Approximations for Non-symmetric Truncated Sequential and Repeated Significance Tests

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Abstract

Let $X_1, X_2, \ldots, X_N$ be a random sample from a $N(\mu, 1)$ population. Using the B-value process $\{B(t); 0 \le t \le 1\}$ introduced by Lan and Wittes (1988) one can design a sequential test for $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$; for some functions $\gamma_1(t) > \gamma_2(t)$: if $B(t) > \gamma_1(t)$ stop and reject H_0, if $B(t) < \gamma_2(t)$ stop and do not reject H_0, otherwise take one more observation. When $t = 1$, stop anyway rejecting H_0 if $B(1) > \gamma_1(1)$. We suggest some approximations which are usually upper bounds for the significance level, power of the tests and lower bounds for the expected stopping time. Simulations show that the approximations get better as $N \to \infty$ and/or $\mu \to \infty$. The exact values are difficult to obtain while the suggested approximations are very easy to compute in the case of linear and RST type boundaries and simulation results show that they are quite reasonable in several cases, even for moderate N.

Key words: B-value, Brownian motion, simulation, stopping time.

1 Introduction

Several sequential and non-sequential experiments deal with examination of data many times before the end of the experiment. Usually the observed Z-value is the statistic used for decision.

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Lan and Wittes (1988) suggest the use of the B-value as a tool for monitoring data (for a review see Jennison and Turnbull (1990)). The proposed B-value has three important properties described below that allow the use of Brownian motion approximation. In this case, we can use the well known sequential probability tests for Brownian motion (see for example, Siegmund (1985)) to approximate the significance level and power of the "monitoring tests". However, the computation of these quantities is usually very difficult. The objective of this work is to propose approximations for these quantities that are very easy to get. Simulations show that, although they are not exact, they are very good in several situations.

For a one-sample location parameter problem, suppose $X_1, X_2, \ldots, X_N$ form a sample from a normal population with mean μ and variance 1. We want to test $H_0 : \mu = 0$ versus $H_1 : \mu > 0$. After n observations are made, let $t = n/N$ be the proportion of data observed at the time of monitoring. The B-value (Lan and Wittes, 1988) is defined as:

$$B(t) = Z_n t^{1/2} \tag{1.1}$$

where $Z_n = \sum_{i=1}^{n} X_i / n^{1/2}$. The process $\{B(t); t = 0, 1/N, 2/N, \ldots, 1\}$ has the following properties:

(i) $B(t)$ and $B(1) - B(t)$ are independent and normally distributed random variables;

(ii) $\mathbb{E}[B(t)] = \theta t$ and $\mathbb{E}[B(1) - B(t)] = \theta(1 - t)$, where $\theta = \mathbb{E}[Z_N] = \mu N^{1/2}$;

(iii) $\mathrm{Var}[B(t)] = t$ and $\mathrm{Var}[B(1) - B(t)] = 1 - t$.

By properties (i), (ii) and (iii), we can assume the existence of an underlying Brownian motion $\{B(t), 0 \leq t \leq 1\}$ such that the B-value defined by (1.1) is the observation of the Brownian motion at the discrete points $t = 0, 1/N, \ldots, 1$. Several authors have studied sequential tests for Brownian motion, Siegmund (1985) has a general discussion of the subject in the case of testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$. He says : "Suppose that for μ considerably larger than μ_0, say $\mu > \mu_1 > \mu_0$, the collection of data is expensive and a small sample is desirable, but if $\mu < \mu_0$ data are relatively inexpensive, so a large (perhaps fixed) sample would be appropriate". In this work we want to study some tests in the situation where the data collection is expensive not only under H_1, but also under H_0. Therefore we want criteria to stop sampling as soon as the evidence is sufficient either for rejecting H_0 or not rejecting it. Similarly to Siegmund (1985) we could use a test of the form, for some functions $\gamma_1(t) > \gamma_2(t)$: if $B(t) > \gamma_1(t)$ stop and reject

H_0, if $B(t) < \gamma_2(t)$ stop and do not reject H_0, otherwise take one more observation, when $t = 1$ stop anyway, rejecting H_0 if, and only if $B(1) > \gamma_1(1)$.

1.1 Straight Line Boundary

For every $t \in [0, 1]$, define

$$\gamma_1(t) = b_1 - \eta_1 t$$

and

$$\gamma_2(t) = -b_2 + \eta_2 t.$$

A sensible decision rule would be: For $0 < t < 1$, stop at t and reject H_0 if $B(t) > b_1 - \eta_1 t$; stop at t and do not reject H_0 if $B(t) < -b_2 + \eta_2 t$; otherwise continue sampling. For $t = 1$, stop and reject H_0 if $B(1) > b_1 - \eta_1$, stop and do not reject H_0 otherwise.

Let

$$\tau_1 = \inf\{t; B(t) \geq b_1 - \eta_1 t\} \tag{1.2}$$

and

$$\tau_2 = \inf\{t; B(t) \leq -b_2 + \eta_2 t\}. \tag{1.3}$$

We have,

$$\begin{aligned}\mathbb{P}[\text{ reject } H_0|\mu] &= \mathbb{P}[\text{ hit upper boundary } |\mu] \\ &= \mathbb{P}[\tau_1 < \tau_2 \text{ and } \tau_1 \leq 1]\end{aligned}$$

Notice that,

(i) For $\mu \gg 0$, $\mathbb{P}[\text{ reject } H_0|\mu] \approx \mathbb{P}[\tau_1 \leq 1|\mu]$ since $\mathbb{P}[\tau_1 < \tau_2|\mu] \approx 1$;

(ii) For $\mu \ll 0$, $\mathbb{P}[\text{ reject } H_0|\mu] \approx 0$ since $\mathbb{P}[\tau_1 < \tau_2|\mu] \approx 0$.

Moreover, if $\eta_1 > 0$ and $\eta_2 > 0$, then τ_1 and τ_2 are bounded by $(b_1 + b_2)/(\eta_1 + \eta_2)$ and we can apply the optional sampling theorem to the martingale $M(t) = B(t) - \mu t$. In this case, $\mathbb{E}[M(\tau_1 \wedge \tau_2 \wedge n)] = 0$ and consequently, as $n \to \infty$

$$\mathbb{P}[\tau_1 < \tau_2] = \frac{b_2 - (\eta_2 - \mu)\mathbb{E}[\tau_2]}{(b_1 - (\eta_1 - \mu)\mathbb{E}[\tau_1]) + (b_2 - (\eta_2 - \mu)\mathbb{E}[\tau_2])}. \tag{1.4}$$

In the particular case of symmetric boundaries, $b_1 = b_2 = b$ and $\eta_1 = \eta_2 = \eta$ and let $\tau = \min\{\tau_1, \tau_2\} = \inf\{t; |B(t)| > b - \eta t\}$. For $\mu = 0$ by symmetry $\mathbb{E}[\tau_1] = \mathbb{E}[\tau_2]$ and

$$\mathbb{P}[\tau_1 < \tau_2] = \mathbb{P}[\tau_1 > \tau_2] = 1/2. \tag{1.5}$$

Therefore, by symmetry of the arguments $\mathbb{P}[$ rejecting $H_0|\mu = 0] \approx (1/2)\mathbb{P}[\tau_1 \leq 1]$.

Equation (1.4) becomes

$$\mathbb{P}[B(\tau) = b - \eta\tau] = \frac{b - (\eta - \mu)\mathbb{E}[\tau]}{2b - 2(\eta - \mu)\mathbb{E}[\tau]}. \tag{1.6}$$

For $\mu = 0$ (1.6) becomes $\mathbb{P}[B(\tau) = b - \eta\tau] = 1/2$.

Suppose now that $b_1 \neq b_2$ and $\eta_1 \neq \eta_2$. As above, for $\mu = 0$, (1.4) becomes

$$\mathbb{P}[B(\tau) = b_1 - \eta_1\tau] = \frac{b_2 - \eta_2\mathbb{E}[\tau]}{b_1 + b_2 - (\eta_1 + \eta_2)\mathbb{E}[\tau]}. \tag{1.7}$$

The expression for $\mathbb{E}[\tau]$ is not easily computed (see Hall (1997)). However, our simulations show that approximating (1.7) by

$$\frac{b_2 - \eta_2}{b_1 + b_2 - (\eta_1 + \eta_2)} \tag{1.8}$$

gives good results.

Note that, if $\eta_1 = \eta_2 = 0$, then

$$\mathbb{P}[B(\tau) = b_1] \quad = \quad \frac{b_2}{b_1 + b_2}.$$

Formulae (3.15) and (3.17) from Siegmund(1985) yield the following approximations:

$$\mathbb{P}[\text{ reject } H_0|\mu] = \frac{b_2 - \eta_2}{b_1 + b_2 - (\eta_1 + \eta_2)}\mathbb{P}[\tau \leq 1]I_{[\mu=0]} + \mathbb{P}[\tau_1 \leq 1]I_{[\mu>0]} \tag{1.9}$$

where

$$\mathbb{P}[\tau_1 \leq 1] = 1 - \Phi(b_1 - (\eta_1 + \mu)) + \Phi(-b_1 - (\mu - \eta_1))e^{2b_1(\mu+\eta_1)}.$$

If $b_1 = b_2 = b$ and $\eta_1 = \eta_2 = \eta$ then

$$\mathbb{P}[\tau \leq 1] \approx 2 - 2\Phi(b - (\eta + \mu)) + \Phi(-b - (\mu - \eta))e^{2b(\mu+\eta)}.$$

In general,

$$\begin{aligned}\mathbb{P}[\tau \leq 1] \quad &= \quad 2 - \Phi(b_1 - (\eta_1 + \mu)) + \Phi(-b_1 - (\mu + \eta_1))e^{2b_1(\mu+\eta_1)} \\ &\quad -\Phi(b_2 - (\eta_2 + \mu)) + \Phi(b_2 - (\mu + \eta_2))e^{2b_2(\mu+\eta_2)}.\end{aligned}$$

The expected sample size is

$$
\begin{aligned}
\mathbb{E}[\tau \wedge 1|\mu > 0] &\approx \mathbb{E}[\tau_1 \wedge 1|\mu > 0] \\
&\approx \frac{b_1}{\mu+\eta_1}\mathbb{P}[\tau_1 \leq 1] + \mathbb{P}[\tau_1 > 1] - \frac{2b_1}{\mu+\eta_1}\Phi(-b_1-\mu-\eta_1)e^{2b_1(\mu+\eta_1)} \quad (1.10)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}[\tau \wedge 1|\mu < 0] &\approx \mathbb{E}[\tau_2 \wedge 1|\mu < 0] \\
&\approx \frac{b_2}{\mu+\eta_2}\mathbb{P}[\tau_2 \leq 1] + \mathbb{P}[\tau_2 > 1] - \frac{2b_2}{\mu+\eta_2}\Phi(-b_2-\mu-\eta_2)e^{2b_2(\mu+\eta_2)}. \quad (1.11)
\end{aligned}
$$

Letting,

$$
p^* = \frac{b_2 - \eta_2}{(b_1+b_2) - (\eta_1+\eta_2)}
$$

we have,

$$
\begin{aligned}
\mathbb{E}[\tau \wedge 1|\mu = 0] &= p^*\mathbb{E}[\tau_1 \wedge 1|\mu = 0] + (1-p^*)\mathbb{E}[\tau_2 \wedge 1|\mu = 0] \\
&= p^*\{[\Phi(\Delta_1) + \Phi(-\Delta_1)e^{2b_1\eta_1}] + 2b_1[\phi(b_1) - b_1\phi(-b_1)]\} \\
&\quad +(1-p^*)\{[\Phi(\Delta_2) + \Phi(-\Delta_2)e^{2b_2\eta_2}] + 2b_2[\phi(b_2) - b_2\phi(-b_2)]\} \quad (1.12)
\end{aligned}
$$

where $\Delta_i = b_i - \eta_i, i = 1, 2$.

1.2 RST Boundary

If $\gamma_i(t) = b_i t^{1/2}$ for $b_i > 0$ and $i = 1, 2$, our rule would turn out to be: For $0 < t < 1$, stop at t and reject H_0 if $B(t) > b_1 t^{1/2}$; stop at t and do not reject H_0 if $B(t) < -b_2 t^{1/2}$; otherwise continue sampling. For $t = 1$, stop and reject H_0 if $B(1) > b_1$, stop and do not reject H_0 otherwise. Therefore, the conditional probability of rejecting H_0 given that it hits the boundary is $1 - \Phi([z_\alpha - b_i t^{1/2}](1-t)^{-1/2})$. The motivation for using the RST type boundary could be given by noting that:

$$
\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n > z_\alpha n^{-1/2}, \text{ if, and only if } n\bar{X}_n N^{-1/2} > z_\alpha (n/N)^{1/2}
$$

Since $B(n/N) = Z_n(n/N)^{1/2} = n\bar{X}_n N^{-1/2}$ it follows that $\bar{X}_n > z_\alpha n^{-1/2}$ if, and only if, $B(t) > z_\alpha t^{1/2}$, which is the RST boundary with $b_1 = z_\alpha$.

Essentially we want to generalize the RST type test described in Siegmund(1985) to the problem without symmetric boundaries where the lower boundary leads to non rejection of H_0 and $m_0 = 1/N$.

Let

$$\tau_1 = \inf\{t > 1/N; B(t) \geq b_1 t^{1/2}\}$$

and

$$\tau_2 = \inf\{t > 1/N; B(t) \leq -b_2 t^{1/2}\},$$

then

$$\mathbb{P}[\text{ reject } H_0|\mu] = \mathbb{P}[\text{ hit upper boundary } |\mu] = \mathbb{P}[\tau_1 < \tau_2, \tau_1 \leq 1|\mu].$$

Similarly as it was argued in the previous section,

(i) For $\mu \gg 0$, $\mathbb{P}[\text{ reject } H_0|\mu] \approx \mathbb{P}[\tau_1 \leq 1|\mu]$ since $\mathbb{P}[\tau_1 < \tau_2|\mu] \approx 1$;

(ii) For $\mu \ll 0$, $\mathbb{P}[\text{ reject } H_0|\mu] \approx 0$ since $\mathbb{P}[\tau_1 < \tau_2|\mu] \approx 0$;

(iii) For $\mu = 0$, define $\tau = \tau_1 \wedge \tau_2$, then $\mathbb{P}[\text{ reject } H_0|\mu] \approx b_2(b_1 + b_2)^{-1}\mathbb{P}[\tau \leq 1]$ since for $\mu = 0$, $B(t)$ is a martingale and by the Optional Sampling Theorem applied to $\tau \wedge n$ and $B(\tau \wedge n)$ we have:

$$0 = \mathbb{E}[B(0)] = \mathbb{E}[B(\tau \wedge n)]$$

For large n, using the following approximation

$$0 = \mathbb{E}[B(\tau)] = b_1\mathbb{E}[\tau_1^{1/2}]\mathbb{P}[\tau = \tau_1] - b_2\mathbb{E}[\tau_2^{1/2}]\mathbb{P}[\tau = \tau_1]$$

with

$$\mathbb{P}[\tau = \tau_1] = \frac{b_2}{b_1 + b_2}$$

we do get good results in the simulation.

Therefore, using formulae (4.20), (4.21) and (4.27) from Siegmund(1985) we get:

$$\mathbb{P}[\text{ reject } H_0|\mu] \approx \frac{b_2}{b_1 + b_2}\mathbb{P}_0[\tau_1 \leq 1]I_{[\mu=0]} + \mathbb{P}_\mu[\tau_1 \leq 1]I_{[\mu>0]} \tag{1.13}$$

where

$$\mathbb{P}_0[\tau_1 \leq 1] \approx \frac{1}{2(b_1 - b_1)}\phi(b_1)\log(N) + \frac{4}{b_1}\phi(b_1)$$

and

$$\mathbb{P}_\mu[\tau_1 \leq 1] \approx 1 - \Phi((b_1 - \mu)) + \mu^{-1}\phi(b_1 - \mu), \mu > 0.$$

Also, the expected sample size is:

$$\mathbb{E}[\tau \wedge 1|\mu] \approx \begin{cases} \mathbb{E}[\tau_2 \wedge 1|\mu], & \text{if } \mu < 0 \\ 1, & \text{if } \mu = 0 \\ \mathbb{E}[\tau_1 \wedge 1|\mu], & \text{if } \mu > 0 \end{cases} \tag{1.14}$$

where

$$\begin{aligned} \mathbb{E}[\tau_i \wedge 1|\mu] \quad \approx \quad & (b_i^2 - 1)\mu^{-2} - \frac{1}{\mu - b_i/2}\{[\phi(\Delta_i) - \Delta_i\phi(-\Delta_i)][1 - b_i^{-2}] \\ & -[(1/8)b_i(\mu - 2b_i)^2][\Phi(-\Delta_i)(1 + \Delta_i^2) - \Delta_i\phi(\Delta_i)] \\ & -b_i^{-2}\phi(\Delta_i) + b_i[(2 + \Delta_i)^2\phi(\Delta_i) - \Delta_i(\Delta_i^2 + 3)\Phi(-\Delta_i)]\} \end{aligned}$$

where $\Delta_i = \mu - b_i$.

2 Approximations and Simulations. Comments.

In order to verify the approximations, we run 10000 simulations and compare the approximated and simulated values for the power function and expected sample size. All the computations were done by using a SUN 20 workstation. A simple fortran program was developed to run the simulations and S-plus code was used to get the approximations. Typical examples are shown in Figures 2.1 to 2.4.

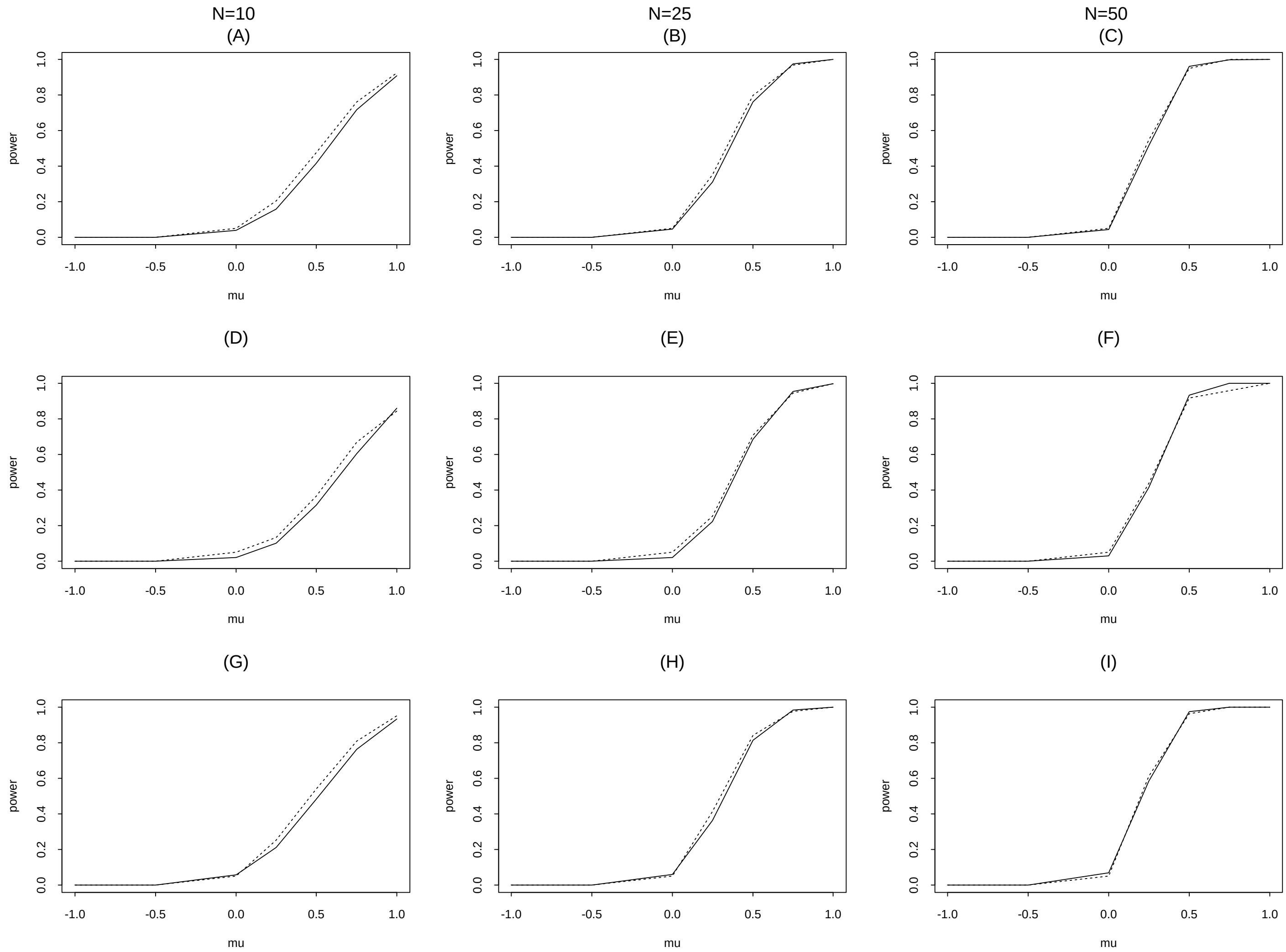

Figure 2.1: Simulated (solid) and approximated (dotted) power functions for straight line boundaries for sample sizes N=10, 25 and 50. (A), (B) and (C): $b_1 = b_2 = 1.99$, $\eta_1 = \eta_2 = 0.1$, (D), (E) and (F): $b_1 = b_2 = 2.25$, $\eta_1 = 0.1$, $\eta_2 = 1.0$, (G), (H) and (I): $b_1 = 1.84$, $b_2 = 2.84$, $\eta_1 = \eta_2 = 0.1$.

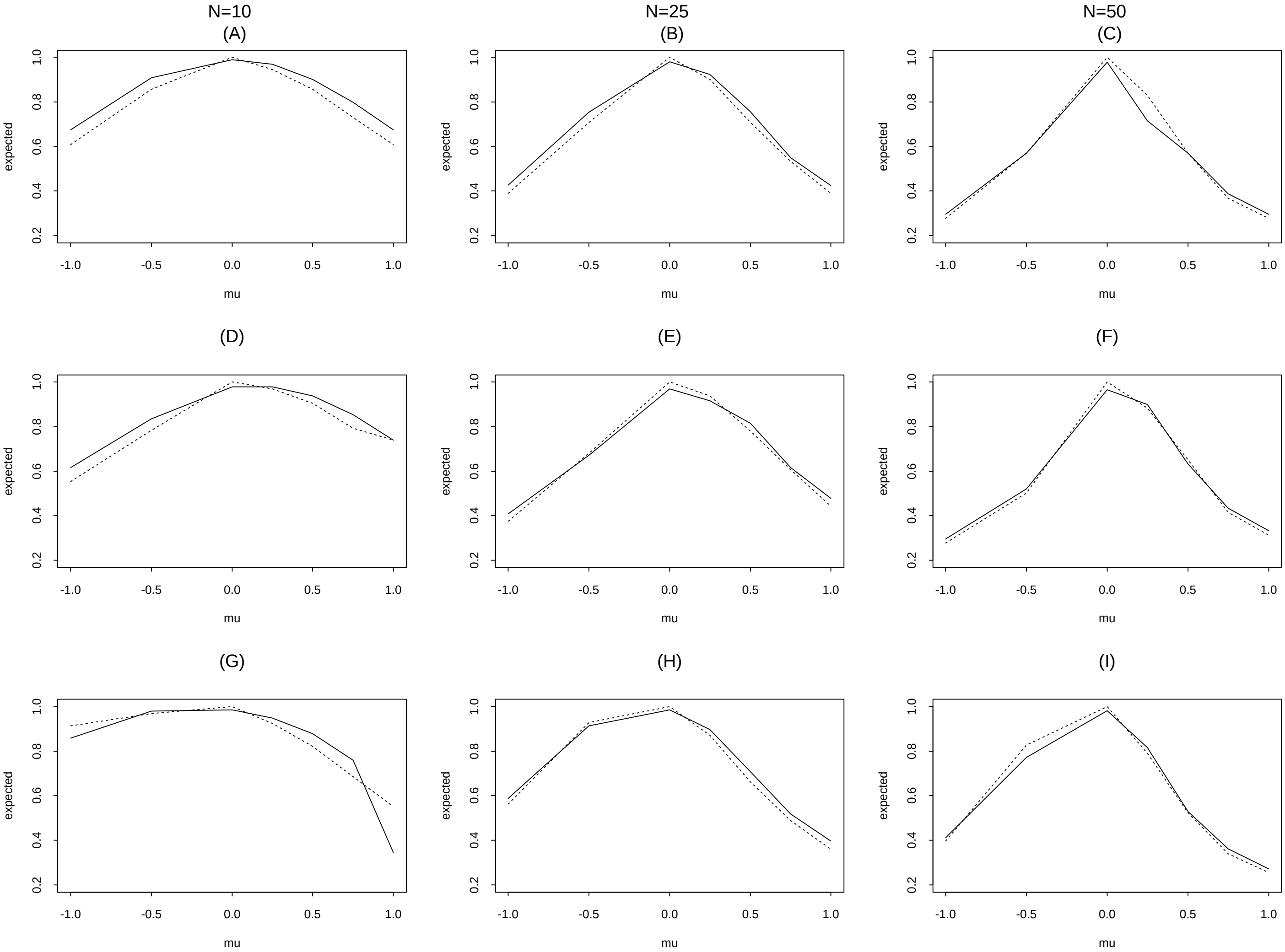

Figure 2.2: Simulated (solid) and approximated (dotted) expected stopping times for straight line boundaries for sample sizes N=10, 25 and 50. (A), (B) and (C): $b_1 = b_2 = 1.99$, $\eta_1 = \eta_2 = 0.1$, (D), (E) and (F): $b_1 = b_2 = 2.25$, $\eta_1 = 0.1$, $\eta_2 = 1.0$, (G), (H) and (I): $b_1 = 1.84$, $b_2 = 2.84$, $\eta_1 = \eta_2 = 0.1$.

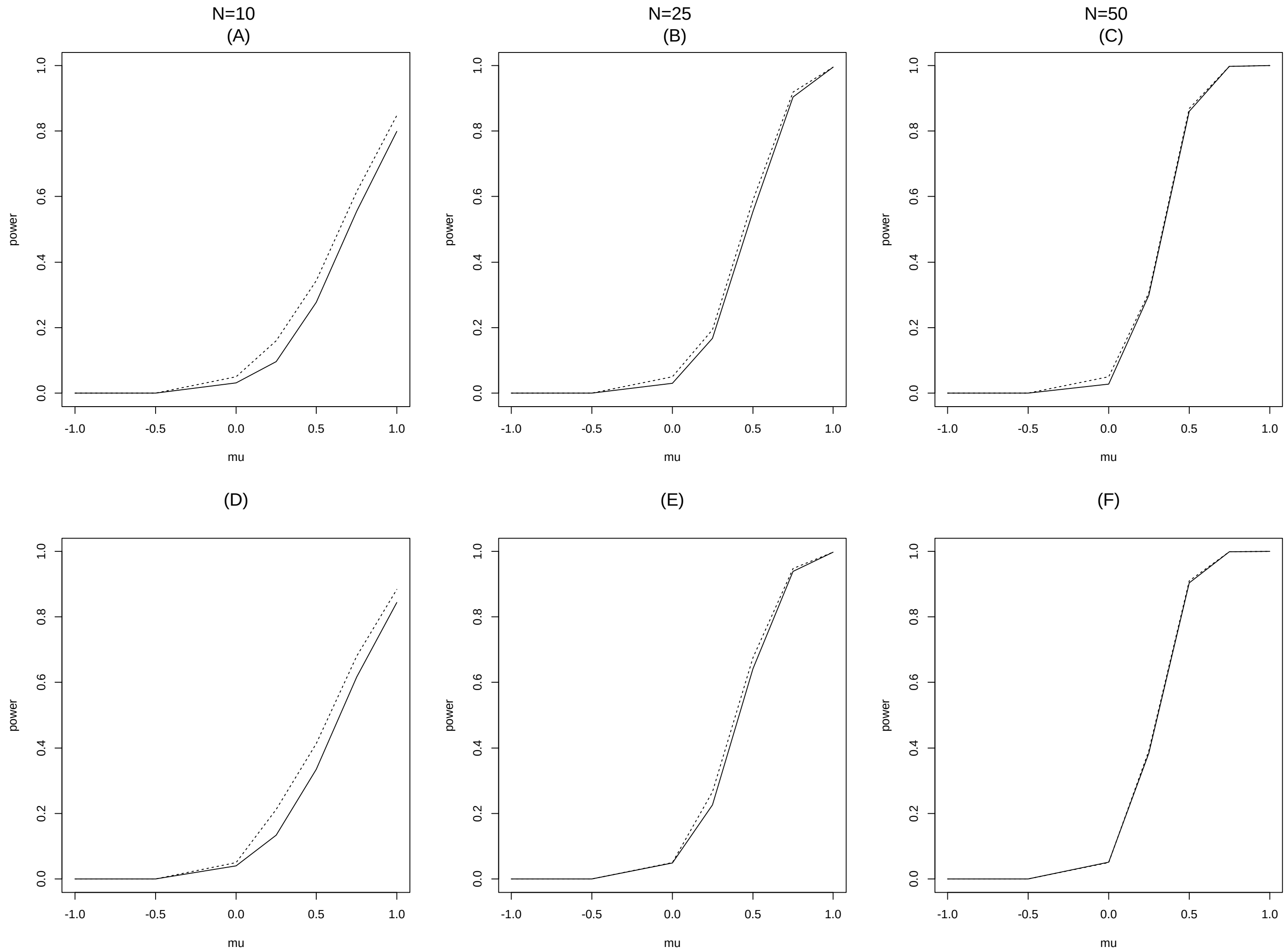

Figure 2.3: Simulated (solid) and approximated (dotted) power functions for RST boundaries for sample sizes N=10, 25 and 50. (A) $b_1 = b_2 = 2.50$, (B) $b_1 = b_2 = 2.67$, (C)$b_1 = b_2 = 2.74$, (D) $b_1 = 2.34$, $b_2 = 3.34$, (E) $b_1 = 2.46$, $b_2 = 3.46$ and (F) $b_1 = 2.53$, $b_2 = 3.53$.

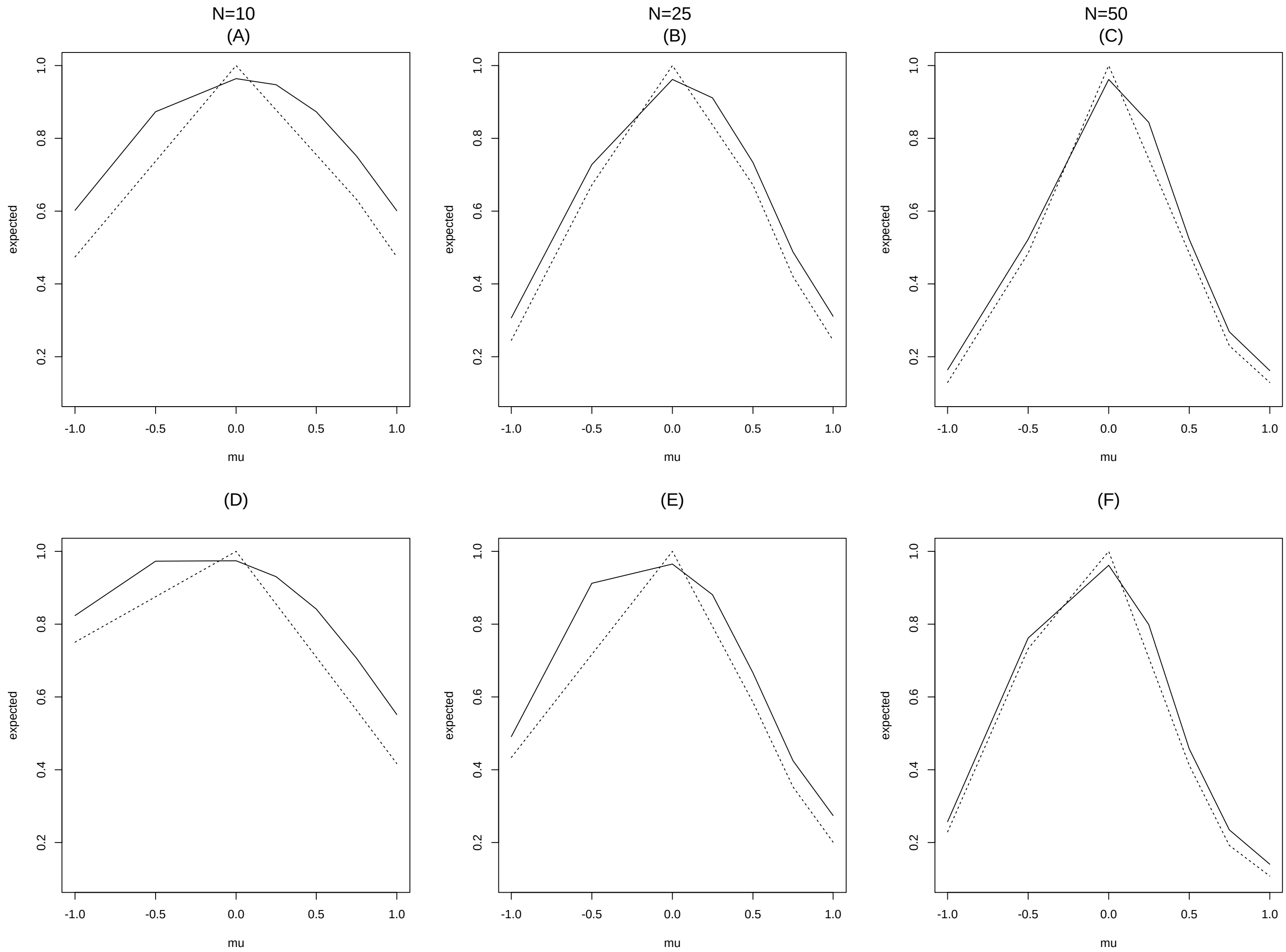

Figure 2.4: Simulated (solid) and approximated (dotted) expected stopping times for RST boundaries for sample sizes N=10, 25 and 50. (A) $b_1 = b_2 = 2.50$, (B) $b_1 = b_2 = 2.67$, (C)$b_1 = b_2 = 2.74$, (D) $b_1 = 2.34$, $b_2 = 3.34$, (E) $b_1 = 2.46$, $b_2 = 3.46$ and (F) $b_1 = 2.53$, $b_2 = 3.53$.

Figures 2.1 to 2.4 show that the approximations are very good. They improve as the sample size increases and/or μ gets farther from zero. Note that the drift of the Brownian motion $B(t)$ is $\mu N^{1/2}$, and as N is increases the drift increases making better approximations. Also, when b_1 and/or b_2 increases the neglected terms become negligible and the approximations are good.

As it was expected the approximations for the straight line boundaries are better than the RST boundaries. This comes from the fact that exact formulae for the power of the one-sided tests were used in the approximation for the straight line case, whereas for the RST case we used asymptotic expressions derived when m and m_0 tend to infinity and they were used with $m = 1$ and $m_0 = 1/N$. It is remarkable that we can get so good approximations in some cases (e.g. $N = 25$ and $N = 50$).

For the case of straight line boundaries we can see that approximations become better as b_1 or b_2 increases, equivalently, as η_1 or η_2 increases (when η_i increases b_i increases to maintain constant size of the test, see (1.9)).

For RST boundaries the approximations are better for $b_2 = b_1 + 1$ than $b_2 = b_1$. As N increases, the proximity is remarkable, taking into account that several terms were neglected in an already approximated formula for the power.

For $\mu N^{1/2} < b_1$ the RST approximation formulae gives non-dependable values, in this case Figure 2.4 shows linear interpolations for these points.

Formulae (1.9)–(1.14) are very easy to implement and can be very useful for practical purposes.

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