Mr. Stoelting's second point is indeed important. In developing the years per stroke numbers in the section titled "Arrester Discharge Current Magnitude," an isokeraunic level of 30 was assumed; that is, it was assumed that 100 strokes per 100 miles per year would impinge on the line. This assumption is also used in developing the number of strokes and outages noted in the disscussion in the first and second paragraphs under the section titled "Results."

We agree with Mr. Stoelting that the degree of protection is a function of the storm frequency or isokeraunic level. As pointed out, in areas of low isokeraunic level, protection of equipment may be uneconomical.

Power Generation Scheduling by Integer Programming—Development of Theory

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Summary: Power generation scheduling involves the selection of units to be placed in operation and the allocation of the load between these operating units. This paper presents the formulation of the economic scheduling problem as an integer program taking into account the discontinuous input–output characteristics and start-up costs of the generators. A recently developed method for solving integer programs, linear programs with whole number answers required, is successfully applied to the solution of the scheduling problem.

THE ECONOMIC scheduling of electric generators is taken here to include both the selection of the units to operate at any time and the allocation of the power demand among these operating units. Knowledge of the minimum cost schedule for a day would provide a measure of the performance of the present selection and dispatching methods and suggest ways to improve future schedules to reduce the cost of electric energy.

The actual allocation of the load among the operating units, often termed the economic dispatch problem, has been extensively investigated. At first only the incremental costs of power from the generators were considered.[1–3] More recently the effects of transmission losses in the form of penalty factors have been included.[4] The mathematical theory of this portion of the scheduling problem is well developed.

At present the selection of the generators to operate during any time interval is done, not by analytical methods, but by using a priority list which ranks the generators in the order in which they are to be started. This list is based on the average cost of energy from each unit and other individual considerations such as location, type of steam cycle, etc. The shape of the daily load cycle only can be kept in mind as the priority list is made up. Recent investigations of generator start-up and shutdown include the use of digital computers to search over the many possible combinations of generators and start-up rules to seek a minimum cost schedule.[5]

This paper will describe how the economic scheduling problem may be formulated as an integer program. An integer program is a plan of operation in which all the quantities must be whole numbers. For this problem it means that whole numbers of generators will be scheduled in the solution. The integer program formulation includes the following:

1. The discontinuous power output characteristics of the generators, i.e., either there is no power output, or a minimum output.
2. The costs of starting and shutting down each unit.
3. The dispatch of the load by incremental costs.

The recent work of R. E. Gomory on the solution of integer programs provides a direct algorithm for the analytical solution of the generator scheduling problem.[6] In this paper his all-integer method will be referred to as the dual Euclidean method. The basic steps of this method will be described in the following paragraphs.

Mathematical Formulation

Steam-turbine generating units are subject to at least three types of nonlinearities in the dollar-per-hour input to power output characteristics. First, a cost is incurred when the turbine is started; second, the output is constrained to lie between a minimum and maximum rating; and third, the input–output relation is complicated by the presence of valve loops. In order to demonstrate how the economic scheduling of machines with these characteristics may be formulated as an integer programming problem, an illustrative example will be used.

Assume that we are given a power system with just two generators. The start-up costs, minimum output costs, and incremental costs for these two units are given in Table I. Fig. 1 is a graph of the incremental cost characteristics given in Table I. Such step char-

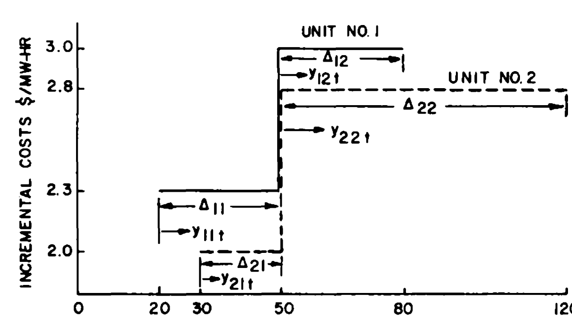

Fig. 1. Incremental cost characteristics for example problem

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Table I. Generator Cost Characteristics

Symbol	Name	Generator 1 $i=1$	Generator 2 $i=2$
U_i	Maximum rating	80 mw	120 mw
L_i	Minimum rating	20 mw	30 mw
c_{iL}	Cost at minimum rating	\$60/hr*	\$85/hr
c_{i1}	1st incremental cost step	\$2.3/mw-hr†	\$2.0/mw-hr
Δ_{i1}	Power available at c_{i1}	30 mw	20 mw
c_{i2}	2nd incremental cost step	\$3.0/mw-hr	\$2.8/mw-hr
Δ_{i2}	Power available at c_{i2}	30 mw	70 mw
c_{iu}	Start-up cost	\$20	\$30
c_{id}	Shutdown cost	\$10	\$15

* Hour.
† Megawatt-hour.

Table II. Capacity Requirements

Symbol	Name	First Hour $t=1$	Second Hour $t=2$
D_t	Estimated load	50 mw	100 mw
	Spinning reserve	15 mw	20 mw
R_t	Total requirement	65 mw	120 mw

acteristics have been found to give a lower cost schedule than connected straight line segments when the effects of valve loops are considered.[7] Assume further that these two units are presently operating and that the estimated loads for the next 2 hours will be 50 mw (megawatts) and then 100 mw. Table II summarizes the load and spinning reserve requirements.

Before proceeding we will need to define the following variables:

(*a*) The on–off indicator for generator i in time period t will be:

$x_{it}=1$ if the generator is scheduled to operate

$x_{it}=0$ otherwise

(*b*) The start-up indicator for generator i in period t will be:

$v_{it}=1$ if the generator is scheduled to start

$v_{it}=0$ otherwise

(*c*) The shutdown indicator for generator i in period t will be:

$w_{it}=1$ if the generator is scheduled to be shut down

$w_{it}=0$ otherwise

(*d*) The output from generator i at incremental cost step j in period t will be:

$y_{ijt}=$ mw

Fig. 1 illustrates the output variable, y_{ijt}. The solution must satisfy the following requirements:

1. All variables must be nonnegative.

2. All variables must assume only integer values.

3. The operating units must have a capacity greater than or equal to the total capacity required for each period, R_t.

$$80x_{11}+120x_{21}\geqq 65$$

$$80x_{12}+120x_{22}\geqq 120$$

4. The sum of the power output from each unit must be equal to the estimated load, D_t.

$$20x_{11}+y_{111}+y_{121}+30x_{21}+y_{211}+y_{221}=50$$

$$20x_{12}+y_{112}+y_{122}+30x_{22}+y_{212}+y_{222}=100$$

5. The power output from each generator at each incremental cost step must be less than, or equal to, the maximum available capacity, and must be zero if the generator is not operating.

$$30x_{11}\geqq y_{111},\quad 30x_{12}\geqq y_{112}$$

$$30x_{11}\geqq y_{121},\quad 30x_{12}\geqq y_{122}$$

$$20x_{21}\geqq y_{211},\quad 20x_{22}\geqq y_{212}$$

$$70x_{21}\geqq y_{221},\quad 70x_{22}\geqq y_{222}$$

6. When a generator is started its start indicator for that period, v_{it}, must be 1. When a generator is shut down its shutdown indicator for that period, w_{it}, must be 1.

$$x_{11}-x_{10}=v_{11}-w_{11}$$

$$x_{21}-x_{20}=v_{21}-w_{21}$$

$$x_{12}-x_{11}=v_{12}-w_{12}$$

$$x_{22}-x_{21}=v_{22}-w_{22}$$

Note that in these equations if $x_{11}=0$ and $x_{10}=1$, then $v_{11}=0$ and $w_{11}=1$, i.e., generator one was initially operating and is scheduled to be shut down in the first hour.

7. The on–off indicator for each generator in each period must not be greater than 1.

$$x_{11}\leqq 1,\quad x_{21}\leqq 1$$

$$x_{12}\leqq 1,\quad x_{22}\leqq 1$$

8. The object is to minimize the cost function which is the sum of the costs of starting, stopping, producing power at minimum output, and producing power above the minimum output for each time period. Let the total cost be represented by the symbol z.

$$\begin{aligned}z=&20v_{11}+30v_{21}+10w_{11}+15w_{21}+60x_{11}+\\&85x_{21}+2.3y_{111}+3.0y_{121}+2.0y_{211}+\\&2.8y_{221}+20v_{12}+30v_{22}+10w_{12}+\\&15w_{22}+60x_{12}+85x_{22}+2.3y_{112}+\\&3.0y_{122}+2.0y_{212}+2.8y_{222}\end{aligned}$$

This completes the formulation of the generator scheduling example. Appendix I contains the mathematical formulation in generalized notation. This simplified example does not include start-up costs which vary with the length of time a generator has been shut down. Additional equations may be used to bring in this effect. Also any special generator requirements, such as keeping one unit on the line at a station, could be entered as a further constraint without affecting the method of solution.

Method of Solution

In the previous section the economic scheduling of generators is mathematically formulated in terms of linear constraints with the further requirement that all variables assume only integer values. Without the integer requirement the problem is a linear programming problem solvable by the simplex method or dual method.[8–11] In 1958 the first adequate method for analytically solving linear programs with the additional integer requirement was presented by R. E. Gomory.[12] The key to his method was the derivation of additional constraints from those already present in the problem. These additional constraints allow all of the integer solutions to the original problem, but eliminate some of the noninteger solutions.

Attempts to solve the generator scheduling problem as formulated using Gomory's original integer algorithm and hand calculations led to the discovery of a more general algorithm which has been named the Euclidean escalator method. The name combines the words "Euclidean" because it is based on Euclid's algorithm for finding the greatest common divisor of two integers, and "escalator" because it escalates, or increases, the number of constraints in the problem.[13] This method when combined with the dual method of linear programming gives an effective method for solving integer programs. It is referred to as the dual Euclidean method and is identical to the method presented by Gomory in 1960.[14] It is straightforward and can best be understood by way of a simple example.

Let a small portion of the previous example be abstracted, including requirements 1, 2, 3, 6, and a modified version of 8. The problem is to minimize the cost, z:

$$\begin{aligned}z=&20v_{11}+30v_{21}+10w_{11}+15w_{21}+60x_{11}+\\&85x_{21}+20v_{12}+30v_{22}+10w_{12}+\\&15w_{22}+60x_{12}+85x_{22}\end{aligned}\quad(1)$$

Subject to:

Table III. Initial Matrix for the Abstracted Example Problem

	P_0	v_{11}	v_{21}	w_{11}	w_{21}	x_{11}	x_{21}	v_{12}	v_{22}	w_{12}	w_{22}	x_{12}	x_{22}	s_1	s_2
z	25	−30	−45	0	0	−60	−85	−30	−45	0	0	−50	−70	0	0
s_1	−65					−80	−120							1	
s_2	−120											−80	−120		1
w_{11}	1	−1		1		1									
w_{21}	1		−1		1		1								
w_{12}	0					−1		−1		1		1			
w_{22}	0						−1		−1		1		1		

Table IV. Matrix After One Linear Programming Iteration

	P_0	v_{11}	v_{21}	w_{11}	w_{21}	x_{11}	x_{21}	v_{12}	v_{22}	w_{12}	w_{22}	x_{12}	x_{22}	s_1	s
z	70	−30	−45	0	0	−10/3	0	−30	−45	0	0	−50	−70	−85/120	0
x_{21}	65/120					80/120	1							−1/120	
s_2	−120											−80	−120		1
w_{11}	1	−1		1		1									
w_{21}	55/120		−1		1	−8/12	0							1/120	
w_{12}	0					−1		−1		1		1			
w_{22}	65/120					8/12	0		−1		1		1	−1/120	

Table V. Initial Matrix Plus One Escalation Row

	P_0	v_{11}	v_{21}	w_{11}	w_{21}	x_{11}	x_{21}	v_{12}	v_{22}	w_{12}	w_{22}	x_{12}	x_{22}	s_1	s_2	s_3
z	25	−30	−45	0	0	−60	−85	−30	−45	0	0	−50	−70	0	0	0
s_1	−65					−80	−120							1		
s_2	−120											−80	−120		1	
w_{11}	1	−1		1		1										
w_{21}	1		−1		1		1									
w_{12}	0					−1		−1		1		1				
w_{22}	0						−1		−1		1		1			
s_3	−1					−1	−1									1

Table VI. Matrix After One Integer Programming Iteration

	P_0	v_{11}	v_{21}	w_{11}	w_{21}	x_{11}	x_{21}	v_{12}	v_{22}	w_{12}	w_{22}	x_{12}	x_{22}	s_1	s_2	s_3
z	85	−30	−45	0	0	0	−25	−30	−45	0	0	−50	−70	0	0	−60
s_1	15						−40							1		−80
s_2	−120											−80	−120		1	
w_{11}	0	−1		1			−1									1
w_{21}	1		−1		1		1									
w_{12}	1						1	−1		1		1				−1
w_{22}	0						−1		−1		1		1			
x_{11}	1					1	1									−1

$$65 \leqq 80x_{11}+120x_{21} \quad (2)$$

$$120 \leqq 80x_{12}+120x_{22} \quad (3)$$

$$0 = x_{11}-1^{*}-v_{11}+w_{11} \quad (4)$$

$$0 = x_{21}-1^{*}-v_{21}+w_{21} \quad (5)$$

$$0 = x_{12}-x_{11}-v_{12}+w_{12} \quad (6)$$

$$0 = x_{22}-x_{21}-v_{22}+w_{22} \quad (7)$$

All variables must be nonnegative integers when the solution is attained.

The requirements and cost function are first put into the matrix form used for linear programs, the simplex tableau, ready to begin the dual method of solution. Table III gives the simplex tableau for this example. Four steps were taken to form this matrix:

1. All inequalities, such as in equations 2 and 3, were changed to equalities by introducing nonnegative integer variables, s_1 and s_2, termed slack variables. Thus equations 2 and 3 become:

$$65 = 80x_{11}+120x_{21}-s_1 \quad (8)$$

$$120 = 80x_{12}+120x_{22}-s_2 \quad (9)$$

2. Each equation is now associated with a positive variable appearing in the one equation only. These positive variables are termed basis variables. Thus equations 4 through 9 become:

$$1 = x_{11}-v_{11}+(w_{11}) \quad (10)$$

$$1 = x_{21}-v_{21}+(w_{21}) \quad (11)$$

$$0 = x_{12}-x_{11}-v_{12}+(w_{12}) \quad (12)$$

$$0 = x_{22}-x_{21}-v_{22}+(w_{22}) \quad (13)$$

$$-65 = -80x_{11}-120x_{21}+(s_1) \quad (14)$$

$$-120 = -80x_{12}-120x_{22}+(s_2) \quad (15)$$

The basis variable in each equation is enclosed in parentheses.

3. Eliminate each of the basis variables from the cost equation 1 by substitution. For example, solving equation 10 for w_{11} and substituting into equation 1 gives:

$$z = 10+30v_{11}+30v_{21}+15w_{21}+50x_{11}+85x_{21}+20v_{12}+30v_{22}+10w_{12}+15w_{22}+60x_{12}+85x_{22}$$

Continuing the substitution and then placing the constant on the left side of the equation and the variables on the right gives:

$$25 = -30v_{11}-45v_{21}-60x_{11}-85x_{21}-30v_{12}-45v_{22}-50x_{12}-70x_{22}+z \quad (16)$$

4. The coefficients of equations 10 through 16 are now entered into the simplex tableau, Table III, with each row containing the coefficients for one equation and each column the coefficients of one variable. The constant terms are entered in the column headed P_0.

As with linear programming methods, matrix row operations will be performed in Table III to change the values in the P_0 column. The minimum cost solution will be given when all the P_0-column entries become positive and the z-row entries are still zero or negative. Let us investigate what would happen if Table III were to be operated on with the usual dual method rules which are:

(*a*) Select a row with a negative entry in the P_0 column and designate this row P_r. Let us select row s_1 to be designated P_r.

(*b*) For those columns with negative entries in the P_r row compute the ratio (z row entry) (P_r row entry)

Designate the column with the smallest ratio as P_k. This choice will assure that the z-row will continue to contain only nonpositive entries.

For the s_1 row the ratios become for column x_{11}

$$-60/-80 = 18/24$$

and for the x_{21} column

$$-85/-120 = 17/24$$

Thus column x_{21} is designated P_k.

(*c*) Use matrix row operations to produce the number one at the intersection of the P_r-row and P_k column and zeros elsewhere in the P_k column. Recall that a row operation may be the multiplication of the coefficients in any row by a nonzero constant or it may be the columnwise addition of a multiple of the coefficients in one row to the coefficients in another row.

Thus for our example we first multiply row s_1 by $-1/120$ giving the row termed x_{21} in Table IV. Next $85/-120$ times the coefficients in row s_1 are added to row z and the z-row of Table IV resulted. For example in column x_{11}

$$-60+(85/-120)(-80) = -3^{1}/_{3}$$

Similarly the other rows of Table III are modified, and Table IV results. This partial solution indicates why the usual dual method of linear programming cannot be used. In Table IV $x_{21} = 65/120$ which indicated the scheduling of a fraction of unit 2 during the first hour. Since this is physically impossible another method of

* The generators are assumed to be operating initially.

solution is needed. It has been found that the addition of one more step to the dual method rules will allow the solution of a linear program for integers.

(d) If the intersection of the P_r-row and P_k-column does not contain a -1, then an additional row of coefficients must be derived and added to the matrix. This additional row will be derived from the P_r-row and will contain a -1 in the P_k-column and other integer entries elsewhere.

To derive the entries for this additional row we use the same equation Euclid used to determine the greatest common divisor of two integers.[14]

$$a_j = b_j(e) + r, \quad 0 \leqq r < e \tag{17}$$

where

a_j is the integer entry in the jth column of row P_r
b_j is a new integer entry to be used in the jth column of the additional row
e is an appropriately chosen escalation number, greater than 1
r is the remainder

The appropriate choice of the escalation number e is a value which will produce a -1 in the P_k-column. In the example the a_j for the P_k-column is $a_j = -120$ and any value of e greater than or equal to 120 is satisfactory. Let us select $e = 120$. Then the other coefficients become:

For the P_0-column $b_j = -1$ from

$$-65 = -1(120) + 55$$

For x_{11}-column $b_j = -1$ from

$$-85 = -1(120) + 35$$

For the s_1-column $b_j = 0$

$$1 = 0(120) \times 1$$

In Table V the s_8 row containing these coefficients has been adjoined to the bottom of the matrix. Also a new column of zeros and a 1 has been adjoined to complete the equation represented by this row. In Appendix II a more detailed explanation is given of the escalation procedure. The solution now proceeds with row s_8 designated as P_r. Using steps b and c of the dual method column x_{11} is designated P_k and Table VI results from the necessary row operations. Since not all entries in the P_0 column are positive, steps a through d must be repeated. When all P_0 entries are positive the optimum solution is achieved.

The original example problem containing power allocation as well as the reserve requirements are displayed in matrix form in Table VII. This problem has been solved by the dual Euclidean method, whose basic steps have already been outlined as steps a through d. After 15 repeated applications of these rules, of which only seven involved step d, the solution shown in Table VIII resulted. The minimum cost schedule is thus to operate both units for both hours. During the first hour each unit will generate power at its minimum rating. During the second hour units no. 1 and 2 should both produce 50 mw. The total cost of production for the 2 hours is $399.

Conclusions

The generator scheduling problem can be expressed as an integer program taking

Table VII. Initial Matrix for the Complete Example Problem

	P_0	y_{111}	y_{121}	y_{211}	y_{221}	x_{11}	x_{21}	v_{11}	v_{21}	y_{112}	y_{122}	y_{212}	y_{222}	x_{12}	x_{22}	v_{12}	v_{22}
z	25	−2.3	−3.0	−2.0	−2.8	−60	−85	−30	−45	−2.3	−3.0	−2.0	−2.8	−50	−70	−30	−45
s_1	−50	−1	−1	−1	−1	−20	−30										
s_2	0	1				−30											
s_3	0		1			−30											
s_4	0			1			−20										
s_5	0				1		−70										
s_6	−65					−80	−120										
w_{11}	1					1		−1									
w_{21}	1						1		−1								
s_7	1					1											
s_8	1						1										
s_9	−100									−1	−1	−1	−1	−20	−30		
s_{10}	0									1				−30			
s_{11}	0										1			−30			
s_{12}	0											1			−20		
s_{13}	0												1		−70		
s_{14}	−120													−80	−120		
w_{12}	0					−1								1		−1	
w_{22}	0						−1								1		−1
s_{15}	1													1			
s_{16}	1														1		

Table VIII. Solution Matrix for the Complete Example Problem

	P_0	s_{10}	s_{10}	w_{12}	y_{121}	s_1	y_{221}	v_{11}	v_{21}	y_{111}	w_{22}	s_{12}	y_{122}	s_6	s_{16}	v_{12}	v_{22}
z	399	−1	−0.5	−20	−1	−2	−0.8	−30	−45	−0.3	−25	−0.8	−0.2	−2.8	−5	−10	−20
s	30	30		−30						1						30	
s_2	30	30		−30	1											30	
s_4	20	20		−20	−1	1	−1			−1	−50				50	20	50
s_5	70						1				−70				70		70
s_6	135	80		−80							−120				120	80	120
w_{11}	0	−1		1				−1								−1	
w_{21}	0								−1		1				−1		−1
s_7	0	−1		1												−1	
s_8	0										1				−1		−1
y_{211}	0	−20		20	1	−1	1			1	30				−30	−20	−30
x_{11}	1	1		−1												1	
x_{21}	1										−1				1		1
s_{11}	30	30											1				
s_{13}	70	50	1									1	−1	1	120		
s_{14}	80	80													120		
y_{112}	30	30	1														
y_{212}	20											1			20		
y_{222}	0	−50	−1									−1	1	−1	−50		
x_{12}	1	1															
x_{22}	1														1		

into account start-up costs, minimum rating costs, and incremental costs. Although not illustrated here the complications introduced by the time dependence of the start-up costs may be treated by the addition of more variables and equations. Also, scheduling considerations such as the supply of steam, charging reactive volt-amperes or reserve in remote areas of the system may be easily expressed in equation form and added to the matrix.

The solution of the scheduling problem requires the use of the dual method of linear programming modified to produce only integer values. This dual Euclidean method will solve the scheduling problem by matrix operations plus adjoining additional rows to the matrix when necessary. Only arithmetic operations are needed and the answer is achieved exactly in a finite number of steps. A digital computer program to perform this solution is presently being developed.

Appendix I

A summary of the constraints necessary to formulate the generator scheduling problem as an integer program including start-up and shutdown costs.

Minimize:

$$\text{Cost}=\sum_t\Big(\sum_i\Big(c_{iu}v_{it}+c_{id}w_{it}+c_{iL}x_{it}+\sum_j c_{ij}y_{ijt}\Big)\Big) \quad (18)$$

Subject to:

$$P_t=\sum_i\Big(L_ix_{it}+\sum_j y_{ijt}\Big) \quad (19)$$

$$R_t\leqq\sum_i U_ix_{it} \quad (20)$$

$$0\leqq\Delta_{ij}x_{it}-y_{ijt} \quad (21)$$

$$1\geqq x_{it} \quad (22)$$

$$x_t-x_{t-1}=v_t-w_t \quad (23)$$

All variables, v_t, w_t, x_{it}, and y_{ijt}, must be nonnegative integers.

where

v_{it} is the start indicator for unit i in period t
w_{it} is the shutdown indicator for unit i in period t
x_{it} is the on–off indicator for unit i in period t
y_{ijt} is the mw output of unit i at cost level j in period t
c_{iu} is the start-up cost for unit i, in dollars
c_{id} is the shutdown cost for unit i, in dollars
c_{iL} is the cost of generating the minimum output of unit i during period t in dollars
c_{ij} is the incremental cost of a mw from unit i at cost level j in dollars per mw
P_t is the total estimated power demand for period t in mw
R_t is the total spinning capacity estimated for period t in mw
L_i is the minimum output rating of generator i in mw
U_i is the maximum output rating of generator i in mw
Δ_{ij} is the mw of capacity available from generator i at incremental cost step j

Appendix II

The derivation of additional equations that are solved by all the positive integer solutions to an original equation can be accomplished by using equation 17 and the coefficients of the selected row. However, let a new symbol be introduced in place of the term b_j in equation 17,

$$a_j=\left[\frac{a_j}{e}\right]e+r,\quad 0\leqq r<e \quad (24)$$

where the brackets mean the largest integer contained in the bracketed quantity. For example,

$$[-2.3]=-3,\quad [2.3]=2$$

Row s_1 of Table III represents the requirement

$$-65\geqq-80x_{11}-120x_{21} \quad (25)$$

The derived requirements will have the form

$$\left[\frac{-65}{e}\right]\geqq\left[\frac{-80}{e}\right]x_{11}+\left[\frac{-120}{e}\right]x_{21} \quad (26)$$

If e is chosen to be 8 then equation 26 becomes the requirement that

$$\left[\frac{-65}{8}\right]\geqq\left[\frac{-80}{8}\right]x_{11}+\left[\frac{-120}{8}\right]x_{21}$$

$$-9\geqq-10x_{11}-15x_{21} \quad (27)$$

Any positive integer solution to requirement 25 is also a solution to 27. However, the converse is not true.

In the example it was necessary to derive a constraint in which the coefficient of x_{21} was -1. With $e=120$ the requirement 26 becomes

$$-1\geqq-x_{11}-x_{21} \quad (28)$$

and putting 28 in equation form by adding a positive integer variable, s_3,

$$-1=-x_{11}-x_{21}+s_3 \quad (29)$$

The coefficients of this equation were used as the elements of the additional row adjoined to the matrix in Table V.

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Discussion

C. W. Watchorn (Pennsylvania Power & Light Company, Allentown, Pa): The paper proposes to expedite the complex problem of determining the units that should be scheduled for maximum economy operation. This problem is one of long standing; in the past a great deal of time was required for the manual methods then available to calculate for the various conditions that might arise from time to time, and one procedure was to develop standard scheduling patterns which could either be used directly, or provide a starting point subject to modification, in accordance with changing conditions. Although reasonably good results were obtainable, it is very desirable to find a method that could be applied quickly to changing conditions as they occur. It appears that the paper proposes a method that meets this criterion, particularly if the number of units to be scheduled is not too great, as could be the case with the present ever-increasing loads and interconnection of power systems.

The latter situations raise the question whether it is possible to develop a method of combining the generating capacity for two or more areas or systems when they are to be operated together, but after they have been first scheduled separately.

K. M. Dale and **C. A. DeSalvo** (Westinghouse Electric Corporation, East Pittsburgh, Pa.): The author is to be congratulated for developing this new approach and thereby stimulating interest in the problem of unit selection or generator scheduling. The discussers also have been