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Numerical and Monte Carlo verification of V-distribution

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Abstract—It is shown that V-distribution is a valid probability density for all values of parameters and approximates the empirical density of eigenvalues of certain random non symmetrical matrices.

1. INTRODUCTION

Random matrix physics is generally thought to consist of the following subfields:

1. Statistical properties of nuclear and atomic spectra. Time reversal symmetry braking and the field theory of quantum chaos [1-51].
2. Quantum chaology and condensed matter physics.
3. Dissipative quantum dynamics.
4. Another rather broad field of the random matrices theory applications is related to quantum field theory: the large limit of QCD, two dimensional (2D) quantum gravity and bosonic strings.
5. Statistics of resonance poles, phase shifts and time delays in quantum chaotic scattering: Random matrix approach for systems with broken time-reversal invariance. For data of the complete spin and parity, the symmetric and unitary matrix of scattering can be written in the form:

$$S = U\tilde{S}U^{\mathrm{T}}, \quad \tilde{S} = I - \mathrm{i}R,$$
$$R = F(I\varepsilon - H_0 - V + \frac{\mathrm{i}}{2}F^{\mathrm{T}}F)^{-1}F^{\mathrm{T}},$$

where U is a unitary matrix of dimension $N \times N$; V and F are the real random matrices of dimensions $\Lambda \times \Lambda$ and $N \times \Lambda$ with the elements v_{ij} and γ_{ij}, respectively: I is the identity matrix of dimension $N \times N$; the matrices H_0 and V are nonrandom, and H_0, V, F do not depend on the energy ε. Here N is the number of open channels, and Λ is the number of so-called bound states "embedded" in the continuum. In the framework of the shell-model approach to nuclear reactions, the values v_{ij} and γ_{ij} may be identified with those of the matrix elements of residual interactions between the states of the discrete spectrum and states of discrete and continuous spectra, respectively; and H_0 may be identified with the Hamiltonian of the mean field. We shall consider the values v_{ij} and γ_{ij} as independent random values distributed according to the normal law with zero mean and second moments given as:

$$\mathbf{E}(v_{ij}v_{i'j'}) = (\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{i'j})\,\mathbf{E}\,\nu_{ij}^2, \quad \mathbf{E}\nu_{ij}^2 = \delta^2\Lambda^{-1},$$

$$\mathbf{E}(\gamma_{ij}\gamma_{i'j'}) = \delta_{ii'}\delta_{jj'}\mathbf{E}\,\gamma_{ij}^2, \quad \mathbf{E}\,\gamma_{ij}^2 = \sigma^2 N^{-1}.$$

It is possible to consider limit properties of the matrix $\tilde{S}$. The problem is to find the limits of the expressions $\mathbf{E}\tilde{S}_{ii}$ and $\mathbf{E}\left[(\tilde{S}-\mathbf{E}\,\tilde{S})_{ij}(\tilde{S}-\mathbf{E}\tilde{S})_{pl}c_n\right]$ as $N\to\infty$, $\Lambda\to\infty$, where c_n is some sequence of normalizing values. In [52, 53] the limit theorems for these expressions were proved under the condition that

$$\lim_{\substack{\Lambda\to\infty,\\ N\to\infty}} \Lambda N^{-1} = c < 1, \quad \nu_\Lambda(x) \Rightarrow \nu(x),$$

and $\nu_\Lambda(x)$ is the normalized spectral function of the matrix $I\varepsilon - H_0$.

6. Theory of spin glasses. In view of the physical properties of spin glasses, it seems reasonable to consider random interactions. Some rigorous results on the Sherrington-Kirkpatric model of spin glasses with the mean field Hamiltonian

$$H = -\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j N^{-1/2},$$

were obtained in [55], where the spins $\sigma_1, ..., \sigma_N$ take values ± 1, and J_{ij} are independent identically distributed random variables with mean zero and variance J^2. The Edwards-Anderson model is defined on the d-dimensional lattice $\Lambda_N = -[-N,\ N]^d \cap Z^d$. The dual lattice Λ_N^* is defined as usual and can be identified with the set $\Lambda_N^* = \left\{\{i,j\},\ i\in\Lambda_N,\ j\in Z^d,\ |i-j|=1\right\}$. For a family of centred, variance 1, independent, Gaussian random variables $J_{ij},\ i,j\in\Lambda_N^*$ indexed by Λ_N^*, the Hamiltonian is defined by

$$H_N(\sigma) = -\sum_{i,j\in\Lambda_N^*} J_{ij}\sigma_i\sigma_j.$$

7. Theory of neural nets. The long-time behavior od some neural-network models studied by mappng them onto statistical-mechanical problems [56, 57]. Note that for neural networks Ising spin-flip dynamics has been applied to neuron dynamics with asymmetric random matrix of synaptic efficaies. The dynamics of the model is defined by a set of Langevin equations:

$$\Gamma_0^{-1}\frac{\partial}{\partial t}\sigma_i(t) = -r_0\sigma_i(t) - \frac{\delta V(\sigma_i)}{\delta\sigma_i(t)} + \sum_j J_{ij}\sigma_i(t) + h_i(t) + \xi_i(t),$$

where the synaptic connections J_{ij} between pairs of neurons i and j are random variables, each degree of freedom is represented by spin σ_i which can vary continuously from $-\infty$ to $+\infty$. The local potential $V(\sigma_i) + r_0\sigma_i^2/2$ is an even function of σ_i. The term h_i represents a local external field, whereas ξ_i is a white noise with width $\mathbf{E}\xi_i(t)\,\xi_j(s) = 2T\delta(t-s)\,\delta_{ij}\Gamma_0^{-1}$. The parameter Γ_0^{-1} sets the scale of the microscopic processing time, whereas T measures the level of the stochastic noise in the system. For the asymptotic analysis of the solutions of this Langevin equations the spectral theory of random matrices can be applied.

8. Motion of flux lines in superconductors with columnar defects or shiral symmetry braking in quantum chromodynamics.
9. Localization transition of random heteropolymer chains.

10. Behaviour of the conductance of mesoscopic devices as a function of the magnetic field.
11. Spatial structure of anomalously localized states in disordered conductors.
12. Theory of the integer quantum Hall transition. Continuum limit of the Chalker-Coddington model.
13. Quantum chromodynamics.

There have been significant interactions between most of these areas with the spectral theory of non symmetrical random matrices. The most complete overview of random matrix physics the readers can find in [19]. Most investigations in random matrix physics have been devoted to the matrices with zero expectations of its entries. The real situation in the application of random matrices in physics is completely another. Usually the entries of matrices have some nonzero means.

In this article we continue to develop new V-analysis (see 60–75, 82) for such matrices and give the domain where its eigenvalues are distributed. Therefore, the main purpose of this article consists to attract physicists to new analysis of random non symmetrical matrices which appear in many modern problems.

The most important result of articles [71, 72], [82] is the V-relation for normalized spectral function (n.s.f.) of the eigenvalues of non symmetric matrix Ξ with n.s.f. of the eigenvalues of the Hermitian matrix $(\Xi - \tau I)(\Xi - \tau I)^*$, $\tau = t + \mathrm{i}s$:

$$\iint \mathrm{e}^{\mathrm{i}px+\mathrm{i}qy} \mathrm{d}\nu_n(x,y) = \frac{q^2+p^2}{4\mathrm{i}p\pi} \int \left\{ \int \frac{\partial}{\partial t} \left[\int_0^\infty \ln x \,\mathrm{d}\mu_n(x,\tau) \right] \mathrm{e}^{\mathrm{i}tp+\mathrm{i}sq}\,\mathrm{d}t \right\} \mathrm{d}s; \quad p,q \neq 0,$$

and modified V-transform [82]:

$$V(x,y,\varepsilon) = \iint \frac{\varepsilon^2 \mathrm{d}\nu_n(u,v)}{\left[(u-x)^2+\varepsilon^2\right]\left[(v-y)^2+\varepsilon^2\right]}$$
$$= \frac{1}{4\pi n} \iint \ln\det|\Xi_n - (u+\mathrm{i}v)I|^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\varepsilon^2}{\left[(u-x)^2+\varepsilon^2\right]\left[(v-y)^2+\varepsilon^2\right]} \mathrm{d}u\,\mathrm{d}v,$$

where $\varepsilon > 0$.

This transform was found by Girko in 1982 [60] and has the name V-transform.

2. THE V-LAW

In [82] Girko has proved the following assertion.

THEOREM 2.1. (see V -equation in [72]). *Let $\Xi = (\xi_{ij})_{i,j=1}^n$ be a random complex matrix, whose entries $\xi_{ij}^{(n)}$ are independent for every n, given on the common probability space,*

$$\mathbf{E}\xi_{ij}^{(n)} = a_{ij}^{(n)}, \ \mathbf{E}\left|\xi_{ij}^{(n)} - a_{ij}^{(n)}\right|^2 = \sigma_{ij}^{(n)}, \quad \sigma_{ij}^{(n)} n > c > 0, \quad i,j = 1,...,n,$$

$$\sup_n \max_{\substack{i=1,\dots,n,\\ j=1,\dots,n}} \left\{ \sum_{j=1}^n \sigma_{ij}^{(n)} + \sum_{i=1}^n \sigma_{ij}^{(n)} + \sum_{j=1}^n \left|a_{ij}^{(n)}\right| + \sum_{i=1}^n \left|a_{ij}^{(n)}\right| \right\} < \infty,$$

there exist densities $p_{ij}^{(n)}(x)$ *of the real parts (or densities* $q_{ij}^{(n)}(x)$ *of the imaginary parts) of* $\sqrt{n}\xi_{ij}^{(n)}$ *such that:*

$$\sup_n \max_{k,l=1,\dots,n} \int_{-\infty}^{\infty} \left[p_{kl}^{(n)}(x)\right]^{\beta} \mathrm{d}x \le c < \infty$$

or

$$\sup_n \max_{k,l=1,\dots,n} \int_{-\infty}^{\infty} \left[q_{kl}^{(n)}(x)\right]^{\beta} \mathrm{d}x \le c < \infty, \quad \beta > 1,$$

for some $\delta > 0$ $\sup_n \max_{i,j=1,\dots,n} \mathbf{E}\left|(\xi_{ij}^{(n)} - a_{ij}^{(n)})\sqrt{n}\right|^{2+\delta} \le c < \infty$. *Then with probability one*

$$\lim_{\alpha\downarrow 0} \lim_{n\to\infty} |\mu_n(x,y) - F_{n,\alpha}(x,y)| = 0,$$

where $\mu_n(x,y) = n^{-1}\sum_{k=1}^{n} \chi\{\operatorname{Re}\lambda_k < x, \operatorname{Im}\lambda_k < y\}$, λ_k *are the eigenvalues of the matrix* Ξ, *the* V*-density* $p_{n,\alpha}(x,y) = \frac{\partial^2}{\partial x\,\partial y}F_{n,\alpha}(x,y)$ *is equal to*

$$p_{n,\alpha}(x,y) = \begin{cases} -\frac{1}{4\pi}\int_{\alpha}^{\infty}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] m_n(y,\tau)\,\mathrm{d}y; & \textit{if} \ \ t,s \notin G \\ 0, \quad \textit{if} & ,t,s \in G, \end{cases}$$

$m_n(y,\tau) = \frac{1}{n}\mathrm{Tr}\left[C_1 + AC_2^{-1}A^*\right]^{-1}$, $y > 0$, $A = \left(a_{ij}^{(n)} - \tau\delta_{ij}\right)_{i,j=1}^{n}$, $C_1 = (c_{1i}(y)\delta_{ij})_{i,j=1}^{n}$, $C_2 = (c_{2i}(y)\delta_{ij})_{i,j=1}^{n}$, $c_{2i}(y)$ *and* $c_{1i}(y)$ *satisfy the system of* V*-equations:*

$$\begin{cases} c_{1p}(y) = y + \sum_{j=1}^{n} \sigma_{pj}^{(n)} \left\{\left[C_2 + A^*C_1^{-1}A\right]^{-1}\right\}_{jj}, \\ c_{2k}(y) = 1 + \sum_{j=1}^{n} \sigma_{jk}^{(n)} \left\{\left[C_1 + AC_2^{-1}A^*\right]^{-1}\right\}_{jj}, \end{cases}$$

$p,k = 1,\dots,n$, *there exist unique solution of this system of equation in the class of real positive analytic function in* $y > 0$ *and the* V*-region is equal to*

$$G = \left\{t,s:\ \limsup_{\alpha\downarrow 0}\limsup_{n\to\infty} |(\partial/\partial\alpha)\, m_n(\alpha,\tau)| < \infty\right\}.$$

Girko also has considered a particular special case of random matrices, when there exists simple formula for this density, when the matrix $A = (a_i\delta_{ij})_{i,j=1}^{2n}$ is a diagonal matrix with diagonal entries: $a_i = a + \mathrm{i}b$, $i = 1,\dots,n$; $a_i = -a - \mathrm{i}b$, $i = 1+n,\dots,2n$ and entries of matrix Ξ are independent, their expectation equal zero and variances equal n^{-1} [82].

Then using Theorem 2.1 we have that limit density $p(t,s)$ of n.s.f. of this matrix $A + \Xi$ equals [82]

$$p\,(t,s)=\frac{1}{\pi}\left\{1-\frac{2\left(a^2+b^2\right)}{16\left(at+bs\right)^2+1}+\frac{2\left(a^2+b^2\right)}{16\left(at+bs\right)^2+1+\left(16\left(at+bs\right)^2+1\right)^{3/2}}\right\},$$

where $t,s\in G$, $G=\left\{t,s:a^2+b^2+t^2+s^2>\left(a^2+b^2+t^2+s^2\right)^2-4\left(at+sb\right)^2\right\}$.

In particular, when $b=0$ we get

$$p\,(t,s)=\frac{1}{\pi}\left\{1-\frac{2a^2}{16a^2t^2+1}+\frac{2a^2}{16a^2t^2+1+\left(16a^2t^2+1\right)^{3/2}}\right\};\ t,s\in G \tag{2.1}$$

$$G=\left\{t,s:4a^2t^2+a^2+t^2+s^2>\left(a^2+t^2+s^2\right)^2\right\}.$$

Figures 2.1, 2.2, 2.3 and 2.4 show the shape of the V-density (2.1) for different values of a. Observe that, for $a=0$ we have a cylinder and when a approaches to 3, this cylinder splits into two cylinders similar to the original one. Those figures were obtained by using routines written in MATLAB version 5.1.0 and they are available upon request.

Figures 2.5, 2.6 and 2.7 are typical examples of the several simulations that were developed to verify V-distribution. All the simulations codes were written in S-plus language. Specifically, random variables were obtained by using S-plus standard routines as well as the matrix operations. A special program in S-plus were developed to build a two dimensional histogram that were used as nonparametric estimate of the V-distribution. The machine used to get all the numerical results was a SUN 20, 135Mhz with 64MB of memory. The execution time depends much more the sample size that the number of replications. Also the S-plus routines are available upon request.

Figure 2.1: V-density for $a=0$

Figure 2.2. V-density for $a = 0.5$

Figure 2.3: V density for $a = 1.0$

Figure 2.4: V-density for $a = 1.5$

Figure 2.5 exhibits a simulation made by 100 replications of standard normal matrices $(50)^{-1/2}\{\xi_{ij}\}_{i,j=1}^{50}$ of the dimension 50 by 50.

Figure 2.5.

Figure 2.6 shows a simulation made by 100 replications of eigenvalues of random matrices of the dimension 50×50 whose entries are independent and are uniformly distributed on the interval $[-\frac{1}{\sqrt{50}}, \frac{1}{\sqrt{50}}]$.

Figure 2.6

Figure 2.7 exhibits a simulation with 10 replications of eigenvalues of standard normal matrices $(200)^{-1/2}\{\xi_{ij}\}_{i,j=1}^{200}$ of the dimension 200 with addition of a diagonal matrix of the form $(1, \dots, 1, -1, \dots, -1)$.

Figure 2.7

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