of the constraints are positively priced.

Gomory & Baumol also give the result that, if the same set of only n constraints are binding for the IP optimal solution, then the resultant prices will be the same as for the LP case. Obviously this does not apply to the example above. Therefore we adapt that example for a different set of RHS coefficients.

$$\begin{array}{lll} \text{(P1)} & \text{Maximize} & 12x_1 + 4x_2 + 3x_3 \\ & \text{subject to} & 2x_1 + 3x_2 \leqslant 15 \qquad (19) \\ & & x_2 + 4x_3 \leqslant 26 \qquad (20) \\ & & 3x_1 + x_3 \leqslant 31 \qquad (21) \\ & & x_1,\ x_2,\ x_3 \leqslant 0 \qquad (22) \\ & & x_1,\ x_2,\ x_3 \text{ integer.} \qquad (23) \end{array}$$

The optimal solution of the LP relaxation is

$$x_1 = 7\tfrac{1}{2}, \qquad x_2 = 0, \qquad x_3 = 6\tfrac{1}{2}, \qquad \text{profit} = 109\tfrac{1}{2}, \tag{24}$$

with dual values of 6, $\frac{3}{4}$, 0 on (19), (20), (21) and a reduced cost of $14\frac{3}{4}$ on $x_2 \geqslant 0$. The facet-defining constraints for the convex hull of feasible integer solutions are

$$2x_1 + 3x_2 \qquad \leqslant 15 \tag{25}$$
$$x_1 + x_2 \qquad \leqslant 7 \tag{26}$$
$$2x_1 + 3x_2 + x_3 \leqslant 20 \tag{27}$$
$$x_2 + 3x_3 \leqslant 20 \tag{28}$$
$$x_3 \leqslant 6 \tag{29}$$
$$-x_1 \qquad \leqslant 0 \tag{30}$$
$$-x_2 \qquad \leqslant 0 \tag{31}$$
$$-x_3 \leqslant 0. \tag{32}$$

Optimizing the objective with respect to these yields the integer solution

$$x_1 = 7, \qquad x_2 = 0, \qquad x_3 = 6, \qquad \text{profit} = 102, \tag{33}$$

with positive dual values of 12, 3, and 8 on (26), (29), and (31).

The linear expression in (26) arises as half of (19) and half of $-x_2 \leqslant 0$. That on (29) arises as a quarter of (20) and a quarter of $-x_2 \leqslant 0$. Imputing the dual values back gives prices of $12 \times \frac{1}{2} = 6$ on (19), $3 \times \frac{1}{4} = \frac{3}{4}$ on (20), and $8 + 12 \times \frac{1}{2} + 3 \times \frac{1}{4} = 14\frac{3}{4}$ on $-x_2 \leqslant 0$. These are clearly the same as for the LP case.

In this example, product 2 is a 'priced out' by these values on the resources. Also the value of the inputs *used* is equal to the imputed value of the total output. The IP model P3 is therefore a particularly simple model whose solution can be deduced fairly easily. It is sometimes referred to as an IP over a cone, since the only non-redundant constraints are those associated with the cone of constraints which are binding at the LP optimum.

4. The value function of an integer programme and Chvátal's dual

In order to explain Chvátal's dual it is convenient first to describe the *value functions* and *consistency testers* of P1 and P2 for general coefficients. The value function is the optimal objective value as a function of the RHS coefficients, and the consistency tester is the condition for feasibility as a function of the RHS coefficients. These concepts are explained in Blair & Jeroslow [5].

We consider the general model (with variable RHS coefficients) in the form

(P1) Maximize $12x_1 + 4x_2 + 3x_3$

subject to
$$2x_1 + 3x_2 \qquad \leqslant b_1 \tag{34}$$
$$x_2 + 4x_3 \leqslant b_2 \tag{35}$$
$$3x_1 \qquad + x_3 \leqslant b_3 \tag{36}$$
$$-x_1 \qquad \leqslant b_4 \tag{37}$$
$$-x_2 \qquad \leqslant b_5 \tag{38}$$
$$-x_3 \leqslant b_6 \tag{39}$$
$$x_1, \; x_2, \; x_3 \text{ integer}. \tag{40}$$

In order to avoid the unnecessary complexity of an extra rounding operation, we will assume the coefficients b_i are integer. The value function of the LP relaxation is

$$\begin{aligned}\min\{&\tfrac{1}{9}(12b_1+28b_3+b_6),\ \tfrac{1}{2}(3b_1+6b_3+b_5),\\ &\tfrac{1}{4}(24b_1+3b_2+59b_5),\ (4b_2+4b_3+17b_6)\}.\end{aligned} \tag{41}$$

The coefficients in the above terms correspond to the vertices of the dual polytope. The consistency tester is

$$\min\{b_1+2b_4+3b_5,\ b_2+b_5+4b_6,\ b_3+3b_4+b_6\} \geqslant 0. \tag{42}$$

The coefficients in these terms correspond to the extreme rays of the dual polytope.

It can be checked that, for the LP relaxation of P4, setting $b_1 = 7$, $b_2 = 8$, $b_3 = 5$, and $b_4 = b_5 = b_6 = 0$ gives consistency and an optimal objective value of $24\frac{8}{9}$. This arises from the first term in (41). The coefficients of b_1, b_3, and b_6 are the dual values of (1), (3), and the reduced cost of x_3 respectively. For the LP relaxation of P4, setting $b_1 = 15, b_2 = 26, b_3 = 31, b_4 = b_5 = b_6 = 0$ gives consistency and an optimal objective value of $109\frac{1}{2}$. This arises from the third term in (41). The coefficients of b_1 and b_2 in this term are the dual values, and the coefficient of b_5 is the reduced cost of x_2. Clearly the optimal objective value for the three-variable model depends (by a theorem usually attributed to Carathéodory [6]) on three of the constraints which are therefore binding. Which three are given by the minimum term in (41).

For an IP model, the form of the value function and consistency tester is more complex. The analogous expressions for (41) and (42) are *Gomory functions*. These consist of the minimum of a (finite) set of *Chvátal functions*. Chvátal functions consist of non-negative linear combinations of the arguments together with (for a maximization) the integer round-down operation (denoted by $\lfloor\bullet\rfloor$). In practice the expressions are very complex, and there may be a very large number of them. We therefore only partially illustrate the value function and consistency tester for P4 using some of the component Chvátal functions. There is no significance in those which have been chosen for this illustrative purpose. We emphasize again that producing *all* the Chvátal functions is prohibitive in both space and time. The value function takes the form

$$\begin{aligned}&\min\{\lfloor\tfrac{1}{9}(12b_1+28b_3+b_6)\rfloor,\lfloor\tfrac{1}{2}(3b_1+6b_3+b_5)\rfloor,\\ &\qquad 12\lfloor\tfrac{1}{2}(b_1+b_5)\rfloor+3\lfloor\tfrac{1}{4}(b_2+b_5)\rfloor+8b_5, 4b_2+4b_3+17b_6,\\ &\lfloor\tfrac{1}{9}(3b_1+b_3+b_6)\rfloor+8\lfloor\tfrac{1}{3}(b_3+b_6)\rfloor+\lfloor 3\lfloor\tfrac{1}{4}(b_2+3\lfloor\tfrac{1}{9}(3b_1+b_3+b_6)\rfloor+\lfloor\tfrac{1}{3}(b_3+b_6)\rfloor\rfloor,\\ &\qquad\qquad \ldots\},\end{aligned} \tag{43}$$

and the consistency tester (in this case) is the same as (42).

For a *specific* set of values of b_i, the term in (43) giving the minimum value is the *Chvátal dual* solution. Notice that it is a function of those b_i which correspond to binding constraints. The way in which the optimal objective value changes with changes in the RHS values b (within certain ranges) is defined by this (discontinuous) function. If the integer round-down operators are deleted, then a Chvátal function becomes a linear expression with non-negative multipliers on the b_i. These multipliers are the Gomory–Baumol prices.

For example, in P1, where $b_1 = 7$, $b_2 = 8$, $b_3 = 5$, and $b_4 = b_5 = b_6$, the optimal objective value arises from the fifth term in (43) demonstrating that (34), (35), (36), and (39) are all binding constraints with respective multipliers of $1\frac{1}{12}, \frac{3}{4}, 3\frac{5}{8}$, and $3\frac{5}{8}$. The Chvátal function, however, goes further than the Gomory–Baumol prices. By means of the $\lfloor \bullet \rfloor$ operators it *closes* the duality gap.

For the example P3, where $b_1 = 15$, $b_2 = 26$, $b_3 = 31$, and $b_4 = b_5 = b_6 = 0$, the optimal objective value arises from the third term of (43), demonstrating that only the three constraints (34), (35), and (38) are binding, with respective prices of 6, $\frac{3}{4}$, and $14\frac{3}{4}$. If the $\lfloor \bullet \rfloor$ operators are removed from this term, it 'collapses' into the third term in (41), so demonstrating why the Gomory–Baumol prices, in this case, are the same as the LP dual values.

The first four terms in (43) all reduce to the first four terms in (41) when the $\lfloor \bullet \rfloor$ operators are removed. These correspond to all the dual vertices of the LP relaxation of P4. It is relatively easy to construct the corresponding Chvátal functions in (43) corresponding to these terms, as is done in Williams [20]. The set of n constraints which are binding at the LP solution to an n-variable model forms a cone. It is also shown in [20] that the value function for an IP function over this cone of constraints is the corresponding Chvátal function (not a Gomory function).

To calculate the 'non-cone' terms in the value function (such as the fifth term in (43)) is much more difficult.

5. Conclusions

Although Chvátal's dual has been developed largely for computational reasons and to provide a unifying treatment of cutting planes, we argue that it does have a useful economic interpretation. This is best illustrated through the value-function representation. This illustrates how the optimal objective value changes in a discontinuous way with the RHS coefficients.

A number of qualitative results can be shown as a result of the form of the value function for an IP problem. It can be shown (e.g. Rhodes & Willliams [13]) that, for sufficiently large values of the b_i, the optimal objective value of the IP problem will arise from a Chvátal term in the Gomory function corresponding to the cone of constraints which are binding at the LP optimum. Hence, for sufficiently large b_i, the Gomory–Baumol prices will become the prices of the LP relaxation. What is more, the average rate of change of the objective value beyond these b_i will be given by these prices. This result is well known (e.g. Garfinkel & Nemhauser [8]). The optimal objective value is also ultimately uniformly shift-periodic with respect to each of the RHS coefficients; i.e. it will alter in a cyclic manner.

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